Conharmonic Curvature Tensor on K-Contact Manifolds

By

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Abstract

The object of the present paper is to characterize K-contact manifolds satisfying certain curvature conditions on the conharmonic curvature tensor.

Keywords and Phrases: K-contact manifold, conharmonic curvature tensor, Quasi-conharmonically flat manifold, ξ-conharmonically flat manifold, ϕ-conharmonically flat manifold.

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1. Introduction

Let $M$ and $\tilde{M}$ be two Riemannian manifolds with $g$ and $\tilde{g}$ being their respective metric tensors related through the equation

$$\tilde{g}(X,Y) = e^{2\sigma}g(X,Y),$$

(1.1)

where $\sigma$ is a real function. Then $M$ and $\tilde{M}$ are called conformally related manifolds and the correspondence between $M$ and $\tilde{M}$ is known as conformal transformation [2].

It is known that a harmonic function is defined as a function whose Laplacian vanishes. In general a harmonic function is not transformed in a harmonic function. The condition under which a harmonic function remains invariant have been studied by Ishii [11] who introduced the conharmonic transformation as a subgroup of the conformal transformation (1.1) satisfying the condition

$$\sigma^i_{\alpha} + \sigma^i_{\beta} \sigma^\beta_{\alpha} = 0,$$

(1.2)

where 'comma' denotes the covariant differentiation with respect to the metric $g$.

A rank four tensor $\tilde{C}$ that remains invariant under conharmonic transformation for an n-dimensional Riemannian manifold $M^n$, is given by

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\[ \tilde{C}(X, Y, Z, W) = \tilde{R}(X, Y, Z, W) \]
\[ + \frac{1}{2} \{ g(Y, Z)S(X, W) - g(X, Z)S(Y, W) \} \]
\[ + S(Y, Z)g(X, W) - S(X, Z)g(Y, W) \]  \hspace{1cm} (1.3)

where \( \tilde{R} \) denotes the Riemannian curvature tensor of type (0,4) defined by
\[ \tilde{R}(X, Y, Z, W) = g(R(X, Y)Z, W), \]
where \( R \) is the Riemannian curvature tensor of type (1,3) and \( S \) denotes Ricci tensor of type (0,2) respectively.

The curvature tensor defined by the equation (1.3) is known as conharmonic curvature tensor. A manifold whose conharmonic curvature vanishes at every point of the manifold is called conharmonically flat manifold. Thus this tensor represents the deviation of the manifold from conharmonic flatness. It satisfies all the symmetric properties of the Riemannian curvature tensor \( R \). There are many physical applications of the tensor \( C \). For example, in [1], Abdussattar showed that sufficient condition for a space-time to be conharmonic to a flat space-time is that the tensor \( C \) vanishes identically. A conharmonically flat space-time is either empty in which case it is flat or filled with a distribution represented by energy momentum tensor \( T \) possessing the algebraic structure of an electromagnetic field and conformal to a flat space-time [1]. Also he described the gravitational field due to a distribution of pure radiation in presence of disordered radiation by means of spherically symmetric conharmonically flat space-time. Conharmonic curvature tensor have been studied by S. A. Siddiqui and Z. Ahsan [16], C. Özgür [13] and many others. Motivated by the above studies we like to study the properties of conharmonic curvature tensor in a K-contact manifold.

The paper is organised as follows:

After preliminaries in section 3, we consider conharmonically flat K-contact manifold and prove that a compact conharmonically flat K-contact manifold is Sasakian with vanishing scalar curvature. In the next section we have studied quasi-conharmonically flat, \( \xi \)-conharmonically flat and \( \phi \)-conharmonically flat K-contact manifolds. We prove that a compact quasi-conharmonically flat K-contact manifold is Sasakian with vanishing scalar curvature. For \( \xi \)-conharmonically flat K-contact manifolds the scalar curvature vanishes. Finally for compact \( \phi \)-conharmonically flat K-contact manifolds we obtain similar result as obtained for compact quasi-conharmonically flat K-contact manifolds.
2. Preliminaries

By a contact manifold we mean a \((2n + 1)\)-dimensional differentiable manifold \(M^{2n+1}\) which carries a global 1-form \(\eta\), there exists a unique vector field \(\xi\), called the characteristic vector field such that, \(\eta(\xi) = 1\) and \(d\eta(\xi, X) = 0\). A Riemannian metric \(g\) on \(M^{2n+1}\) is said to be an associated metric if there exists a \((1,1)\) tensor field \(\phi\) such that

\[
d\eta(X, Y) = g(X, \phi Y), \quad \eta(X) = g(X, \xi), \quad \phi^2 = -I + \eta \otimes \xi. \tag{2.1}
\]

From these equations we have

\[
\phi \xi = 0, \quad \eta \circ \phi = 0, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y). \tag{2.2}
\]

The manifold \(M\) equipped with the contact structure \((\phi, \xi, \eta, g)\) is called a contact metric manifold. If \(\xi\) is a Killing vector field, then \(M^{2n+1}\) is said to be a K-contact manifold\[4\], \[5\], \[14\]. K-contact manifolds have been studied by several authors such as S. Tanno \[17\], \[18\], \[19\], S. Sasaki \[14\], \[15\], Y. Hatakeyama, Y. Ogawa and S. Tanno \[10\], M. C. Chaki and D. Ghosh \[7\], U. C. De and S. Biswas\[9\] and many others.

A contact metric structure is said to be normal(Sasakian) if the almost complex structure \(J\) on \(M \times \mathbb{R}\) defined by \(J(X, f\frac{\partial}{\partial f}) = (\phi X - f\xi, \eta(X)\frac{\partial}{\partial f})\), \(f\) being a function on \(M^{2n+1}\), is integrable. A contact metric manifold is Sasakian if and only if

\[
R(X, Y)\xi = \eta(Y)X - \eta(X)Y. \tag{2.3}
\]

Every Sasakian manifold is K-contact, but the converse need not be true, except in dimension three \[12\]. K-contact metric manifold are not too well known, because there is no such a simple expression for the curvature tensor as in the case of Sasakian manifold. For details we refer to \[4\], \[12\], \[14\].

Besides the above relations in K-contact manifold the following relations hold \[4\], \[12\], \[14\]:

\[
\nabla_X \xi = -\phi X. \tag{2.4}
\]

\[
\tilde{R}(\xi, X, Y, \xi) = \eta(R(\xi, X)Y) = g(X, Y) - \eta(X)\eta(Y). \tag{2.5}
\]

\[
R(\xi, X)\xi = -X + \eta(X)\xi; \tag{2.6}
\]

\[
S(X, \xi) = 2n\eta(X). \tag{2.7}
\]

\[
(\nabla_X \phi)Y = R(\xi, X)Y, \tag{2.8}
\]

for any vector fields \(X, Y\).
Again a K-contact manifold is called Einstein if the Ricci tensor $S$ is of the form $S = \lambda g$, where $\lambda$ is a constant and $\eta$--Einstein if the Ricci tensor $S$ is of the form $S = a g + b \eta \otimes \eta$, where $a$, $b$ are smooth functions on $M$. It is well known [12] that in a K-contact manifold $a$ and $b$ are constants. Also it is known [6] that a compact $\eta$-Einstein $K$--contact manifold is Sasakian provided $a \geq -2$.

In a $(2n + 1)$-dimensional almost contact metric manifold, if $\{e_1, ..., e_{2n}, \xi\}$ is a local orthonormal basis of vector fields, then $\{\phi e_1, ..., \phi e_{2n}, \xi\}$ is also a local orthonormal basis. It is easy to verify that

$$\sum_{i=1}^{2n} g(e_i, e_i) = \sum_{i=1}^{2n} g(\phi e_i, \phi e_i) = 2n.$$  \hspace{1cm} (2.9)

$$\sum_{i=1}^{2n} g(e_i, Z)S(Y, e_i) = \sum_{i=1}^{2n} g(\phi e_i, Z)S(Y, \phi e_i) = S(Y, Z) - S(Y, \xi)\eta(Z),$$  \hspace{1cm} (2.10)

for $Y, Z \in T(M)$. In particular in view of $\eta \circ \phi = 0$, we get

$$\sum_{i=1}^{2n} g(\phi e_i, Z)S(Y, e_i) = \sum_{i=1}^{2n} g(\phi e_i, \phi Z)S(Y, \phi e_i) = S(Y, \phi Z),$$  \hspace{1cm} (2.11)

for $Y, Z \in T(M)$. If $M$ is a K-contact manifold then it is known that

$$R(X, \xi)\xi = X - \eta(X)\xi, \quad X \in T(M)$$  \hspace{1cm} (2.12)

and

$$S(\xi, \xi) = 2n.$$  \hspace{1cm} (2.13)

Moreover $M$ is Einstein if and only if

$$S = 2ng.$$  \hspace{1cm} (2.14)

From (2.13), we get

$$\sum_{i=1}^{2n} S(e_i, e_i) = \sum_{i=1}^{2n} S(\phi e_i, \phi e_i) = r - 2n.$$  \hspace{1cm} (2.15)

where $r$ is the scalar curvature. In a K-contact manifold we also have

$$\tilde{R}(\xi, Y, Z, \xi) = g(\phi Y, \phi Z), \quad Y, Z \in T(M).$$  \hspace{1cm} (2.16)

Consequently

$$\sum_{i=1}^{2n} \tilde{R}(e_i, Y, Z, e_i) = \sum_{i=1}^{2n} \tilde{R}(\phi e_i, Y, Z, \phi e_i) = S(Y, Z) - g(\phi Y, \phi Z).$$  \hspace{1cm} (2.17)

For more details we refer to [5].
3. Conharmonically flat K-contact manifold

In this section we study conharmonically flat K-contact manifold.

**Definition 3.1.** A K-contact manifold is said to be conharmonically flat if
\[ g(\tilde{C}(X,Y)Z,W) = 0. \]  

Let a \((2n+1)\)-dimensional K-contact manifold \(M\) be conharmonically flat. Then using (3.1) in (1.3) we have
\[ R(X,Y,Z,W) = \sum_{i,j=1}^{2n} [S(Y,Z)g(X,W) - S(X,Z)g(Y,W) \right. \]
\[ + S(Y,W)g(X,Z) - S(X,W)g(Y,Z)]. \]  

If \(\{e_1, \ldots, e_{2n}, \xi\}\) is a local orthonormal basis of vector fields in \(M\), then putting \(X = W = e_i\) in (3.2) and summing up from \(i = 1\) to \(2n\) we have
\[ \sum_{i=1}^{2n} \tilde{R}(e_i, Y, Z, e_i) = \frac{1}{2n-1} \left[ \sum_{i=1}^{2n} S(Y,Z)g(e_i, e_i) \right. \]
\[ - \sum_{i=1}^{2n} S(e_i, Z)g(Y, e_i) + \sum_{i=1}^{2n} S(e_i, X)g(Y, Z) \right. \]
\[ - \sum_{i=1}^{2n} S(Y, e_i)g(e_i, Z)] \]  

Using (2.1), (2.9), (2.10), (2.15) and (2.17) in (3.3) we get
\[ S(Y,Z) = \frac{1}{2n-1} \left[ (2n-2)S(Y,Z) \right. \]
\[ + (r-2n)g(Y,Z) \right. \]
\[ + S(Z, \xi)\eta(Y) - S(Y, \xi)\eta(Z)]. \]  

Using (2.2) and (2.7) in (3.4) we obtain
\[ S(Y,Z) = (r-1)g(Y,Z) + (2n+1)\eta(Y)\eta(Z). \]  

In view of (3.5) we have the following:

**Proposition 3.1.** A conharmonically flat K-contact manifold is an \(\eta\)-Einstein manifold.

Putting \(Y = Z = \xi\) in (3.5) we have
\[ r = 0. \]  

Now using \(r = 0\) in (3.5), we get
\[ S(Y,Z) = -g(Y,Z) + (2n+1)\eta(Y)\eta(Z). \]  

It is known [6] that a compact \(K\)-contact \(\eta\)-Einstein manifold with \(\alpha \geq -2\) is Sasakian. So in view of (3.6) and (3.7) we state the following:
Theorem 3.1. A compact conharmonically flat K-contact manifold is Sasakian with vanishing scalar curvature.

4. Some structure theorems

In this section we study quasi-conharmonically flat, $\xi$-conharmonically flat and $\phi$-conharmonically flat K-contact manifolds.

Let $C$ be the Weyl conformal curvature tensor of a $(2n + 1)$-dimensional manifold $M$. Since at each point $p \in M$ the tangent space $T_p(M)$ can be decomposed into the direct sum $T_p(M) = \phi(T_p(M)) \oplus L(\xi_p)$, where $L(\xi_p)$ is an 1-dimensional linear subspace of $T_p(M)$ generated by $\xi_p$. Then we have a map:

$$C : T_p(M) \times T_p(M) \times T_p(M) \rightarrow \phi(T_p(M)) \oplus L(\xi_p).$$

It may be natural to consider the following particular cases:

1. $C : T_p(M) \times T_p(M) \times T_p(M) \rightarrow L(\xi_p)$, i.e., the projection of the image of $C$ in $\phi(T_p(M))$ is zero.

2. $C : T_p(M) \times T_p(M) \times T_p(M) \rightarrow \phi(T_p(M))$, i.e., the projection of the image of $C$ in $L(\xi_p)$ is zero.

