CHAPTER-I
PRELIMINARIES

In this chapter, we collect the basic definitions and theorems which are needed for the subsequent chapters. For graph theoretic terminology, we refer to [3]. For domination concepts we refer to [4].

1.1 Basic Concepts

Definition 1.1.1: A graph is a finite non-empty set of objects called vertices or nodes together with a set of unordered pairs of distinct vertices of G, called edges or lines. The vertex set and the edge set of G are denoted by V(G) and E(G) respectively. If \( e = (u, v) \) is an edge, we write \( e = uv \), we say that ‘e’ joins the vertices u and v or u and v are adjacent vertices. If two vertices are not joined, then we say that they are non adjacent. If two distinct edges are incident with a common vertex, then they are said to be adjacent edges.

Definition 1.1.2: Two vertices of a graph which are adjacent are said to be neighbours. The set of all neighbours of a vertex v of G is called the neighbourhood set of v. It is denoted by N(v) or N[v] and they are respectively known as open and closed neighbourhood set where as \( N(u) = \{u \in V(G) / u \text{ is adjacent to } u \text{ and } u \neq v\} \) and \( N[u] = N(u) \cup \{v\} \).

Definition 1.1.3: The cardinality of the vertex set of a graph G is called the order of G and is denoted by ‘p’. The cardinality of the edge set of G is called the size of G and is denoted by ‘q’. A graph with p vertices and q edges is called a (p, q) graph. A graph G is said to be connected, if a path joins any two distinct vertices of G, otherwise G is said to be disconnected.
**Definition 1.1.4:** Let u and v be (not necessarily distinct) vertices of a graph G. A **u-v walk** of G is a finite alternating sequence \( u = u_0e_1u_1e_2 ... e_nu_n = v \), of vertices and edges beginning with vertex u and ending with vertex v such that, \( e_i = u_{i-1}u_i \), \( i = 1, 2, 3... n \). It is important to mention that the vertices need not be distinct and the same holds for the edges. The number ‘n’ is called the **length** of the walk. A graph G is have **girth** ‘g’ if its shortest cycle is of length g.

**Definition 1.1.5:** The walk is said to be **open**, if u and v are distinct vertices, it is **closed** otherwise. A walk in which all the edges are distinct is called a **trail**. A walk in which all the vertices are distinct is called a **path**. A path on p vertices is denoted by \( P_p \). A **cycle** is a closed walk in which all the vertices are distinct except \( u = v \). A **cycle** on p vertices is denoted by \( C_p \).

**Definition 1.1.6:** A graph \( G_1 \) is isomorphic to a graph \( G_2 \), if there exists a bijection \( \phi \) from \( V(G_1) \) to \( V(G_2) \) such that \( uv \in E(G_1) \) if and only if \( \phi(u)\phi(v) \in E(G_2) \). In other words, two graphs \( G_1 \) and \( G_2 \) are **isomorphic** (symbolically denoted by \( G_1 \cong G_2 \)) if there exists a one to one correspondence between their vertex sets, which preserves adjacency.

**Definition 1.1.7:** A graph H is called a **sub graph** of G, if \( V(H) \subseteq V(G) \) and \( E(H) \subseteq E(G) \). A **spanning sub graph** of G is a sub graph H with \( V(H) = V(G) \). Any sets of vertices of G, the **induced sub graph** < S > is the maximal sub graph of G with vertex set S. Thus two vertices of S are adjacent in < S > if and only if they are adjacent in G.
**Definition 1.1.8:** The **degree** of a vertex \( v \) in a graph \( G \) is the number of edges of \( G \) incident with \( v \) and is denoted by \( \text{deg}(v) \) or \( d(v) \). The minimum and maximum degrees of vertices of \( G \) are denoted by \( \delta(G) \) and \( \Delta(G) \) respectively.

**Definition 1.1.9:** A vertex of degree zero in \( G \) is called an **isolated vertex**. A vertex of degree one is called a **pendant vertex** or an **end vertex** of \( G \). Any vertex that is adjacent to a pendant vertex is called a **support**. A **full vertex** of a graph \( G \) is a vertex which is adjacent to all other vertices of \( G \). An apex graph is the graph that can be made planar by the removal of a single vertex. The deleted vertex is called an **apex** of the graph.

**Definition 1.1.10:** The edge \( e = uv \) is called an **isolated edge** if \( \text{deg} e = 0 \) and **pendant edge** if either \( u \) or \( v \) is pendant vertex not both. A maximal connected subgraph of a graph \( G \) is called a **component** of \( G \).