3. $C : \phi(T_p(M)) \times \phi(T_p(M)) \times \phi(T_p(M)) \rightarrow L(\xi_p)$, i.e., when $C$ is restricted to $\phi(T_p(M)) \times \phi(T_p(M)) \times \phi(T_p(M))$, the projection of the image of $C$ in $\phi(T_p(M))$ is zero. This condition is equivalent to

$$\phi^2 C(\phi X, \phi Y, \phi Z) = 0.$$  

The cases (1) and (2) were considered in [8] and [20] respectively. The case (3) was considered in [21] for the case $M$ is a K-contact manifold. Furthermore in [3], the authors studied $(k, \mu)$-contact metric manifolds satisfying (3). Now we define the following:

Definition 4.1. A K-contact manifold is said to be quasi-conharmonically flat if

$$g(\bar{C}(X, Y)Z, \phi W) = 0.$$  

Definition 4.2. A K-contact manifold is said to be $\xi$-conharmonically flat if

$$\bar{C}(X, Y) \xi = 0.$$  

Definition 4.3. A K-contact manifold is said to be $\phi$-conharmonically flat if

$$g(\bar{C}(\phi X, \phi Y)Z, \phi W) = 0.$$
where $X, Y, Z, W \in T(M)$.

Let us first prove the following Lemma:

Lemma 4.1. Let $M$ be a $(2n+1)$-dimensional K-contact manifold. If $M$ satisfies

$$g(\tilde{C}(\phi X,Y)Z, \phi W) = 0 \quad \text{for all } X, Y, Z, W \in T(M).$$

(4.4)

Then $M$ is a Sasakian manifold with vanishing scalar curvature.

Proof. Let $M$ be a $(2n+1)$-dimensional K-contact manifold satisfying (4.4). Then using (4.4) in (1.3) we obtain

$$\tilde{R}(\phi X,Y,Z,\phi W) = \frac{1}{2n-1} S(Y,Z) g(\phi X,\phi W) - S(\phi X,Z) g(Y,\phi W) + S(\phi X,\phi W) g(Y,Z) - S(Y,\phi W) g(\phi X,Z).$$

(4.5)

Let $\{e_1, \ldots, e_{2n+1}\}$ be a local orthonormal basis of vector fields in $M$. Putting $X = W = e_i$ and summing up over 1 to $2n$ both sides of (4.5) we have

$$\sum_{i=1}^{2n} \tilde{R}(\phi e_i,Y,Z,\phi e_i) = \frac{1}{2n-1} \left[ \sum_{i=1}^{2n} S(Y,Z) g(\phi e_i,\phi e_i) - \sum_{i=1}^{2n} S(\phi e_i,Z) g(Y,\phi e_i) + \sum_{i=1}^{2n} S(\phi e_i,\phi e_i) g(Y,Z) - \sum_{i=1}^{2n} S(Y,\phi e_i) g(\phi e_i,Z) \right].$$

(4.6)

Using (2.9), (2.10), (2.15) and (2.17) in (4.6) we get

$$S(Y,Z) - g(\phi Y,\phi Z) = \frac{1}{2n-1} \left[ 2n S(Y,Z) - S(Y,Z) + S(Z,\xi) \eta(Y) + (r - 2n) g(Y,Z) - S(Y,Z) + S(Y,\xi) \eta(Z) \right].$$

(4.7)

Using (2.2) and (2.7) in (4.7) we obtain

$$S(Y,Z) = (r - 1) g(Y,Z) + (2n + 1) \eta(Y) \eta(Z).$$

(4.8)

Putting $Z = \xi$ in both sides of (4.8) and using (2.1), (2.7) and $\eta(\xi) = 1$ we obtain

$$r \eta(Y) = 0.$$  

(4.9)

i.e.,

$$r = 0.$$  

(4.10)

Using $r = 0$ in (4.8) we have

$$S(Y,Z) = - g(Y,Z) + (2n + 1) \eta(Y) \eta(Z).$$

(4.11)

It is well known [6] that a compact K-contact $\eta$-Einstein manifold with $a \geq -2$ is Sasakian. Therefore the relations (4.10) and (4.11) together proves the Lemma.
Now in view of the Lemma 4.1 we state the following:

**Theorem 4.1.** A compact quasi-conharmonically flat K-contact manifold is Sasakian with vanishing scalar curvature.

Next we consider a $\xi$-conharmonically flat K-contact manifold $M$. Then using (4.2) in (1.3) we get

$$R(X, Y, \xi, W) = \frac{1}{2n-1}[S(Y, \xi)g(X, W) - S(X, \xi)g(Y, W) + S(X, W)g(Y, \xi) - S(Y, W)g(X, \xi)].$$

Let $\{e_1, \ldots, e_{2n}, \xi\}$ be a local orthonormal basis of vector fields in $M$. Putting $X = W = e_i$ in (4.12) and summing up from 1 to $2n$ we have

$$\sum_{i=1}^{2n} R(e_i, Y, \xi, e_i) = \frac{1}{2n-1} \sum_{i=1}^{2n} [S(Y, \xi)g(e_i, e_i) - S(e_i, \xi)g(Y, e_i) + S(e_i, e_i)g(Y, \xi) - S(Y, e_i)g(e_i, \xi)].$$

Using (2.9), (2.10), (2.15), (2.17) and $\eta(\xi) = 1$ in (4.13) we get

$$\eta(Y) = 0.$$ (4.14)

Which implies that

$$r = 0.$$ (4.15)

In view of (4.15) we state the following:

**Theorem 4.2.** For a $\xi$-conharmonically flat K-contact manifold the scalar curvature vanishes.

Finally we study $\phi$-conharmonically flat $K$-contact manifolds. Thus we use (3.3) in (1.3) and obtain

$$\tilde{R}(\phi X, \phi Y, \phi Z, \phi W) = \frac{1}{2n-1}[S(\phi Y, \phi Z)g(\phi X, \phi W) - S(\phi X, \phi Z)g(\phi Y, \phi W) + S(\phi X, \phi W)g(\phi Y, \phi Z) - S(\phi Y, \phi W)g(\phi X, \phi Z)].$$

Let $\{e_1, \ldots, e_{2n}, \xi\}$ be a local orthonormal basis of vector fields in $M$. Putting $X = W = e_i$ in (4.16) and summing up from 1 to $2n$ we have

$$\sum_{i=1}^{2n} \tilde{R}(\phi e_i, \phi Y, \phi Z, \phi e_i) = \frac{1}{2n-1} \sum_{i=1}^{2n} [S(\phi Y, \phi Z)g(\phi e_i, \phi e_i) - S(\phi e_i, \phi Z)g(\phi Y, \phi e_i) + S(\phi e_i, \phi e_i)g(\phi Y, \phi Z) - S(\phi Y, \phi e_i)g(\phi e_i, \phi Z)].$$
Using (2.9), (2.10), (2.15) and (2.17) in (4.17) we get

\[ S(\phi Y, \phi Z) - g(\phi^2 Y, \phi^2 Z) = \frac{2n-3}{2n-1} S(\phi Y, \phi Z) + \frac{2n}{2n-1} g(\phi Y, \phi Z). \]  

i.e.,

\[ S(\phi Y, \phi Z) = (r - 1) g(\phi Y, \phi Z). \]  

Substituting \( Y \) by \( \phi Y \) and \( Z \) by \( \phi Z \) in (4.19) and using (2.2) we have

\[ S(\phi^2 Y, \phi^2 Z) = (r - 1) g(\phi Y, \phi Z). \]  

Using (2.1) (2.2) and (2.7) in (4.20) we get

\[ S(\phi Y, \phi Z) = (r - 1) g(Y, Z) + (2n + 1 - r) \eta(Y) \eta(Z). \]  

Contacting (4.21) we have

\[ r = 0. \]  

In view of (4.21) and (4.22) we have the following:

**Proposition 4.1.** A \( \phi \)-conharmonically flat \( K \)-contact manifold is an \( \eta \)-Einstein manifold with vanishing scalar curvature.

Putting the value of \( r \) in (4.21) we have

\[ S(Y, Z) = -g(Y, Z) + (2n + 1) \eta(Y) \eta(Z). \]  

It is well known [6] that a compact \( K \)-contact \( \eta \)-Einstein manifold with \( \alpha \geq -2 \) is Sasakian. Therefore in view of (4.23) we have the following:

**Theorem 4.3.** A compact \( \phi \)-conharmonically flat \( K \)-contact manifold is a Sasakian manifold with vanishing scalar curvature.

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Research Article

Conharmonic Curvature Tensor on $N(K)$-Contact Metric Manifolds

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The object of the present paper is to characterize $N(k)$-contact metric manifolds satisfying certain curvature conditions on the conharmonic curvature tensor. In this paper we study conharmonically symmetric, $\phi$-conharmonically flat, and $\xi$-conharmonically flat $N(k)$-contact metric manifolds.

1. Introduction

Let $M$ and $\overline{M}$ be two Riemannian manifolds with $g$ and $\overline{g}$ being their respective metric tensors related through

$$\overline{g}(X, Y) = e^{2\sigma} g(X, Y),$$

(1.1)

where $\sigma$ is a real function. Then $M$ and $\overline{M}$ are called conformally related manifolds and the correspondence between $M$ and $\overline{M}$ is known as conformal transformation [1].

It is known that a harmonic function is defined as a function whose Laplacian vanishes. A harmonic function is not invariant, in general. The condition under which a harmonic function remains invariant have been studied by Ishii [2] who introduced the conharmonic transformation as a subgroup of the conformal transformation (1.1) satisfying the condition

$$\sigma'' + \sigma' \sigma' = 0,$$

(1.2)

where comma denotes the covariant differentiation with respect to the metric $g$.
A rank-four tensor $\tilde{C}$ that remains invariant under conharmonic transformation for a $(2n + 1)$-dimensional Riemannian manifold $M$ is given by

$$
\tilde{C}(X, Y, Z, W) = \tilde{R}(X, Y, Z, W)
$$

where $\tilde{R}$ denotes the Riemannian curvature tensor of type $(0, 4)$ defined by

$$
\tilde{R}(X, Y, Z, W) = g(\tilde{R}(X, Y)Z, W),
$$

(1.4)

where $R$ is the Riemannian curvature tensor of type $(1, 3)$ and $S$ denotes Ricci tensor of type $(0, 2)$, respectively.

The curvature tensor defined by (1.3) is known as conharmonic curvature tensor. A manifold whose conharmonic curvature vanishes at every point of the manifold is called conharmonically flat manifold. Thus this tensor represents the deviation of the manifold from conharmonic flatness. It satisfies all the symmetric properties of the Riemannian curvature tensor $R$.

There are many physical applications of the tensor $\tilde{C}$. For example, in [3], Abdussattar showed that sufficient condition for a space-time to be conharmonic to a flat space-time is that the tensor $\tilde{C}$ vanishes identically. A conharmonically flat space-time is either empty in which case it is flat or filled with a distribution represented by energy momentum tensor $T$ possessing the algebraic structure of an electromagnetic field and conformal to a flat space-time [3]. Also he described the gravitational field due to a distribution of pure radiation in presence of disordered radiation by means of spherically symmetric conharmonically flat space-time. Conharmonic curvature tensor have been studied by Siddiqi and Ahsan [4], Özyürt [5], and many others.

Let $M$ be an almost contact metric manifold equipped with an almost contact metric structure $(\phi, \xi, \eta, g)$. At each point $p \in M$, decompose the tangent space $T_pM$ into direct sum $T_pM = \phi(T_pM) \oplus \{\xi_p\}$, where $\{\xi_p\}$ is the 1-dimensional linear subspace of $T_pM$ generated by $\xi_p$. Thus the conformal curvature tensor $C$ is a map

$$
C : T_pM \times T_pM \times T_pM \rightarrow \phi(T_pM) \oplus \{\xi_p\}, \quad p \in M.
$$

(1.5)

It may be natural to consider the following particular cases:

1. $C : T_p(M) \times T_p(M) \times T_p(M) \rightarrow L(\xi_p)$, that is, the projection of the image of $C$ in $\phi(T_p(M))$ is zero;
2. $C : T_p(M) \times T_p(M) \times T_p(M) \rightarrow \phi(T_p(M))$, that is, the projection of the image of $C$ in $L(\xi_p)$ is zero;
3. $C : \phi(T_p(M)) \times \phi(T_p(M)) \times \phi(T_p(M)) \rightarrow L(\xi_p)$, that is, when $C$ is restricted to $\phi(T_p(M)) \times \phi(T_p(M)) \times \phi(T_p(M))$, the projection of the image of $C$ in $\phi(T_p(M))$ is zero. This condition is equivalent to

$$
\phi^2 C(\phi X, \phi Y, \phi Z) = 0.
$$

(1.6)
Here cases 1, 2, and 3 are synonymous to conformally symmetric, \( \xi \)-conformally flat, and \( \phi \)-conformally flat.