**Definition 1.1.11:** Let \( G_1 = (V_1, E_1) \) and \( G_2 = (V_2, E_2) \) be any two graphs. The **union** of \( G_1 \) and \( G_2 \) is the graph \( G = G_1 \cup G_2 \) with vertex set \( V = V_1 \cup V_2 \) and edge set \( E = E_1 \cup E_2 \). The **join** of \( G_1 \) and \( G_2 \) is the graph \( G = G_1 + G_2 \) with vertex set \( V = V_1 \cup V_2 \) and edge set \( E = E_1 \cup E_2 \cup \{uv : u \in V_1, v \in V_2\} \).

**Definition 1.1.12:** A **subdivision** of an edge \( e = uv \) of a graph \( G \) is the replacement of the edge \( e \) by a path \((u, w, v)\). The graph obtained from \( G \) by subdividing each edge of \( G \) exactly once is called the **subdivision graph** of \( G \) denoted by \( S(G) \).

**Definition 1.1.13:** A subset \( M \) of \( E \) is called a **matching** in \( G \) if its elements are links and no two are adjacent in \( G \); the two ends of an edge in \( M \) are said to be matched under \( M \).
**Definition 1.1.14:** A vertex ‘v’ of a graph G is called a **cut-vertex** of a graph G, if the removal of ‘v’ increases the number of components. An edge ‘e’ of a graph G is called a **cut-edge or bridge** if the removal of ‘e’ increases the number of components. A **block** of a graph is a maximal connected non-trivial sub graph without cut-vertices.

**Definition 1.1.15:** The **connectivity** $\kappa$ of a graph G is the minimum number of vertices whose removal results in a disconnected graph or the trivial graph.

**Definition 1.1.16:** A colouring of a simple graph is the assignment of a colour to each vertex of the graph so that no two adjacent vertices are assigned the same colour. The **chromatic number** of a graph is the least number of colours needed for a colouring of the graph G and is denoted by $\chi(G)$.

**Definition 1.1.17:** A graph G is **regular** of degree ‘r’ if and only if every vertex of G has degree r. Such graphs are called **r-regular** graphs. Any 3-regular graph is called a **cubic** graph.

**Definition 1.1.18:** A graph G is **complete** if every pair of its vertices is adjacent. A **complete graph** on p vertices is denoted by $K_p$. A **clique** of a graph G is a maximal complete sub graph of G.

**Definition 1.1.19:** A **bipartite graph** is a graph whose vertex V (G) can be partitioned into two non-empty subsets $V_1$ and $V_2$ such that every edge of G has one end in $V_1$ and other end in $V_2$; ($V_1, V_2$) is called a **bipartition** of G.

**Definition 1.1.20:** A graph is **acyclic**, if it has no cycles. A **tree** is a connected acyclic graph. A spanning sub graph of G, which is a tree, is called a **spanning tree** of G. A **Caterpillar** is a tree in which all the vertices are within distance 1 of a central path. That is, the removal of its end vertices leaves a path.
Definition 1.1.21: Every vertex of \( V_1 \) is joined to every vertex of \( V_2 \), and then \( G \) is called a **complete bipartite** graph. The complete bipartite graph with bipartition \((V_1, V_2)\) such that \(|V_1| = m\) and \(|V_2| = n\) is denoted by \( K_{m,n} \). A **star** is a complete bipartite graph \( K_{1,p} \). A graph \( G \) is **claw-free** if it does not contain any induced subgraph isomorphic to \( K_{1,3} \).

Definition 1.1.22: For \( p \geq 4 \), a **wheel** \( W_p \) is defined to be the graph \( K_1 + C_{p-1} \). By attaching a pendant edge at each vertex of the \( p \)-cycle forms a **helm graph**, a **closed helm** is the graph obtained from a helm by joining each pendant vertex to form a cycle.

Definition 1.1.23: The **friendship graph**, denoted by \( F_p \) can be constructed by identifying \( p \) copies of the cycle \( C_3 \) at a common vertex. By joining each pendant vertex to the central vertex of the helm forms a **flower graph**. By adding pendant edges to the central vertex of the wheel forms a **sunflower graph**. The **Lotus inside a circle** \( LC_p \) is a graph obtained from the cycle \( C_p \) and a star \( K_{1,p} \) with central vertex \( v_0 \) and the end vertices \( v_1v_2 \ldots v_n \) by joining each \( v_i \) to \( u_i \) and \( u_{i+1} \) (mod \( n \)). The graph \( P_p + 2K_1 \) is called a **double fan** \( DF_p \). **Crown graph** is equivalent to the complete bipartite graph with horizontal edges removed.

![Figure 1.1 Lotus inside circle](image1)  
**LC_4**  
![Figure 1.2 Crown graph](image2)  
**Cr_{3,3}**  
![Figure 1.3 Double fan](image3)  
**DF_4**
**Definition 1.1.24:** Prism graph is the graph obtained by the Cartesian product of \( C_p \) and \( K_2 \) and is denoted by \( Y_p \). Crossed prism is the graph obtained by the taking two disjoint cycle graphs \( C_p \) and adding edges \((v_{k},v_{2k+1})\)and \((v_{k+1},v_{2k})\) for \( k = 1, 3, 5 \ldots p – 1 \) and is denoted by \( CY_p \).

**Definition 1.1.25:** Book graph is the graph obtained by the Cartesian product of \( K_{1,p} \) and \( K_2 \). That is, consists of \( p \) quadrilaterals sharing common edge and is denoted by \( B_p \). Two copies of \( K_{1,p} \) are connected a path is bi star graph. The Turán graph \( T(n,r) \) is a complete multipartite graph formed by partitioning a set of \( n \) vertices into \( r \) subsets, with sizes as equal as possible, and connecting two vertices by an edge whenever they belong to different subsets.

**Definition 1.1.26:** A graph is said to be a power of cycle, denoted by \( C_n^k \), if

\[
V(C_n^k) = (v_0 (= v_n), v_1, v_2, \ldots v_{n-1}) \quad \text{and} \quad E(C_n^k) = E^1 \cup E^2 \cup \ldots \cup E^k, \]

where

\[
E^j = v_j, v_{(j+i) \mod n} : 0 \leq j \leq n - 1 \quad \text{and} \quad 1 \leq k \leq \left\lfloor \frac{n-1}{3} \right\rfloor.
\]

The \((m, n)\) Lollipop graph is obtained by joining \( K_m \) to the path \( P_n \) with the bridge denoted by \( L_{m,n} \). The \( n\)-barbell graph is the simple graph obtained by connecting two copies of a complete graph \( K_n \) by a bridge. Tadpole graph is obtained by joining a cycle graph \( C_m \) to a path \( P_n \) with the bridge denoted by \( T_{m,n} \).

![Figure 1.4 Power of cycle \( C_8^3 \)](image)

![Figure 1.5 Lollipop graph \( L_{6,3} \)](image)
Definition 1.1.27: The **gear graph**, also sometimes known as a bipartite wheel graph is a wheel graph with a graph vertex added between each pair of adjacent graph vertices of the outer cycle. The gear graph $G_p$ has $2p + 1$ vertices and $3p$ edges. **Cocktail graph** is the graph consisting of two rows of paired nodes in which all nodes but the paired ones are connected with a graph edge. It is the graph complement of the ladder graph where as ladder graph is the Cartesian product of $P_2$ and $P_p$.

Definition 1.1.28: **Triangular snake** is a connected graph in which all blocks are triangles and the block-cut-point graph is a path and is denoted by $T_{Sp}$. Addition of edges in the triangular graph is denoted by $T_{Sp} + (p – 1)e$. The **Goldner – Harary** is a simple undirected graph with 11 vertices and 27 edges. The Moser spindle is the 7-node unit distance graph.
Definition 1.1.29: The windmill graph $W_d(p,n)$ is an undirected graph constructed for $p \geq 2$ and $n \geq 2$ by joining $n$ copies of the complete graph $K_p$ at a shared vertex.

Definition 1.1.30: Let $G$ be a graph. For each vertex $v$ of a graph $G$, take a new vertex $u$. Join $u$ to those vertices of $G$ adjacent to $v$. The graph thus obtained is called the splitting graph of $G$ and thus $N(u) = N(v)$ and is denoted by $Spl(G)$.

The $m$-splitting graph, $Spl_m(G)$ of a graph $G$ is obtained by adding to each vertex $v$ of $G$ new $m$ vertices, say $v_1, v_2, v_3, ..., v_m$, such that $v_i, 1 \leq i \leq m$ is adjacent to every vertex, that is, adjacent to $v$ in $G$. By definitions, the 2-shadow graph is the known shadow graph $D_2(G)$ and the 1-splitting graph is the known splitting graph.