In [6], it is proved that a conformally symmetric \( K \)-contact manifold is locally isometric to the unit sphere. In [7], it is proved that a \( K \)-contact manifold is \( \xi \)-conformally flat if and only if it is an \( \eta \)-Einstein Sasakian manifold. In [8], some necessary conditions for a \( K \)-contact manifold to be \( \phi \)-conformally flat are proved. In [9], a necessary and sufficient condition for a Sasakian manifold to be \( \phi \)-conformally flat is obtained. In [10], projective curvature tensor in \( K \)-contact and Sasakian manifolds has been studied. Moreover, the author [11] considered some conditions on conharmonic curvature tensor \( \bar{\mathcal{C}} \), which has many applications in physics and mathematics, on a hypersurface in the semi-Euclidean space \( E^{n+1}_s \). He proved that every conharmonically Ricci-symmetric hypersurface \( M \) satisfying the condition \( \bar{\mathcal{C}} \cdot R = 0 \) is pseudo-symmetric. He also considered the condition \( \bar{\mathcal{C}} \cdot \bar{\mathcal{C}} = L_e Q(\xi, \bar{\mathcal{C}}) \) on hypersurfaces of the semi-Euclidean space \( E^{n+1}_s \).

Motivated by the studies of conformal curvature tensor in (see [6-9]) and the studies of projective curvature tensor in \( K \)-contact and Sasakian manifolds in [10] and Lorentzian para-Sasakian manifolds in [5], in this paper we study conharmonic curvature tensor in \( N(k) \)-contact metric manifolds.

Analogous to the considerations of conformal curvature tensor, we give following definitions.

**Definition 1.1.** A \((2n-1)\)-dimensional \( N(k) \)-contact metric manifold is said to be conharmonically symmetric if \((\nabla_w \bar{\mathcal{C}})(X, Y)Z = 0\), where \( X, Y, Z, W \in TM \).

**Definition 1.2.** A \((2n + 1)\)-dimensional \( N(k) \)-contact metric manifold is said to be \( \xi \)-conharmonically flat if \( \bar{\mathcal{C}}(X, Y)\xi = 0 \) for \( X, Y \in TM \).

**Definition 1.3.** A \((2n + 1)\)-dimensional \( N(k) \)-contact metric manifold is said to be \( \phi \)-conharmonically flat if \( \bar{\mathcal{C}}(\phi X, \phi Y, \phi Z, \phi W) = 0 \), where \( X, Y, Z, W \in TM \).

The paper is organized as follows. After preliminaries in Section 2, in Section 3 we consider conharmonically symmetric \( N(k) \)-contact metric manifolds. In this section we prove that if an \( n \)-dimensional \( N(k) \)-contact metric manifold is conharmonically symmetric, then it is locally isometric to the product \( E^{n+1}_s(0) \times S^n(4) \). Section 4 deals with \( \xi \)-conharmonically flat \( N(k) \)-contact metric manifolds and we prove that an \( n \)-dimensional \( N(k) \)-contact metric manifold is \( \xi \)-conharmonically flat if and only if it is an \( \eta \)-Einstein manifold. Besides these some important corollaries are given in this section. Finally, in Section 5, we prove that a \( \phi \)-conharmonically flat \( N(k) \)-contact metric manifold is a Sasakian manifold with vanishing scalar curvature.

## 2. Preliminaries

A \((2n + 1)\)-dimensional differentiable manifold \( M \) is said to admit an almost contact structure if it admits a tensor field \( \phi \) of type \((1, 1)\), a vector field \( \xi \), and a 1-form \( \eta \) satisfying (see [12, 13])

\[
\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta \circ \phi = 0.
\]
An almost contact metric structure is said to be normal if the almost induced complex structure $J$ on the product manifold $M \times \mathbb{R}$ defined by

$$J(X, f \frac{d}{dt}) = \left( \phi X - f \xi, \eta(X) \frac{d}{dt} \right)$$

(2.2)

is integrable, where $X$ is tangent to $M$, $t$ is the coordinate of $\mathbb{R}$, and $f$ is a smooth function on $M \times \mathbb{R}$. Let $g$ be the compatible Riemannian metric with almost contact structure $(\phi, \xi, \eta)$ that is,

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y).$$

(2.3)

Then $M$ becomes an almost contact metric manifold equipped with an almost contact metric structure. From (2.1) it can be easily seen that

$$g(X, \phi Y) = -g(\phi X, Y), \quad g(X, \xi) = \eta(X),$$

for any vector fields $X, Y$ on the manifold. An almost contact metric structure becomes contact metric structure if $g(X, \phi Y) = \lambda g(X, Y)$, for all vector fields $X, Y$.

A contact metric manifold is said to be Einstein if $\mathcal{S}(X, Y) = \lambda g(X, Y)$, where $\lambda$ is a constant and $\eta$-Einstein if $\mathcal{S}(X, Y) = \alpha g(X, Y) + \beta \eta(X)\eta(Y)$, where $\alpha$ and $\beta$ are smooth functions.

A normal contact metric manifold is a Sasakian manifold. An almost contact metric manifold is Sasakian if and only if

$$(\nabla_X \phi) Y = g(X, Y)\xi - \eta(Y)X,$$

(2.4)

$X, Y \in TM$, where $\nabla$ is the Levi-Civita connection of the Riemannian metric $g$. A contact metric manifold $M^{2m+1}(\phi, \xi, \eta, g)$ for which $\xi$ is a Killing vector field is said to be a $K$-contact metric manifold. A Sasakian manifold is $K$-contact but not conversely. However, a 3-dimensional $K$-contact manifold is Sasakian [14].

It is well known that the tangent sphere bundle of a flat Riemannian manifold admits a contact metric structure satisfying $\mathcal{R}(X, Y)\xi = 0$ [15]. Again on a Sasakian manifold [16] we have

$$\mathcal{R}(X, Y)\xi = \eta(Y)X - \eta(X)Y.$$ 

(2.5)

As a generalization of both $\mathcal{R}(X, Y)\xi = 0$ and the Sasakian case, Blair et al. [17] introduced the $(k, \mu)$-nullity distribution on a contact metric manifold and gave seven reasons for studying it. The $(k, \mu)$-nullity distribution $N(k, \mu)$ [17] of a contact metric manifold $M$ is defined by

$$N(k, \mu) : p \to N_p(k, \mu) = \{W \in \tau_p^*M : \mathcal{R}(X, Y)W = (kI + \mu h)(g(Y, W)X - g(X, W)Y)\}.$$ 

(2.6)
for all $X, Y \in TM$, where $(k, \mu) \in \mathbb{R}^2$. A contact metric manifold $M$ with $\xi \in N(k, \mu)$ is called a $(k, \mu)$-contact metric manifold. If $\mu = 0$, the $(k, \mu)$-nullity distribution reduces to $k$-nullity distribution [18]. The $k$-nullity distribution $N(k)$ of a Riemannian manifold is defined by [18]

$$N(k) : p \to N_p(k) = \{Z \in T_pM : R(X, Y)Z = k[g(Y, Z)X - g(X, Z)Y]\}, \quad (2.8)$$

with $k$ being a constant. If the characteristic vector field $\xi \in N(k)$, then we call a contact metric manifold as $N(k)$-contact metric manifold [19]. If $k = 1$, then the manifold is Sasakian, and if $k = 0$, then the manifold is locally isometric to the product $E^{n+1}(0) \times S^n(4)$ for $n > 1$ and flat for $n = 1$ [15].

Given a non-Sasakian $(k, \mu)$-contact manifold $M$, Boeckx [20] introduced an invariant

$$I_M = \frac{1 - \mu/2}{\sqrt{1 - k}} \quad (2.9)$$

and showed that, for two non-Sasakian $(k, \mu)$-manifolds $M_1$ and $M_2$, we have $I_{M_1} = I_{M_2}$ if and only if, up to a $D$-homothetic deformation, the two manifolds are locally isometric as contact metric manifolds.

Thus we see that from all non-Sasakian $(k, \mu)$-manifolds of dimension $(2n + 1)$ and for every possible value of the invariant $I$, one $(k, \mu)$-manifold $M$ can be obtained with $I_M = 1$. For $I > -1$ such examples may be found from the standard contact metric structure on the tangent sphere bundle of a manifold of constant curvature $c$, where we have $I = (1 + c)/(1 - c)$. Boeckx also gives a Lie algebra construction for any odd dimension and value of $I < -1$.

Using this invariant, Blair et al. [19] constructed an example of a $(2n + 1)$-dimensional $N(1 - 1/n)$-contact metric manifold, $n > 1$. The example is given in the following.

Since the Boeckx invariant for a $(1 - 1/n, 0)$-manifold is $\sqrt{n} > -1$, we consider the tangent sphere bundle of a manifold of constant curvature $c$ so choosing that the resulting $D$-homothetic deformation will be a $(1 - 1/n, 0)$-manifold. That is, for $k = c(2 - c)$ and $\mu = -2c$ we solve

$$1 - \frac{1}{n} = \frac{k + a^2 - 1}{a^2}, \quad 0 = \frac{\mu + 2a - 2}{a} \quad (2.10)$$

for $a$ and $c$. The result is

$$c = \frac{\sqrt{n} \pm 1}{n - 1}, \quad a = 1 + c, \quad (2.11)$$

and taking $c$ and $a$ to be these values we obtain $N(1 - 1/n)$-contact metric manifold.

However, for a $N(k)$-contact metric manifold $M$ of dimension $(2n + 1)$, we have [19]

$$\nabla X\phi \cdot Y = g(X + hX, Y)\phi - \eta(Y)(X + hX), \quad (2.12)$$
where \( h = (1/2)\xi \phi \),

\[
h^2 = (k - 1)\phi^2,
\]

\[
R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y],
\]

\[
S(X, Y) = 2(n - 1)g(X, Y) + 2(n - 1)g(hX, Y)
\]
\[
+ [2nk - 2(n - 1)]\eta(X)\eta(Y), \quad n \geq 1,
\]

\[
S(Y, \xi) = 2nk\eta(X),
\]

\[
(\nabla_X \eta)(Y) = g(X, hX, \phi Y),
\]

\[
(\nabla_X \eta)(Y) = (1 - k)g(X, \phi Y) + g(X, h\phi Y) \xi + \eta(Y) [h(\phi X + \phi hX)],
\]

In a \((2n + 1)\)-dimensional almost contact metric manifold, if \( \{e_1, \ldots, e_{2n}, \xi\} \) is a local orthonormal basis of the tangent space of the manifold, then \( \{\phi e_1, \ldots, \phi e_{2n}, \xi\} \) is also a local orthonormal basis. It is easy to verify that

\[
\sum_{i=1}^{2n} g(e_i, e_i) = 2n,
\]

\[
\sum_{i=1}^{2n} S(e_i, e_i) = 2nS(\phi e_i, \phi e_i) = r - 2nk,
\]

\[
\sum_{i=1}^{2n} g(e_i, Z)S(Y, e_i) = 2nS(\phi e_i, Z)S(Y, \phi e_i) = S(Y, Z) - 2nk\eta(Z),
\]

for \( Y, Z \in T(M) \). In particular in view of \( \eta \circ \phi = 0 \), we get

\[
\sum_{i=1}^{2n} g(e_i, \phi Z)S(Y, e_i) = \sum_{i=1}^{2n} g(\phi e_i, \phi Z)S(Y, \phi e_i) = S(Y, \phi Z).
\]

Here we state a lemma due to Baikoussis and Koufogiorgos [21] which will be used in this paper.

**Lemma 2.1.** Let \( M^{2n+1} \) be an \( \eta \)-Einstein manifold of dimension \((2n + 1)(n \geq 1)\). If \( \xi \) belongs to the \( k \)-nullity distribution, then \( k = 1 \) and the structure is Sasakian.
3. Conharmonically Symmetric $N(k)$-Contact Metric Manifolds

In this section we study conharmonically symmetric $N(k)$-contact metric manifolds. Differentiating (1.3) covariantly with respect to $W$, we obtain

$$(\nabla_W \tilde{C})(X, Y)Z = (\nabla_W R)(X, Y)Z - \frac{1}{2n-1} [g(Y, Z)(\nabla_W Q)X - g(X, Z)(\nabla_W Q)Y + (\nabla_W S)(Y, Z)X - (\nabla_W S)(X, Z)Y].$$

(3.1)

Therefore for conharmonically symmetric $N(k)$-contact metric manifolds we have

$$(\nabla_W R)(X, Y)Z = \frac{1}{2n-1} [g(Y, Z)(\nabla_W Q)X - g(X, Z)(\nabla_W Q)Y + (\nabla_W S)(Y, Z)X - (\nabla_W S)(X, Z)Y].$$

(3.2)

Differentiating (2.12) covariantly with respect to $W$ and using (2.15) we obtain

$$(\nabla_W R)(X, Y)Z = k [g(W, \phi Y)X + g(hW, \phi Y)X - g(W, \phi X)Y - g(hW, \phi X)Y].$$

(3.3)

Again, differentiating (2.14) covariantly with respect to $W$ and using (2.16) and (2.17) we have

$$(\nabla_W S)(Y, Z) = 2(n-1) [(1 - k)g(W, \phi Y)\eta(Z) + g(W, h\phi Y)\eta(Z) + g(h\phi W, Z)\eta(Y)]
+ g(h\phi hW, Z)\eta(Y)
+ [2(1 - n) + 2nk] [g(W, \phi Y)\eta(Z) + g(hW, \phi Y)\eta(Z) + g(W, \phi Z)\eta(Y)]
+ g(hW, \phi Z)\eta(Y).$$

(3.4)

Therefore we have

$$(\nabla_W Q)(Y) = 2k [g(W, \phi Y)\xi - (\phi W)\eta(Y)] + 2nk [g(W, h\phi Y) + (h\phi W)\eta(Y)].$$

(3.5)

Putting $Z = \xi$ in (3.2) and using (3.3), (3.4), and (3.5) we obtain

$$(2n-1)k [g(W, \phi Y)X + g(hW, \phi Y)X - g(W, \phi X)Y - g(hW, \phi X)Y]
= 2k [g(W, \phi X)\phi^2 Y - g(W, \phi Y)\phi^2 X]
+ 2nk [g(W, h\phi X)\phi^2 Y - g(W, h\phi Y)\phi^2 X].$$

(3.6)
Taking inner product of (3.6) with \( \xi \) and using (2.1) we obtain

\[
(2n-1)k\left[g(W,\phi Y)\eta(X) + g(hW,\phi Y)\eta(X) - g(W,\phi X)\eta(Y) - g(hW,\phi X)\eta(Y)\right] = 0. \tag{3.7}
\]

From (3.7) we get, either \( k = 0 \) or

\[
g(W,\phi Y)\eta(X) + g(hW,\phi Y)\eta(X) - g(W,\phi X)\eta(Y) - g(hW,\phi X)\eta(Y) = 0. \tag{3.8}
\]

Putting \( hY \) instead of \( Y \) in (3.8) and using (2.12) we obtain

\[
g(W,\phi hY)\eta(X) = (k-1)g(W,\phi Y)\eta(X). \tag{3.9}
\]

Using (3.9) in (3.7) yields

\[
k\left[g(W,\phi Y)\eta(X) - g(W,\phi X)\eta(Y)\right] = 0. \tag{3.10}
\]

The relation (3.10) gives \( k = 0 \), since \( g(W,\phi Y)\eta(X) - g(W,\phi X)\eta(Y) = 0 \) gives \( g(W,\phi Y) = 0 \) (by putting \( X = \xi \)), which is not the case for a \( N(k) \)-contact metric manifold, in general.