![Figure 1.13 Splitting graph of $P_3^+$](image)

![Figure 1.14 Splitting graph of $Spl(P_3)$](image)

Definition 1.1.31: The shadow graph $D_2(G)$ of a connected graph $G$ is constructed by taking two copies of $G$ say $G_1$ and $G_2$. Join each vertex $u$ in $G_1$ to the neighbours of the corresponding vertex $v$ in $G_2$. The $m$-shadow graph, $D_m(G)$ of a connected graph $G$ is constructed by taking $m$ copies of $G$, say $G_1, G_2, G_3, ..., G_m$ then join each vertex $u$ in $G_i$ to the neighbours of the corresponding vertex $v$ in $G_j$, $1 \leq i, j \leq m$.

![Figure 1.15 Shadow graph of path $D_3(P_4)$](image)
**Definition 1.1.32:** The corona $G_1 \odot G_2$ is defined as the graph $G$ obtained by taking one copy of $G_1$ of order $p_1$ and $p_1$ copies of $G_2$ and then joining the $i^{th}$ vertex of $G_1$ to every vertex in the $i^{th}$ copy of $G_2$.

**Definition 1.1.33:** A two dimensional grid graph is an $m \times n$ graph $G_{m,n}$ is the Cartesian product, $P_m \square P_n$ of path on $m$ and $n$ vertices. The generalized grid graph is denoted by $G_{m,n,...}$. Special cases include the path $P_p = G_{p,1}$, ladder graph $L_p = G_{2,p}$, that is, Cartesian product of $P_2$ and $P_p$ is called ladder graph. Cycle $C_4 = G_{2,2}$. A grid graph $G_{p,q}$ has $pq$ vertices and $2pq - m - n$ edges. The grid graph $G_{2,3} = P_2 \square P_3$ is called domino graph or $L_3$.

**Definition 1.1.34:** The **horizontal merging** of two cycles gives a complete grid graph $P_2 \square P_3 = G_{2,3} = G_{2,2} \sqcup G_{2,2}$. By the same way, the vertical merging of two cycles gives $P_3 \square P_2 = G_{3,2} = G_{2,2} \sqcup G_{2,2}$. For the complete grid graph $P_m \square P_n$, if $m = n = 2$, $G_{2,2} = P_2 \square P_2$ is a cycle with 4 vertices. Let $C_1$ and $C_2$ be two cycles each having 4 vertices. Suppose $C_1$ has vertex set $(u_1, u_2, u_3, u_4)$ and $C_2$ has vertex set $(v_1, v_2, v_3, v_4)$ then **H-merging** of $C_1$ and $C_2$ having the vertex $(u_1, u_2 = v_1, u_3, u_4 = v_3, v_2, v_4)$ edge set including all the edges $C_1$ and $C_2$ and $(u_2, u_4) = (v_1, v_3)$. The same way is used for finding **V-merging**. From H-merging and V-merging, $G_{2,3} = G_{2,2} \sqcup G_{2,2}$ and $G_{3,2} = G_{2,2} \sqcup G_{2,2}$. Generally, $G_{m,p} \square G_{n,p} = G_{m+n-1, p}$ or $G_{p,m} \square G_{p,n} = G_{p, m+n-1}$.

**Definition 1.1.35:** For a given graph $G = (V, E)$, by subdividing each edge exactly once and joining all the non adjacent vertices of $G$, the graph obtained by this process is called **central graph** and is denoted by $C(G)$.

Here, the vertex set of $K_{1,p}$ is $(v_1, v_2, ..., v_p)$ and vertices are indicated within square box.
Definition 1.1.36: The middle graph of $G$ is denoted by $M(G)$ is defined as, $M(G)$ is a graph whose vertex set is $V(G) \cup E(G)$ and in which two vertices are adjacent if and only if either they are adjacent edges of $G$ or one is a vertex of $G$ and the other an edge incident with it.

Definition 1.1.37: The total graph $T(G)$ of a graph $G$ is the graph whose vertex set is $V(G) \cup E(G)$ and two vertices are adjacent whenever they are either adjacent or incident in $G.$
**Definition 1.1.38:** Double star $K_{1,p,p}$ is a tree obtained from the star $K_{1,p}$ by adding a new pendant edge of the existing $n$ pendant vertices. It has $2p + 1$ vertices and $2p$ edges. Let $V(K_{1,p,p}) = (v_0) \cup (v_1, v_2, ..., v_p) \cup (u_1, u_2, ..., u_p)$ and $E(K_{1,p,p}) = (e_1, e_2, ..., e_p) \cup (s_1, s_2, s_3, ..., s_p)$.