Therefore in either case we obtain \( k = 0 \).

Hence we have the following.

**Theorem 3.1.** A conharmonically symmetric \( n \)-dimensional \( N(k) \)-contact metric manifold is locally isometric to the product \( E^{m+1}(0) \times S^n(4) \).

**Remark 3.2.** The converse of the above theorem is not true in general. However if \( k = 0 \), then we get \( R(X,Y)\xi = 0 \), and hence from the definition of the conharmonic curvature tensor we obtain \( C(X,Y)\xi = 0 \), that is, the manifold under consideration is \( \xi \)-conharmonically flat.

Thus if an \( N(k) \)-contact manifold is locally isometric to \( E^{m+1}(0) \times S^n(4) \), then the manifold is \( \xi \)-conharmonically flat.

### 4. \( \xi \)-Conharmonically Flat \( N(k) \)-Contact Metric Manifolds

In this section we consider a \( (2n+1) \)-dimensional \( \xi \)-conharmonically flat \( N(k) \)-contact metric manifolds. Then from (1.3) we obtain

\[
R(X,Y)\xi = \frac{1}{2n-1} [g(\eta Y,\xi)QX - g(\xi Y,\xi)QY + S(\xi X)X - S(\xi Y)Y]. \tag{4.1}
\]

Using (2.1), (2.13), and (2.15) in (4.1) we obtain

\[
[\eta(Y)QX - \eta(X)QY] + k[\eta(Y)X - \eta(X)Y] = 0. \tag{4.2}
\]

Putting \( Y = \xi \) in (4.2) and using (2.1) and (2.15) we get

\[
QX = -kX + (2n+1)k\eta(X)\xi. \tag{4.3}
\]
Taking inner product with $W$ of (4.3) yields

$$S(X, W) = -kg(X, W) + (2n + 1)k\eta(X)\eta(W).$$  \hspace{1cm} (4.4)

From relation (4.4), we conclude that the manifold is an $\eta$-Einstein manifold. Conversely, we assume that a $(2n+1)$-dimensional $N(k)$-contact manifold satisfies the relation (4.4). Then we easily obtain from (1.3) that $\tilde{C}(X, Y)\xi = 0$.

In view of the above discussions we state the following.

**Theorem 4.1.** A $(2n+1)$-dimensional $N(k)$-contact metric manifold is $\xi$-conharmonically flat if and only if it is an $\eta$-Einstein manifold.

Hence in view of Lemma 2.1 we state the following.

**Corollary 4.2.** Let $M$ be a $(2n+1)$-dimensional $\xi$-conharmonically flat $N(k)$-contact metric manifold, then $k = 1$ and the structure is Sasakian.

Let $\{e_1, e_2, \ldots, e_n, e_{n+1}, \ldots, e_{2n}, e_{2n+1} = \xi\}$ be a local orthonormal basis of the tangent space of the manifold. Putting $X = W = e_1$ in (4.4) and summing up from 1 to $2n + 1$ we obtain in view of (2.18) and (2.19) that

$$r = 0. \hspace{1cm} (4.5)$$

Therefore we have the following corollary.

**Corollary 4.3.** In a $(2n+1)$-dimensional $\xi$-conharmonically flat $N(k)$-contact metric manifold, the scalar curvature $r$ vanishes.

**5. $\phi$-Conharmonically Flat $N(k)$-Contact Metric Manifolds**

This section deals with a $(2n + 1)$-dimensional $\phi$-conharmonically flat $N(k)$-contact metric manifold. Then we have from (1.3) that

$$\tilde{R}(\phi X, \phi Y, \phi Z, \phi W)$$

$$= \frac{1}{2n-1} [g(\phi Y, \phi Z)S(\phi X, \phi W) - g(\phi X, \phi Z)S(\phi Y, \phi W) + S(\phi Y, \phi Z)g(\phi X, \phi W)$$

$$- S(\phi X, \phi Z)g(\phi Y, \phi W)].$$  \hspace{1cm} (5.1)
Let \( \{e_1, e_2, \ldots, e_{2n}, \xi \} \) be a local orthonormal basis of the tangent space of the manifold. Then \( \{\phi e_1, \phi e_2, \ldots, \phi e_{2n}, \xi \} \) is also a local orthonormal basis of the tangent space. Putting \( X = W = e_i \) in (5.1) and summing up from 1 to \( 2n \) we have

\[
\sum_{i=1}^{2n} R(\phi e_i, \phi Y, \phi Z, \phi e_i)
= \frac{1}{2n-1} \sum_{i=1}^{2n} [g(\phi Y, \phi Z)S(\phi e_i, \phi e_i) - g(\phi e_i, \phi Z)S(\phi Y, \phi e_i) + S(\phi Y, \phi Z)g(\phi e_i, \phi e_i)]
- S(\phi e_i, \phi Z)g(\phi Y, \phi e_i)].
\] (5.2)

Using (2.18), (2.19), (2.20), and (2.21) in (5.2) we obtain

\[
S(\phi Y, \phi Z) = (r - k)g(\phi Y, \phi Z).
\] (5.3)

Replacing \( Y \) and \( Z \) by \( \phi Y \) and \( \phi Z \) in (5.3) and using (2.1) we have

\[
S(Y, Z) = (r - k)g(Y, Z) + [(2n + 1)k - r]g(Y, Z).
\] (5.4)

Putting \( Y = Z = e_i \) in (5.4) and taking summation over \( i = 1 \) to \( 2n + 1 \) we get by using (2.18) and (2.19) that

\[ r = 0. \] (5.5)

In view of the above discussions we have the following.

**Proposition 5.1.** A \((2n + 1)\)-dimensional \( \phi \)-conharmonically flat \( N(k) \)-contact metric manifold is an \( \eta \)-Einstein manifold with vanishing scalar curvature.

Therefore in view of the Lemma 2.1 we state the following theorem.

**Theorem 5.2.** A \((2n + 1)\)-dimensional \( \phi \)-conharmonically flat \( N(k) \)-contact metric manifold is a Sasakian manifold with vanishing scalar curvature.

**Definition 5.3.** In a \((2n + 1)\)-dimensional \( N(k) \)-contact metric manifold, if the Ricci tensor \( S \) satisfies \((\nabla_X S)(\phi Y, \phi Z) = 0\), then the Ricci tensor is said to be \( \eta \)-parallel.

The notion of \( \eta \)-parallel Ricci tensor for Sasakian manifold was introduced by Kon [22].

Putting \( r = 0 \) in (5.4) we have

\[
S(Y, Z) = -kg(Y, Z) + (2n + 1)k\eta(Y)\eta(Z).
\] (5.6)

Replacing \( Y \) and \( Z \) by \( \phi Y \) and \( \phi Z \) in (5.6) and using (2.1) we obtain

\[
S(\phi Y, \phi Z) = -kg(\phi Y, \phi Z).
\] (5.7)
Relation (5.7) yields
\((\nabla_X S)(\phi Y, \phi Z) = 0\), \((5.8)\)
since \(k\) is a constant. Therefore we have the following corollary.

**Corollary 5.4.** A \((2n+1)\)-dimensional \(\phi\)-conharmonically flat \(N(k)\)-contact metric manifold satisfies \(\eta\)-parallel Ricci tensor.

**References**


On $\phi$-Quasiconformally Symmetric $(k,\mu)$-Contact Manifolds

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Abstract—The object of the present paper is to study locally and globally $\phi$-quasiconformally symmetric $(k,\mu)$-contact manifolds.

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1. INTRODUCTION

Let $(M^n, g)$, $n \geq 3$, be a Riemannian manifold. The notion of the quasiconformal curvature tensor was given by Yano and Sawaki [15]. According to them a quasiconformal curvature tensor was given by

$$
\bar{C}(X,Y)Z = aR(X,Y)Z + b(S(Y,Z)X - S(X,Z)Y) + g(Y,Z)QX - g(X,Z)QY
$$

(1.1)

where $a$ and $b$ are constants and $R$, $S$, $Q$ and $r$ are the Riemannian curvature tensor of type (1,3), the Ricci tensor of type (0,2), the Ricci operator defined by $S(X,Y) = g(QX,Y)$ and scalar curvature of the manifold respectively. If $a = 1$ and $b = -\frac{1}{n-1}$, then (1.1) takes the form

$$
\bar{C}(X,Y)Z = R(X,Y)Z - \frac{1}{n-2}[S(Y,Z)X - S(X,Z)Y] + g(Y,Z)QX - g(X,Z)QY
$$

(1.2)

where $C$ is the conformal curvature tensor [16]. In [5], De and Matsuyama studied a quasiconformally flat Riemannian manifold satisfying certain condition on the Ricci tensor. From Theorem 5 of [5], it can be proved that a 4-dimensional quasiconformally flat semi-Riemannian manifold is the Robertson-Walker space-time. Robertson-Walker space-time is the warped product $I \times F M^n$, where $M^n$ is a space of constant curvature and $I$ is an open interval [11]. This means that quasiconformal curvature tensor has some applications in general theory of relativity. From (1.1) we obtain

$$
(\nabla_W \bar{C})(X,Y)Z = a(\nabla_W R)(X,Y)Z + b((\nabla_W S)(Y,Z)X - (\nabla_W S)(X,Z)Y
$$

(1.3)

$$
+ g(Y,Z)(\nabla_W Q)(X) - g(X,Z)(\nabla_W Q)(Y)) - \frac{dr(W)}{n} \left[\frac{a}{n-1} + 2b\right] g(Y,Z)X - g(X,Z)Y.
$$

If the condition $VR = 0$ holds on $M$, then $M$ is called locally symmetric, where $\nabla$ denotes the Levi-Civita connection on $M$. The notion of locally $\phi$-symmetric Sasakian manifolds was introduced

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by T. Takahashi [13]. He studied several interesting properties of such a manifold in the context of Sasakian geometry. Recently U.C. De, A.A. Shaikh and Sudipta Biswas [6] introduced the notion of \( \phi \)-recurrent Sasakian manifolds which generalizes the notion of \( \phi \)-symmetric Sasakian manifolds. Also in another paper U.C. De and Aboul Kalam Gazi [7] introduced the notion of \( \phi \)-recurrent \( \text{N}(k) \)-contact metric manifolds. In a recent paper [9] J.B. Jun, A. Yildiz and U.C. De studied \( \phi \)-recurrent \( (k, n) \)-contact metric manifolds. Here we study globally and locally \( \phi \)-quasiconformally symmetric \( (k, \mu) \)-contact metric manifolds.

A \( (k, \mu) \)-contact manifold is said to be globally \( \phi \)-quasiconformally symmetric manifold if
\[
\phi^2(\nabla_X \xi)(X, Y)Z = 0
\]
for any \( X, Y, Z \in \chi(M) \).

If \( X, Y, Z \) are horizontal vector fields then the manifold is said to be locally \( \phi \)-quasiconformally symmetric. A horizontal vector field means the vector field which is perpendicular to \( \xi \).

In this paper we study globally and locally \( \phi \)-quasiconformally symmetry in \( (k, \mu) \)-contact manifolds.