![Figure 1.19 Double graph of total graph of star graph of $K_{1,p}$: $T(K_{1,p,p})$](image)

**Definition 1.1.39:** The Square of a graph $G$ is denoted by $G^2$ has the same vertices as in $G$ and the two vertices $u$ and $v$ in vertex of $G$ are joined in $G^2$ if and only if they are joined in $G$ by a path of length one or two. For example, the square graph of $C_6$ is given by,

![Figure 1.20 A graph and its Square graph](image)
Definition 1.1.40: With every non empty ordinary graph there is associated a graph \( L(G) \), called the line graph of \( G \) whose points are in one-to-one correspondence with the lines of \( G \) and such that two points are adjacent in \( L(G) \) if and only if the corresponding lines of \( G \) are adjacent. For example, the line graph of \( K_4 \) is given by,

![Figure 1.21 Line graph of \( K_4 \)](image)

1.2 Domination Concepts

Definition 1.2.1[4]: A set \( S \subseteq V(G) \) is said to be a dominating set in \( G \), if every vertex in \( V - S \) is adjacent to at least one vertex in \( S \). The domination number of \( G \) is the minimum cardinality taken over all dominating sets in \( G \) and is denoted by \( \gamma(G) \).

Definition 1.2.2[13]: A graph \( G \) is said to be triple connected if any three vertices lie on a path in \( G \). If \( S \) is a dominating set and \( < S > \) is triple connected then \( S \) said to be triple connected dominating set of \( G \). The minimum cardinality taken over all triple connected dominating sets is called the triple connected domination number and is denoted by \( \gamma_{tc}(G) \).

Definition 1.2.3[34]: A dominating set is said to be restrained dominating set if every vertex in \( V - S \) is adjacent to at least one vertex in \( S \) as well as another vertex in \( V - S \). The minimum cardinality taken over all restrained dominating sets is called the restrained domination number and is denoted by \( \gamma_r(G) \).
**Definition 1.2.4[21]:** A set $S \subseteq V(G)$ is called a **[1, 2] dominating set**, if every vertex $v \in V - S$, $1 \leq |N(v) \cap S| \leq 2$, that is, every vertex in $V - S$ is adjacent to at least one vertex but not more than two vertices in $S$. The minimum cardinality taken over all [1,2] dominating set is called [1,2] domination number and is denoted by $\gamma_{[1,2]}(G)$.

**Definition 1.2.5[33]:** A set $S \subseteq V(G)$ is a **strong dominating set** of $G$ if for every vertex $y \in V(G) - S$ there is a vertex $y \in S$ with $xy \in E(G)$ and $d(y, G) \geq d(x, G)$. The strong domination number $\gamma_s(G)$ is defined as the minimum cardinality of a strong dominating set.

**Definition 1.2.6[33]:** A set $S \subseteq V(G)$ is a **weak dominating set** of $G$ if for every vertex $y \in V(G) - S$ there is a vertex $x \in S$ with $xy \in E$ and $d(y, G) \geq d(x, G)$. The weak domination number $\gamma_w(G)$ is defined as the minimum cardinality of a weak dominating set.

**Definition 1.2.7[4]:** A dominating set $S \subseteq V(G)$ is a **connected dominating set** if the induced subgraph $<S>$ is connected. The **connected domination number** $\gamma_c(G)$ of a graph $G$ is the minimum cardinality of a connected dominating set.

**Theorem 1.3.8[31]:** A tree is triple connected graph $G$ if and only if $T \cong P_p, P \geq 3$.

**Theorem 1.3.9[4]:** For any Graph $G$, $\chi(G) \leq \Delta(G) + 1$.

**Theorem 1.3.10[4]:** If $G$ is a graph of order $p$, with maximum degree $\Delta$, then $\gamma \geq \lfloor p/(\Delta + 1) \rfloor$

**Definition 1.2.11[37]:** The total number of vertices that dominates all the pairs of vertices and evaluates the average of this value is called **medium domination number**.
For $G = (V, E)$ and for all $u,v \in V(G)$, if $u$ and $v$ are adjacent they dominate each other then $\text{dom}(u,v) = 1$. The total number of vertices that dominate every pair of vertices is defined by $\text{TDV}(G) = \sum \text{dom}(u,v)$, for all $u,v \in V(G)$. For any connected graph $G$ of order $n$, then the **medium domination number** of $G$ is defined as

$$\text{MD}(G) = \frac{\text{TDV}(G)}{nC_2}.$$

**Theorem 1.2.12[37]:** For $G$ has $n$ vertices, $q$ edges and for $\deg(v_i) \geq 2$,

$$\text{TDV}(G) = q + \left\{ \sum_{u \in V} \binom{\deg(v_i)}{2} \right\}.$$