The paper is organized as follows:

In Section 2, we give some Preliminaries of \( (k, \mu) \)-contact manifolds. In Section 3, we study globally \( \phi \)-quasiconformally symmetric \( (k, \mu) \)-contact manifolds and prove that the manifold is an Einstein manifold. Also we prove that a \( (k, \mu) \)-contact manifold is globally \( \phi \)-quasiconformally symmetric if and only if it is \( \phi \)-symmetric. Besides this we also show that the globally \( \phi \)-quasiconformally symmetric \( (k, \mu) \)-contact manifold is 3-dimensional and flat. In the next section, we deal with 3-dimensional locally \( \phi \)-quasiconformally symmetric \( (k, \mu) \)-contact manifolds. We prove that a 3-dimensional \( (k, \mu) \)-contact manifold is locally \( \phi \)-quasiconformally symmetric if and only if the scalar curvature is constant. Finally we give an example of a 3-dimensional locally \( \phi \)-quasiconformally symmetric \( (k, \mu) \)-contact manifold.

2. PRELIMINARIES

By a contact manifold we mean an \( n \)-dimensional differentiable manifold \( M^n \) which carries a global 1-form \( \eta \); there exists a unique vector field \( \xi \), called the characteristic vector field such that, \( \eta(\xi) = 1 \) and \( d\eta(\xi, X) = 0 \). A Riemannian metric \( g \) on \( M^n \) is said to be an associated metric if there exists a \((1,1)\) tensor field \( \phi \) such that
\[
d\eta(X, Y) = g(X, \phi Y), \quad \eta(X) = g(X, \xi), \quad \phi^2 = -I + \eta \otimes \xi.
\]
From these equations we have
\[
\phi \xi = 0, \quad \eta \circ \phi = 0, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y).
\]

The manifold \( M \) equipped with the contact structure \( (\phi, \xi, \eta, g) \) is called a contact metric manifold [1].

Given a contact metric manifold \( M^n(\phi, \xi, \eta, g) \) we define a \((1,1)\) tensor field \( h \) by \( h = L_\phi \xi \), where \( L \) denotes the Lie differentiation. Then \( h \) is symmetric and satisfies \( h\phi = -\phi h \). Thus, if \( \lambda \) is an eigenvalue of \( h \) with eigenvector \( X \), \( -\lambda \) is also an eigenvalue with eigenvector \( \phi X \). Also we have \( Tr.h = Tr.\phi h = 0 \) and \( h\xi = 0 \). Moreover, if \( \nabla \) denotes the Riemannian connection of \( g \), then the following relation holds:
\[
\nabla_X \xi = -\phi X - \phi h X.
\]

A contact metric manifold is said to be Einstein if \( S(X, Y) = ag(X, Y) \), where \( a \) is a constant and \( \eta \)-Einstein if \( S(X, Y) = ag(X, Y) + 2h(\eta(X))\eta(Y) \), where \( a \) and \( h \) are smooth functions. A normal contact metric manifold is a Sasakian manifold. An almost contact metric manifold is Sasakian if and only if
\[
(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X,
\]
where \( M \) is the Levi-Civita connection of the Riemannian metric \( g \). A contact metric manifold \( M^n(\phi, \xi, \eta, g) \) for which \( \xi \) is a Killing vector field is said to be a \( K \)-contact metric manifold. A Sasakian manifold is \( K \)-contact but not conversely. However a 3-dimensional \( K \)-contact manifold is Sasakian [8]. It is well known that the tangent sphere bundle of a flat Riemannian manifold admits a contact metric structure satisfying \( R(X, Y)\xi = 0 \) [9]. On the other hand, on a Sasakian manifold the following relation holds:
\[
R(X, Y)\xi = \eta(Y)X - \eta(X)Y.
\]
It is well known that there exists contact metric manifolds for which the curvature tensor $R$ and the sectional of the characteristic vector field $\xi$ satisfying $R(X, Y)\xi = 0$ for any vector fields $X$ and $Y$. For example, tangent sphere bundle of a flat Riemannian manifold admits such a structure.

As a generalisation of $R(X, Y)\xi = 0$ and the Sasakian case: D.E. Blair, T. Koufogiorgos, and J.J. Papantoniou [2] considered the $(k, \mu)$-nullity condition on a contact metric manifold and gave several reasons for studying it. The $(k, \mu)$-nullity distribution $N(k, \mu)$ [2, 12] of a contact metric manifold is defined by

$$N(k, \mu) : p \rightarrow N_p(k, \mu) = \{W \in T_pM \mid R(X, Y)W = (kl + \mu h)(g(Y, W)X - g(X, W)Y)\}$$

for all $X, Y \in TM$, where $(k, \mu) \in R^2$. A contact metric manifold $M^n$ with $\xi \in N(k, \mu)$ is called a $(k, \mu)$-contact metric manifold. Thus we have

$$R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY]. \quad (2.6)$$

Applying a $D$-homothetic deformation to a contact metric manifold with $R(X, Y)\xi = 0$, we obtain a contact metric manifold satisfying (2.6). In [2], it is proved that the standard contact metric structure on the tangent sphere bundle $T\pi(M)$ satisfies the condition that $\xi$ belongs to the $(k, \mu)$-nullity distribution if and only if the base manifold is the space of constant curvature. There exists examples in all dimensions and the condition that $\xi$ belongs to the $(k, \mu)$-nullity distribution is invariant under $D$-homothetic deformations; in dimension greater than 5, the condition determines the curvature completely; dimension 3 includes the 3-dimensional unimoduler Lie groups with the left invariant metric.

On $(k, \mu)$-contact metric manifold, $k \leq 1$. If $k = 1$, the structure is Sasakian ($h = 0$ and $\mu$ is indeterminant) and if $k < 1$, the $(k, \mu)$-nullity condition completely determines the curvature of $M^n$ [2]. In fact, for a $(k, \mu)$-contact manifold, the condition of being Sasakian manifold, a $K$-contact manifold, $k = 1$ and $h = 0$ are all equivalent. Again a $(k, \mu)$-contact manifold reduces to an $N(k)$-contact manifold if and only if $\mu = 0$.

In a $(k, \mu)$ contact manifold $M^n$ where $n = 2m + 1, (m \geq 1)$, the following relations hold ([2, 4]):

$$h^2 = (k - 1)\phi^2, \quad k \leq 1, \quad (2.7)$$

$$(\nabla_X \phi)Y = g(X + hX, Y)\xi - \eta(Y)(X + hX), \quad (2.8)$$

$$R(\xi, X)Y = k[g(X, Y)\xi - \eta(Y)X] + \mu[g(hX, Y)\xi - \eta(Y)hX], \quad (2.9)$$

$$S(\xi, X, \xi) = 2m\kappa_{\eta}(X), \quad (2.10)$$

$$S(X, Y) = [2(m - 1) - m\mu]g(X, Y) + [2(m - 1) + \mu]g(hX, Y) + [2(1 - \mu) + m(2k + \mu)]\eta(X)\eta(Y), \quad m \geq 1, \quad (2.11)$$

$$r = 2m(2m - 2 + k - m\mu), \quad (2.12)$$

$$S(\phi X, \phi Y) = S(X, Y) - 2m\kappa_{\eta}(X)\eta(Y) - 2(2m - 2 + \mu)g(hX, Y), \quad (2.13)$$

where $S$ is the Ricci tensor of type $(0, 2)$ and $r$ is the scalar curvature of the manifold. From (2.3) it follows that

$$(\nabla_X \eta)Y = g(X + hX, \phi Y). \quad (2.14)$$

$$(\nabla_X \phi)Y = [(1 - k)g(X, \phi Y) + g(X, h\phi Y)]\xi + \eta(Y)[h(\phi X + \phi hX)] - \mu \eta(X)\phi\phi Y. \quad (2.15)$$

Also in a $(k, \mu)$-contact manifold, the following holds

$$\eta(R(X, Y)Z) = k[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)] + \mu[g(hY, Z)\eta(X) - g(hX, Z)\eta(Y)]. \quad (2.16)$$

Especially for the case $\mu = 2(1 - m)$, from (2.11) it follows that the manifold is $\eta$-Einstein.

Now we prove a Lemma:

**Lemma 2.1.** An Einstein $(k, \mu)$-contact manifold $M^n, n = 2m + 1$, is three dimensional and flat.
Proof: For an Einstein manifold we have \( S(X, Y) = a g(X, Y) \), where \( a \) is a constant. Comparing this value of \( S(X, Y) \) with those given in (2.11) we have

\[
ag(X, Y) = 2(m - 1) - m\mu \eta(X, Y) + 2(1 - m) + \mu \eta(hX, Y)
\]

(2.17)

\[
+ [2(1 - m) + m(2k + \mu)]\eta(X)\eta(Y).
\]

Putting \( X = Y = \xi \) in (2.17) and applying \( g(X, \xi) = \eta(X), \eta(\xi) = 1 \) and \( h\xi = 0 \) we obtain \( a = 2mk \).

Therefore the relation (2.17) becomes

\[
2mkX = [2(m - 1) - m\mu]X + 2[2(m - 1) + \mu]hX + [2(1 - m) + m(2k + \mu)]\eta(X)\xi,
\]

(2.18)

i.e,

\[
[2(m - 1) - m(2k + \mu)]X - \eta(X)\xi + [2(1 - m) + \mu]hX = 0.
\]

(2.19)

Equating the coefficients of \( X \) and \( hX \) from both sides of (2.19) we obtain

\[
2(m - 1) + \mu = 0 \quad \text{and} \quad 2(m - 1) - m(2k + \mu) = 0.
\]

(2.20)

Using (2.20) in (2.18) we obtain

\[
2mk = 2(m^2 - 1).
\]

(2.21)

Therefore \( k = \frac{m^2 - 1}{m} \leq 1 \), so \( r = 1 \) is the only case. This gives \( \mu = 0 \) which with \( m = 1 \) gives \( k = 0 \).

Applying these in (2.6) we get \( R(X, Y)\xi = 0 \).

In [3] D.E. Blair proved that a \((2m + 1)\)-dimensional contact metric manifold satisfying \( R(X, Y)\xi = 0 \) is locally isometric to \( E^{m+1}(0) \times S^m(4) \) for \( m > 1 \) and flat if \( m = 3 \).

Therefore we conclude that the manifold under consideration is three-dimensional and flat. This proves the Lemma.

3. GLOBALLY \( \phi \)-QUASICONFORMALLY SYMMETRIC \((k, \mu)\)-CONTACT MANIFOLDS

Here we consider an \( n \)-dimensional \((k, \mu)\)-contact manifold which is globally \( \phi \)-quasiconformally symmetric. Then using (2.1) in (1.4) we obtain

\[
-(\nabla_{\nabla}C)(X, Y)Z + \eta(\nabla_{\nabla}C)(X, Y)Z)\xi = 0.
\]

(3.1)

Using (1.3) in (3.1) and taking inner product with \( U \) we obtain

\[
-ag((\nabla \nabla R)(X, Y)Z, U) - b((\nabla \nabla S)(Y, Z)g(X, U) - (\nabla \nabla S)(X, Z)g(Y, U)
\]

(3.2)

\[
+ g(Y, Z)g((\nabla \nabla Q)(X, Z)U) - g(X, Z)g((\nabla \nabla Q)(Y, U)) + \frac{dr(W)}{n} \left[ \frac{a}{n - 1} + 2b \right]
\]

\[
\times [g(Y, Z)g(X, U) - g(X, Z)g(Y, U)] + \eta((\nabla \nabla R)(X, Y)Z)\eta(U) + b((\nabla \nabla S)(Y, Z)\eta(X)\eta(U)
\]

\[
- (\nabla \nabla S)(X, Z)\eta(Y)\eta(U) + g(Y, Z)\eta((\nabla \nabla Q)(X)\eta(U) - g(X, Z)\eta((\nabla \nabla Q)(Y))\eta(U)]
\]

\[
- \frac{dr(W)}{n} \left[ \frac{a}{n - 1} + 2b \right] [g(Y, Z)\eta(X)\eta(U) - g(X, Z)\eta(Y)\eta(U)] = 0.
\]

Putting \( X = U = e_i \) in (3.2), where \( \{e_i\}, i = 1, 2, \ldots, n, \) is an orthonormal basis of the tangent space at each point of the manifold and taking summation over \( i \), we get

\[
-(a + nb - 2b)(\nabla \nabla S)(Y, Z) - \left[ \sum_{i=1}^{n} \eta((\nabla \nabla Q)e_i, Y)Z)\eta(e_i) - b((\nabla \nabla S)(\xi, Z)\eta(Y) - b\eta((\nabla \nabla Q)(Y)\eta(Z)
\]

(3.3)

\[
- b \sum_{i=1}^{n} \eta((\nabla \nabla Q)e_i, Y)Z)\eta(e_i) + \frac{dr(W)}{n} \left( \frac{a}{n - 1} + 2b \right) g(Y, Z) + bg((\nabla \nabla Q)(Y, Z)
\]

\[
+ a \sum_{i=1}^{n} \eta((\nabla \nabla R)e_i, Y)Z)\eta(e_i) - b((\nabla \nabla S)(\xi, Z)\eta(Y) - b\eta((\nabla \nabla Q)(Y)\eta(Z).
\]

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\[ + \frac{dr(W)}{n} \left( \frac{a}{n-1} + 2b \right) \eta(Y) \eta(Z) = 0. \]

Putting \( Z = \xi \) in (3.3) and using (2.1), \( \eta(\xi) = 1 \) we obtain

\[ -(a + nb - 2b)(\nabla_w S)(Y, \xi) = \left[ bdr(W) - \frac{n-1}{n} dr(W) \left( \frac{a}{n-1} + 2b \right) - b \eta(\nabla_w Q)\xi \right] \]

\[ + \frac{dr(W)}{n} \left( \frac{a}{n-1} + 2b \right) \eta(Y) + a g((\nabla_w R)(\xi, Y)\xi, \xi) - b(\nabla_w S)(\xi, \xi) \eta(Y) \]

\[ + \frac{dr(W)}{n} \left( \frac{a}{n-1} + 2b \right) \eta(Y) = 0. \]

Using (2.2), (2.3), (2.10) and \( h\xi = 0 \) we obtain

\[ \eta((\nabla_w Q)\xi) = g((\nabla_w Q), \xi) = (n-1)k g(\nabla_w \xi, \xi) = (n-1)k g(-\phi W - \phi h W, \xi) = 0. \]

Again

\[ g((\nabla_w R)\xi, Y)\xi, \xi) = g(\nabla_w R(\xi, Y)\xi, \xi) - g(R(\nabla_w Q)\xi, \xi) \]

\[ - g(R(\xi, \nabla_w Y)\xi, \xi) - g(R(\xi, Y)\nabla_w \xi, \xi) - g(R(\xi, Y)\nabla_w \xi, \xi). \]

Using (2.1), (2.9), \( h\xi = 0 \) and \( \eta(\xi) = 1 \) we have

\[ g(\xi, Y)\xi, \xi) = 0. \]

Therefore we have

\[ g((\nabla_w R)(\xi, Y)\xi, \xi) = 0. \]

Here by using (2.9) and \( h\xi = 0 \) we have

\[ g(R(\xi, \nabla_w Y)\xi, \xi) = k g(-\eta(\xi)\nabla_w Y + \eta(\nabla_w Y)\xi, \xi) \]

\[ + \mu g(\eta(\nabla_w Y)h\xi - \eta(\xi)h(\nabla_w Y), \xi) = 0. \]

In view of \( g(R(X, Y)Z, W) = -g(R(X, Y)W, Z) \) we have

\[ -g(R(\xi, Y)\nabla_w \xi, \xi) - g(R(\xi, Y)\nabla_w \xi, \xi) = 0 \quad \text{and} \quad g(R(\nabla_w \xi, Y)\xi, \xi) = 0. \]

Using (3.8), (3.9) and (3.10) in (3.6) yields

\[ g((\nabla_w R)(\xi, Y)\xi, \xi) = 0. \]

By using (2.2), (2.3) and (2.10) we have

\[ (\nabla_w S)(\xi, \xi) = \nabla_w S(\xi, \xi) - 2S(\nabla_w \xi, \xi) = -2S(-\phi W - \phi h W, \xi) = 0. \]

Then using (3.5), (3.11) and (3.12) in (3.4) we get

\[ (\nabla_w S)(Y, \xi) = \frac{1}{n} dr(W) \eta(Y). \]

Since \( a + (n - 2)b \neq 0 \), because if \( a + (n - 2)b = 0 \) then from (1.1), it follows that \( \overline{C} = \alpha C \). Therefore we cannot take \( a + (n - 2)b = 0 \). Putting \( Y = \xi \) in (3.13) we get \( dr(W) = 0 \). This implies \( r \) is constant. So from (3.13) we get

\[ (\nabla_w S)(Y, \xi) = 0. \]

i.e.,

\[ \nabla_w S(Y, \xi) - S(\nabla_w Y, \xi) - S(Y, \nabla_w \xi) = 0. \]

Using (2.3) and (2.10) in (3.14) we have

\[ (n - 1)k g(\nabla_w \eta)(Y) + S(Y, \phi W) + S(Y, \phi h W) = 0. \]

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Using (2.14) in (3.15) we obtain
\[(n - 1)kg(\phi Y, W) + S(Y, \phi W) + (n - 1)kg(\phi Y, hW) + S(Y, \phi hW) = 0.\] (3.16)

Replacing \(W\) by \(hW\) in (3.16) and using (2.7) we have
\[(n - 1)kg(\phi Y, hW) + S(Y, \phi hW) = (n - 1)k(k - 1)g(\phi Y, W) + (k - 1)S(Y, \phi W).\] (3.17)

Using (3.17) in (3.16) we obtain
\[(n - 1)k^2g(\phi Y, W) + kS(Y, \phi W) = 0.\] (3.18)

Replacing \(W\) by \(\phi W\) in (3.18) and using (2.1), (2.2) and (2.10) we get
\[\eta[\phi(n - 1)kg(\phi Y, W) - S(Y, W)] = 0.\] (3.19)

Therefore (3.19) gives either \(\eta = 0\) or
\[S(Y, W) = (n - 1)kg(Y, W).\] (3.20)

If we consider \(k \neq 0\) then in view of (3.20) we state the following:

Proposition 3.1. A globally \(\phi\)-quasiconformally symmetric \((k, \mu)\)-contact manifold is an Einstein manifold provided \(k \neq 0\).

Again from (3.20) we have \(Q X = (n - 1)k X\). Therefore from (1.1) we obtain
\[\hat{C}(X, Y)Z = \alpha R(X, Y)Z + \left[2b(n - 1)k - \frac{r}{n} \left(-\frac{a}{n - 1} + 2b\right)\right]g(Y, Z)X - g(X, Z)Y + \frac{r}{2}[g(X, Z)Y - g(Y, Z)X].\] (4.1)

Putting \(Z = \xi\) in (4.1) and using (2.9) we get
\[\left(\frac{r}{2} - k\right)\eta[Y, Y]X = \mu[g(Y)hX - \eta(X)hY] + \eta(Y)Q X - \eta(X)Q Y.\] (4.2)

Putting \(Y = \xi\) in (4.2) and using (2.10) we get
\[Q X = \left(\frac{r}{2} - k\right)X + \left(3k - \frac{r}{2}\right)\eta(X)\xi + \mu h X.\] (4.3)

Therefore
\[S(X, Y) = \left(\frac{r}{2} - k\right)g(X, Y) + \left(3k - \frac{r}{2}\right)\eta(Y) + \mu g(hX, Y).\] (4.4)
Using (4.3) and (4.4) in (4.1) we have
\[ R(X, Y)Z = \left( \frac{r}{2} - 2k \right) [g(Y, Z)X - g(X, Z)Y] + \left( 3k - \frac{r}{2} \right) [g(Y, Z)n(X)\xi - g(X, Z)n(Y)\xi] + \eta(Y)\eta(\xi)X - \eta(\xi)\eta(Y)X + \mu(g(Y, Z)hX - g(hX, Z)Y - g(hY, Z)X - g(hX, Z)Y]. \] (4.5)

Using (4.3), (4.4) and (4.5) in (4.1) we obtain
\[ C(X, Y)Z = \left( \frac{r(a + b)}{3} - 2k(a + b) \right) [g(Y, Z)X - g(X, Z)Y] + \left( 3k - \frac{r}{2} \right) [g(Y, Z)n(X)\xi - g(X, Z)n(Y)\xi] + \eta(Y)\eta(\xi)X - \eta(\xi)\eta(Y)X + (a + b)\mu(g(Y, Z)hX - g(hX, Z)Y - g(hY, Z)X - g(hX, Z)Y). \] (4.6)

Taking the covariant differentiation to both sides of the equation (4.6) and using (2.3), (2.13) and (2.14) we obtain
\[ (\nabla_W C)(X, Y)Z = \left( \frac{a + b}{3} \right) [g(Y, Z)X - g(X, Z)Y] \] (4.7)
\[ \quad - \left( a + b \right) dr(W) \left[ g(Y, Z)n(X)\xi - g(X, Z)n(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Y)Y \right] \]
\[ \quad + \left( a + b \right) \left( 3k - \frac{r}{2} \right) [g(Y, Z)X - g(X, Z)Y] + \left( 3k - \frac{r}{2} \right) [g(Y, Z)n(X)\xi - g(X, Z)n(Y)\xi] + \eta(Y)\eta(\xi)X - \eta(\xi)\eta(Y)X + \mu(g(Y, Z)hX - g(hX, Z)Y - g(hY, Z)X - g(hX, Z)Y). \]

Since \( X, Y, Z, W \) are orthogonal to \( \xi \) and using (2.1) we get from (4.7)
\[ \phi^2(\nabla_W \tilde{C})(X, Y)Z = \left( \frac{a + b}{3} \right) [g(X, Z)Y - g(Y, Z)X]. \] (4.8)

For locally \( \phi \)-quasiconformally symmetric we have from (4.8) either \( a + b = 0 \) or \( dr(W) = 0 \). Now \( a + b = 0 \) gives \( \tilde{C}(X, Y)Z = aC(X, Y)Z \). But in three dimensional Riemannian manifold since \( C = 0 \), we obtain \( \tilde{C} = 0 \). So we consider \( a + b \neq 0 \). Then we have \( dr(W) = 0 \). Hence we have the following Theorem:

**Theorem 4.1.** A 3-dimensional \((k, \mu)\)-contact manifold is locally \( \phi \)-quasiconformally symmetric if and only if the scalar curvature \( r \) is constant.

5. EXAMPLE

Here we consider an example of a 3-dimensional \((k, \mu)\)-contact manifold.

- In [10] J. Milnor gave a complete classification of three dimensional manifolds admitting the Lie algebra structure
\[ [e_2, e_3] = c_1 e_1, \quad [e_3, e_1] = c_2 e_2, \quad [e_1, e_2] = c_3 e_3. \] (5.1)
As in the case of the given example of [2], let us consider \( \eta \) be the dual 1-form to the vector field \( e_1 \). Using (5.1) we get
\[ d\eta(e_2, e_3) = -d\eta(e_3, e_2) = \frac{c_1}{2} \neq 0. \]
and $d\eta(e_i, e_j) = 0$ for $(i, j) \neq (2, 3), (3, 2)$. It is easy to check that $\eta$ is a contact form and $e_1$ is the characteristic vector field. Defining a Riemannian metric $g$ by $g(e_i, e_j) = \delta_{ij}$, then, because we must have $d\eta(e_i, e_j) = g(e_i, \phi e_j)$, $\phi$ has the same metric as $d\eta$ with respect to the basis $e_i$. Moreover, for $g$ to be an associated metric, we must have $\phi^2 = -I + \eta \otimes e_1$. So for $(\phi, e_1, \eta, g)$ to be a contact metric structure we must have $c_1 = 2$. The unique Riemannian connection $\nabla$ corresponding to $g$ is given by

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g([X, Y], Z) - g(Y, [X, Z]) + g(Z, [X, Y]).$$

So using $c_1 = 2$ we easily get

$$\nabla_{e_1} e_1 = 0, \quad \nabla_{e_2} e_2 = 0, \quad \nabla_{e_3} e_3 = 0, \quad \nabla_{e_1} e_2 = \frac{1}{2}(c_2 + c_3 - 2)e_3, \quad \nabla_{e_2} e_1 = \frac{1}{2}(c_2 - c_3 - 2)e_3,$$

$$\nabla_{e_1} e_3 = -\frac{1}{2}(c_2 + c_3 - 2)e_2, \quad \nabla_{e_2} e_3 = \frac{1}{2}(2 + c_2 - c_3)e_2.$$

But we also know that

$$\nabla_{e_2} e_1 = -\phi e_2 - \psi e_3.$$

Comparing now those two relations of $\nabla_{e_2} e_1$ and using $\phi e_1 = 0, \phi e_3 = -e_2$ we conclude that

$$h e_2 = \frac{c_3 - c_2}{2} e_2.$$

And hence

$$h e_3 = -\frac{c_3 - c_2}{2} e_3.$$

Thus $e_i$ are eigenvectors of $h$ with corresponding eigenvalues $(0, \lambda, -\lambda)$ where $\lambda = \frac{c_3 - c_2}{2}$. Moreover by direct calculation we have

$$R(e_2, e_1)e_1 = [1 - \frac{(c_3 - c_2)^2}{4}]e_2 + [2 - c_2 - c_3]he_2,$$

$$R(e_3, e_1)e_1 = [1 - \frac{(c_3 - c_2)^2}{4}]e_3 + [2 - c_2 - c_3]he_3. \quad R(e_2, e_3)e_1 = 0.$$

Putting $k = 1 - \frac{(c_3 - c_2)^2}{4}$ and $\mu = 2 - c_2 - c_3$ we conclude, from these relations that $e_1$ belongs to the $(k, \mu)$-nullity distribution, for any $c_2, c_3$. Here we see that

$$S(e_1, e_1) = 0, \quad S(e_2, e_2) = 1 - \frac{(c_3 - c_2)^2}{4} + \frac{2 - c_2 - c_3(c_3 - c_2)}{2},$$

$$S(e_3, e_3) = 1 - \frac{(c_3 - c_2)^2}{4} - \frac{2 - c_2 - c_3(c_3 - c_2)}{2}.$$

Therefore $r = 2 - \frac{(c_3 - c_2)^2}{2}$, which is a constant, since $c_2, c_3$ are constants.

Therefore in view of Theorem 4.1 we conclude that this manifold is locally $\psi$-quasiconformally symmetric $(k, \mu)$-contact manifold.

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ON A CLASS OF \((k, \mu)\)-CONTACT MANIFOLDS

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Abstract. This paper presents a study of projective curvature tensor satisfying certain curvature conditions in a \((2n + 1)\)-dimensional \((k, \mu)\)-contact metric manifold.

1. Introduction. In 1995 Blair, Koufogiorgos and Papantoniou (1995) introduced the notion of contact metric manifolds with the characteristic vector field \(\xi\) belonging to the \((k, \mu)\)-distributivity distribution and such type of manifolds are called \((k, \mu)\)-contact metric manifolds. They obtained several results of such a manifold and a full classification of this manifold has been given by Boeckx (2000). \((k, \mu)\)-contact manifolds have been studied by several authors such as D.E. Blair, T. Kofogiorgos and R. Sarma (1990) U.C. De, Y. H. Kim and A.A. Shaikh (2005), A.A. Shaikh and K.K. Baishya (2005), J.B. Jun, A. Yildiz and U.C. De (2008) and many others.

The Projective curvature tensor is an important tensor from the differential geometric point of view. Let \(M\) be a \((2n + 1)\)-dimensional Riemannian manifold. If there exists a one-to-one correspondence between each coordinate neighbourhood of \(M\) and a domain in Euclidian space such that any geodesic of the Riemannian manifold corresponds to a straight line in the Euclidean space, then \(M\) is said to be locally projectively flat. For \(n \geq 1\), \(M\) is locally projectively flat if and only if the well known projective curvature tensor \(P\) vanishes. Here \(P\) is defined by

\[
P(X, Y)Z = R(X, Y)Z - \frac{1}{2n}[S(Y, Z)X - S(X, Z)Y],
\]

for \(X, Y, Z \in T(M)\) where \(R\) is the curvature tensor and \(S\) is the Ricci tensor. In fact \(M\) is projectively flat if and only if it is of constant curvature (1953). Thus the projective curvature tensor is the measure of the failure of a Riemannian manifold to be of constant curvature.

In this paper we study projective curvature tensor on \((k, \mu)\)-contact metric manifold. After Preliminaries in section 3, we consider three cases of projective curvature tensor. At first we consider quasi-projectively flat \((k, \mu)\)-contact manifold and we prove that the manifold is a \(\eta\)-Einstein manifold. Then we consider \(\xi\)-projectively flat \((k, \mu)\)-contact manifolds. In this case we prove that the manifold is either Sasakian or \(N(k)\)-contact manifold. Finally we prove that a \(\phi\)-projectively flat \((k, \mu)\)-contact manifold is also a \(\eta\)-Einstein manifold.

2. Preliminaries. By a contact manifold we mean a \((2n + 1)\)-dimensional differentiable manifold \(M^{2n+1}\) which carries a global 1-form \(\eta\), there exists a unique vector field \(\xi\), called the characteristic vector field such that, \(\eta(\xi) = 1\) and \(d\eta(\xi, X) = 0\). A Riemannian metric \(g\) on \(M^{2n+1}\) is said to be an associated metric if there exists a \((1, 1)\) tensor field \(\phi\) such that

\[
d\eta(X, Y) = g(X, \phi Y), \quad \eta(X) = g(X, \xi), \quad \phi^2 = -I + \eta \otimes \xi.
\]
From these equations we have

$$\phi \xi = 0, \ \eta \circ \phi = 0, \ g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y).$$

(2.2)

The manifold $M$ equipped with the contact structure $(\phi, \xi, \eta, g)$ is called a contact metric manifold (Blair, 1976).

Given a contact metric manifold $M^{2n+1} (\phi, \xi, \eta, g)$ we define a $(1,1)$ tensor field $h$ by

$$h = L\phi \xi,$$

where $L$ denotes the Lie differentiation. Then $h$ is symmetric and satisfies $h\phi = -\phi h$. Thus, if $\lambda$ is an eigenvalue of $h$ with eigenvector $X$, $-\lambda$ is also an eigenvalue with eigenvector $\phi X$. Also we have $Tr h = Tr \phi h = 0$ and $h\xi = 0$. Moreover, if $\nabla$ denotes the Riemannian connection of $g$, then the following relation holds:

$$\nabla_X \xi = -\phi X - \phi h X.$$

(2.3)

A contact metric manifold is said to be Einstein if $S(X, Y) = a g(X, Y)$, where $a$ is a constant and $\eta$-Einstein if $S(X, Y) = a g(X, Y) + b \eta(X)\eta(Y)$, where $a$ and $b$ are smooth functions. A normal contact metric manifold is a Sasakian manifold. An almost contact metric manifold is a Sasakian if and only if

$$(\nabla_X \phi) Y = g(X, Y) \xi - \eta(Y) X,$$

(2.4)

$X, Y \in TM$, where $\nabla$ is the Levi-Civita connection of the Riemannian metric $g$. A contact metric manifold $M^{2n+1} (\phi, \xi, \eta, g)$ for which $\xi$ is a Killing vector field is said to be a $K$-contact metric manifold. A Sasakian manifold is $K$-contact but not conversely. However a 3-dimensional $K$-contact manifold is Sasakian (Jun and Kim, 1994). It is well known that the tangent sphere bundle of a flat Riemannian manifold admits a contact metric structure satisfying $R(X, Y) \xi = 0$ (Blair, 1977). On the other hand, on a Sasakian manifold the following holds:

$$R(X, Y) \xi = \eta(Y) X - \eta(X) Y.$$

(2.5)

It is well known that there exists contact metric manifolds for which the curvature tensor $R$ and the direction of the characteristic vector field $\xi$ satisfying $R(X, Y) \xi = 0$ for any vector fields $X$ and $Y$. For example, tangent sphere bundle of a flat Riemannian manifold admits such a structure.

As a generalisation of both $R(X, Y) \xi = 0$ and the Sasakian case: D.E. Blair, T. Koufogiorgos and B.J. Papantoniou (1995) considered the $(k, \mu)$-nullity condition on a contact metric manifold and gave several reasons for studying it. The $(k, \mu)$-nullity distribution $N(k, \mu)$ (Blair, Koufogiorgos and Papantoniou, 1995; Jun, Yildiz and De, 2008) of a contact metric manifold is defined by

$$N(k, \mu) : p \to N_p(k, \mu) = \{W \in T_p M | R(X, Y) W = (k I + \mu h)(g(Y, W) X - g(X, W) Y)\}$$

for all $X, Y \in TM$, where $(k, \mu) \in R^2$. A contact metric manifold $M^{2n+1}$ with $\xi \in N(k, \mu)$ is called a $(k, \mu)$-contact metric manifold. Then we have

$$R(X, Y) \xi = k[\eta(Y) X - \eta(X) Y] + \mu[\eta(Y) h X - \eta(X) h Y].$$

(2.6)
Applying a $D$-homothetic deformation to a contact metric manifold with $R(X, Y)\xi = 0$, we obtained a contact metric manifold satisfying (2.6). In (Blair, Koufogiorgos and Papantoniou, 1995), it is proved that the standard contact metric structure on the tangent sphere bundle $T_1(M)$ satisfies the condition that $\xi$ belongs to the $(k, \mu)$-nullity distribution if and only if the base manifold is the space of constant curvature. There exists examples in all dimensions and the condition that $\xi$ belongs to the $(k, \mu)$-nullity distribution is invariant under $D$-homothetic deformations; in dimension greater than 5, the condition determines the curvature completely; dimension 3 includes the 3-dimensional unimoduler Lie groups with the left invariant metric.

On $(k, \mu)$-contact metric manifold, $k \leq 1$. If $k = 1$, the structure is Sasakian ($h = 0$ and $\mu$ is indeterminant) and if $k < 1$, the $(k, \mu)$-nullity condition completely determines the curvature of $M^{2n+1}$ (Blair, Koufogiorgos and Papantoniou, 1995). In fact, for a $(k, \mu)$-contact manifold, the condition of being Sasakian manifold, a $K$-contact manifold, $k = 1$ and $h = 0$ are all equivalent. Again a $(k, \mu)$-contact manifold reduces to an $N(k)$-contact manifold if and only if $\mu = 0$.

In a $(k, \mu)$ contact manifold, the following relations hold (Blair, Koufogiorgos and Papantoniou, 1995 and Boeckx, 2000):

$$h^2 = (k - 1)\phi^2, \quad k \leq 1,$$

(2.7)

$$(\nabla_X \phi)Y = g(X + hX, Y)\xi - \eta(Y)(X + hX),$$

(2.8)

$$R(\xi, X)Y = k[g(X, Y)\xi - \eta(Y)X] + \mu[g(hX, Y)\xi - \eta(Y)hX],$$

(2.9)

$$S(X, \xi) = 2nk\eta(X),$$

(2.10)

$$S(X, Y) = [2(n - 1) - n\mu]g(X, Y) + (2(n - 1) + \mu)g(hX, Y)$$

$$+ [2(1 - n) + n(2k + \mu)]\eta(X)\eta(Y), \quad n \geq 1,$$

(2.11)

$$r = 2n(2n - 2 + k - n\mu),$$

(2.12)

$$S(\phi X, \phi Y) = S(X, Y) - 2nk\eta(X)\eta(Y) - 2(2n - 2 + \mu)g(hX, Y),$$

(2.13)

where $S$ is the Ricci tensor of type $(0, 2)$ and $r$ is the scalar curvature of the manifold. From (2.5) it follows that

$$\nabla_X \eta Y = g(X + hX, \phi Y).$$

(2.14)

Also in a $(k, \mu)$-manifold, the following holds

$$\eta (R(X, Y)Z) = k[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)] + \mu [g(hY, Z)\eta(X) - g(hX, Z)\eta(Y)],$$

(2.15)

if $Z \in (k, \mu)$-nullity distribution.

Especially for the case $\mu = 2(1 - n)$, from (2.13) it follows that the manifold is $\eta$-Einstein.

In a $(2n + 1)$-dimensional almost contact metric manifold $M(\phi, \xi, \eta, g)$, if $\{e_1, \ldots, e_{2n}\}$ is a local orthonormal basis of vector fields in $M$, then $\{\phi e_1, \ldots, \phi e_{2n}, \xi\}$ is also a local orthonormal basis. It is easy to verify that

$$\sum_{i=1}^{2n} g(e_i, e_i) = \sum_{i=1}^{2n} g(\phi e_i, \phi e_i) = 2n.$$

(2.16)
\[ \sum_{i=1}^{2n} g(e_i, Z)S(Y, e_i) = \sum_{i=1}^{2n} g(\phi e_i, Z)S(Y, \phi e_i) = S(Y, Z) - S(Y, \xi)\eta(Z), \quad (2.17) \]

for \( Y, Z \in T(M) \). In particular in view of \( \eta \circ \phi = 0 \), we get

\[ \sum_{i=1}^{2n} g(e_i, \phi Z)S(Y, e_i) = \sum_{i=1}^{2n} g(\phi e_i, \phi Z)S(Y, \phi e_i) = S(Y, \phi Z), \quad (2.18) \]

for \( Y, Z \in T(M) \). Moreover if \( M \) is \((k, \mu)\)-contact manifold then from (2.10) we have

\[ S(\xi, \xi) = 2nk. \quad (2.19) \]

Then we have

\[ g(R(\xi, X)Y, \xi) = kg(\phi X, \phi Y) + \mu g(hX, Y), \quad X, Y \in TM. \quad (2.20) \]

Consequently we have

\[ \sum_{i=1}^{2n} g(R(e_i, X)Y, e_i) = \sum_{i=1}^{2n} g(R(\phi e_i, X)Y, \phi e_i) = S(X, Y) - kg(\phi X, \phi Y) - \mu g(hX, Y). \quad (2.21) \]

For more details we refer to (Blair, 2002).

3. Some Structure Theorems. In this section we study quasi-projectively flat, \( \xi \)-projectively flat and \( \phi \)-projectively flat \((k, \mu)\)-contact manifolds.

Let \( C \) be the Weyl conformal curvature tensor of a \((2n + 1)\)-dimensional manifold \( M \). Since at each point \( p \in M \) the tangent space \( T_p(M) \) can be decomposed into the direct sum \( T_p(M) = \phi(T_p(M)) \oplus L(\xi_p) \), where \( L(\xi_p) \) is an 1-dimensional linear subspace of \( T_p(M) \) generated by \( \xi_p \). Then we have a map:

\[ C : T_p(M) \times T_p(M) \times T_p(M) \rightarrow \phi(T_p(M)) \oplus L(\xi_p). \]

It may be natural to consider the following particular cases:

1. \( C : T_p(M) \times T_p(M) \times T_p(M) \rightarrow L(\xi_p) \), i.e., the projection of the image of \( C \) in \( \phi(T_p(M)) \) is zero.

2. \( C : T_p(M) \times T_p(M) \times T_p(M) \rightarrow \phi(T_p(M)) \), i.e., the projection of the image of \( C \) in \( L(\xi_p) \) is zero.

3. \( \phi(T_p(M)) \times \phi(T_p(M)) \times \phi(T_p(M)) \rightarrow L(\xi_p) \), i.e., when \( C \) is restricted to \( \phi(T_p(M)) \times \phi(T_p(M)) \times \phi(T_p(M)) \), the projection of the image of \( C \) in \( \phi(T_p(M)) \) is zero. This condition is equivalent to

\[ \phi^2 C(\phi X, \phi Y, \phi Z) = 0. \]
The cases (1) and (2) were considered in (Cabrerizo, Fernandez, Fernandez and Zhen, 1997 and Zhen, 1992) respectively. The case (3) was considered in (Zhen, Cabrerizo, Fernandez and Fernandez, 1999) for the case $M$ is a $K$-contact manifold. Furthermore in (Arslan, Murathan, and Özgür, 2000), the authors studied $(k, \mu)$-contact metric manifolds satisfying (3). Analogous to the consideration of conformal curvature tensor, we give the following definitions:

**DEFINITION 3.1** A $(k, \mu)$-contact manifold is said to be quasi-projectively flat if
\[ g(P(X,Y)Z,\phi W) = 0. \] (3.1)

**DEFINITION 3.2** A $(k, \mu)$-contact manifold is said to be $\xi$-projectively flat if
\[ P(X,Y)\xi = 0. \] (3.2)

**DEFINITION 3.3** A $(k, \mu)$-contact manifold is said to be $\phi$-projectively flat if
\[ g(P(\phi X,\phi Y)\phi Z,\phi W) = 0, \] (3.3)

where $X, Y, Z, W \in T(M)$.

We begin with the following:

**LEMMA 3.1** Let $M$ be a $(2n+1)$-dimensional $(k, \mu)$-contact manifold. if $M$ satisfies
\[ g(P(\phi X, Y)Z, \phi W) = 0, \quad X, Y, Z, W \in T(M), \] (3.4)

then $M$ is $\eta$-Einstein.

**Proof:** Applying (3.4) in (1.1) we have
\[ g(R(\phi X,Y)Z, \phi W) = \frac{1}{2n} [S(Y,Z)g(\phi X, \phi W) - S(\phi X, Z)g(Y, \phi W)]. \] (3.5)

Let $\{e_1, \ldots, e_{2n}, \xi\}$ is a local orthonormal basis of vector fields in $M$, then $\{\phi e_1, \ldots, \phi e_{2n}, \xi\}$ is also a local orthonormal basis. Putting $X = W = e_i$ in both sides of (3.5) and summing over $i = 1$ to $2n$ we have
\[ \sum_{i=1}^{2n} g(R(\phi e_i, Y)Z, \phi e_i) = \frac{1}{2n} \left[ S(Y,Z) \sum_{i=1}^{2n} g(\phi e_i, \phi e_i) \right. \]
\[ \left. - \sum_{i=1}^{2n} S(\phi e_i, Z)g(Y, \phi e_i) \right]. \] (3.6)

Applying (2.16), (2.17) and (2.21) in (3.6) we have
\[ S(Y,Z) - kg(\phi Y, \phi Z) - \mu g(\eta Y, Z) = \frac{1}{2n} [(2n - 1)S(Y,Z) + S(Z, \xi)\eta(Y)]. \] (3.7)
Using (2.10) and (2.2) in (3.7) we obtain
\[ S(Y, Z) = 2nk g(Y, Z) + 2n \mu g(hY, Z). \] (3.8)

Again using (3.8) in (2.11) we have
\[ g(hY, Z) = \frac{2(n - 1) - n \mu - 2nk}{(2n - 1)\mu - 2(n - 1)} g(Y, Z) \]
\[ + \frac{2(1 - n) + n(2k + \mu)}{(2n - 1)\mu - 2(n - 1)} \eta(Y)\eta(Z). \] (3.9)

Now using (3.9) in (3.8) we obtain
\[ S(Y, Z) = \left[ 2nk + \frac{2n\mu[2(n - 1) - n \mu - 2nk]}{(2n - 1)\mu - 2(n - 1)} \right] g(Y, Z) \]
\[ + \frac{2n\mu[2(1 - n) + n(2k + \mu)]}{(2n - 1)\mu - 2(n - 1)} \eta(Y)\eta(Z). \] (3.10)

which implies that
\[ S(Y, Z) = a g(Y, Z) + b \eta(Y)\eta(Z), \] (3.11)

where \( a = \left[ 2nk + \frac{2n(2(n - 1) - n \mu - 2nk)}{(2n - 1)\mu - 2(n - 1)} \right] \) and \( b = \frac{2n\mu[2(1 - n) + n(2k + \mu)]}{(2n - 1)\mu - 2(n - 1)} \). These results show that the manifold is an \( \eta \)-Einstein manifold. This proves the Lemma. In view of Lemma 3.1 we have the following:

**THEOREM 3.1** A quasi-projectively flat \((2n + 1)\)-dimensional \((k, \mu)\)-contact manifold is an \( \eta \)-Einstein manifold.

Now we consider a \((2n + 1)\)-dimensional \((k, \mu)\)-contact metric manifold \(M^{2n+1}\) which is \( \xi \)-projectively flat. Then applying (3.2) in (1.1) we have
\[ R(X, Y) \xi = \frac{1}{2n} [S(Y, \xi)X - S(X, \xi)Y]. \] (3.12)

Using (2.10) in (3.12) we obtain
\[ R(X, Y) \xi = k[\eta(Y)X - \eta(X)Y]. \] (3.13)

With the help of (2.6) we have from (3.13)
\[ \mu[\eta(Y)X - \eta(X)hY] = 0. \] (3.14)

Putting \( X = \xi \) in (3.14) and using \( h\xi = 0 \) we obtain
\[ \mu hY = 0. \] (3.15)

Now (3.15) shows that either \( h = 0 \) or \( \mu = 0 \). If \( h = 0 \), then by (2.7) we have \( k = 1 \). Therefore in this case the manifold is a Sasakian manifold. Again if \( \mu = 0 \), then the \((k, \mu)\)-contact manifold becomes an \( N(k)\)-contact manifold.
Therefore we state the following:

**THEOREM 3.2** A $\xi$-projectively flat $(k, \mu)$-contact manifold is either a Sasakian manifold or an $N(k)$-contact manifold.

Finally we consider a $(k, \mu)$-contact manifold which is $\phi$-projectively flat. Then using (3.3) in (1.1) we obtain

$$g(R(\phi X, \phi Y)\phi Z, \phi W) = \frac{1}{2n}[S(\phi Y, \phi Z)g(\phi X, \phi W)$$
$$-S(\phi X, \phi Z)g(\phi Y, \phi W)].$$

(3.16)

Let $\{e_1, \ldots, e_{2n}, \xi\}$ is a local orthonormal basis of vector fields in $M$, then $\{\phi e_1, \ldots, \phi e_{2n}, \xi\}$ is also a local orthonormal basis. Putting $X = W = e_i$ in both sides of (3.16) and summing up over $i = 1$ to $2n$ we have

$$\sum_{i=1}^{2n} g(R(\phi e_i, \phi Y)\phi Z, \phi e_i) = \frac{1}{2n} \sum_{i=1}^{2n} S(\phi Y, \phi Z)g(\phi e_i, \phi e_i)$$
$$-\sum_{i=1}^{2n} S(\phi e_i, \phi Z)g(\phi Y, \phi e_i).$$

(3.17)

Using (2.16), (2.17) and (2.21) in (3.17) we have

$$S(\phi Y, \phi Z) = 2nk\phi Y, \phi Z) + 2n\mu g(\phi Y, \phi Z).$$

(3.18)

With the help of (2.11), the above equation becomes

$$S(\phi Y, \phi Z) = [2(n - 1) + \mu]g(\phi Y, \phi Z) + [2(n - 1) + \mu]g(\phi Y, \phi Z).$$

(3.19)

From (3.18) and (3.19) we get

$$g(\phi Y, \phi Z) = \frac{2(n - 1) - n\mu - 2nk}{(2n - 1)\mu - 2(n - 1)} g(\phi Y, \phi Z).$$

(3.20)

Using (3.20) in (3.18) we have

$$S(\phi Y, \phi Z) = \left[2nk + \frac{2n\mu[2(n - 1) - n\mu - 2nk]}{(2n - 1)\mu - 2(n - 1)} \right] g(\phi Y, \phi Z).$$

(3.21)

Substituting $Y, Z$ by $\phi Y, \phi Z$ in (3.21) and using (2.1), (2.10) we obtain

$$S(Y, Z) = \left[2nk + \frac{2n\mu[2(n - 1) - n\mu - 2nk]}{(2n - 1)\mu - 2(n - 1)} \right] g(Y, Z)$$
$$+ \frac{2n\mu[n\mu + 2nk - 2(n - 1)]}{(2n - 1)\mu - 2(n - 1)} g(Y)g(Z).$$

(3.22)
i.e.,

\[ S(Y, Z) = a\eta(Y, Z) + b\eta(Y)\eta(Z), \]

(3.23)

where \( a = \left[ 2nk + 2n(n-1) - n(n-2) \right] \) and \( b = \left[ 2n(n-1) - n(n-2) \right] \). Therefore (3.23) shows that the manifold is an \( \eta \)-Einstein manifold. Thus we state the following:

**THEOREM 3.3** A \( \phi \)-projectively flat \((2n + 1)\)-dimensional \((k, \mu)\)-contact manifold is an \( \eta \)-Einstein manifold.

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ON A CLASS OF \((k, \mu)\)-CONTACT METRIC MANIFOLDS

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Abstract. The object of the present paper is to study \(\xi\)-conformally flat, \(\varphi\)-conharmonically flat and \(\xi\)-concircularly flat contact metric manifolds with \(\xi \in (k, \mu)\)-nullity distribution.

1. Introduction

Let \(M\) be a \((2n + 1)\)-dimensional Riemannian manifold with metric \(g\) and let \(T(M)\) be the Lie algebra of differentiable vector fields on \(M\). The Ricci operator \(Q\) of \((M, g)\) is defined by
\[
g(QX, Y) = S(X, Y),
\]
where \(S\) denotes the Ricci tensor of type \((0, 2)\) on \(M\) and \(X, Y \in TM\).

Weyl [18] constructed a generalized curvature tensor on a Riemannian manifold which vanishes, whenever the metric tensor is (locally) conformally equivalent to a flat metric. The Weyl conformal curvature tensor is defined by
\[
C(X, Y)Z = R(X, Y)Z - \frac{1}{2n-1}[S(Y, Z)X - S(X, Z)Y] + g(Y, Z)QX - g(X, Z)QY + \frac{r}{2n(2n-1)}[g(Y, Z)X - g(X, Z)Y],
\]
for \(X, Y, Z \in TM\), where \(R\) and \(r\) denote the Riemannian curvature tensor and the scalar curvature of \(M\) respectively.

Let \(\bar{M}\) and \(\bar{M}\) be two Riemannian manifolds with \(g\) and \(\bar{g}\) being their respective metric tensors related through the equation
\[
\bar{g}(X, Y) = e^{2\sigma}g(X, Y), \quad (1.2)
\]
where \(\sigma\) is a real function. Then \(\bar{M}\) and \(\bar{M}\) are called conformally related manifolds and the correspondence between \(M\) and \(\bar{M}\) is known as conformal transformation [2].

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It is known that a harmonic function is defined as a function whose Laplacian vanishes. A harmonic function is not invariant, in general. The condition under which a harmonic function remains invariant have been studied by Ishii [12] who introduced the conharmonic transformation as a subgroup of the conformal transformation (1.2) satisfying the condition

\[ \sigma_t^i + \sigma_i^t = 0, \]  

(1.3)

where comma denotes the covariant differentiation with respect to the metric \( g \).

A rank four tensor \( \tilde{C} \) that remains invariant under conharmonic transformation for a \((2n + 1)\)-dimensional Riemannian manifold \( M' \), is given by

\[ g(\tilde{C}(X,Y)Z,W) = \tilde{R}(X,Y,Z,W) - \frac{1}{2n-1} [g(Y,Z)S(X,W) - g(X,Z)S(Y,W) \]  

\[ + S(Y,Z)g(X,W) - S(X,Z)g(Y,W)]. \]

where \( \tilde{R} \) denotes the Riemannian curvature tensor of type \((0, 4)\) defined by

\[ \tilde{R}(X,Y,Z,W) = g(R(X,Y)Z,W). \]

The curvature tensor defined by the equation (1.4) is known as conharmonic curvature tensor. A manifold whose conharmonic curvature vanishes at every point of the manifold is called conharmonically flat manifold. Thus this tensor represents the deviation of the manifold from conharmonic flatness. It satisfies all the symmetric properties of the Riemannian curvature tensor \( R \). There are many physical applications of the tensor \( \tilde{C} \). For example, in [1], Abdussattar showed that sufficient condition for a space-time to be conharmonic to a flat space-time is that the tensor \( \tilde{C} \) vanishes identically. A conharmonically flat space-time is either empty in which case it is flat or filled with a distribution represented by energy momentum tensor \( T \) possessing the algebraic structure of an electromagnetic field and conformal to a flat space-time [1]. Also be described the gravitational field due to a distribution of pure radiation in presence of disordered radiation by means of spherically symmetric conharmonically flat space-time. Conharmonic curvature tensor have been studied by S. A. Siddiqui and Z. Ahsan [17], C. Özgür [16] and many others.

A transformation of a \((2n + 1)\)-dimensional Riemannian manifold \( M \), which transforms every geodesic circle of \( M \) into a geodesic circle, is called a concircular transformation ([15],[19]). A concircular transformation is always a conformal transformation [15]. Here geodesic circle means a curve in \( M \) whose first curvature is constant and
second curvature is identically zero. Thus the geometry of concircular transformations, i.e., the concircular geometry, is generalization of inversive geometry in the sense that the change of metric is more general than that induced by a circle preserving diffeomorphism (see also [5]). An interesting invariant of a concircular transformation is the concircular curvature tensor \( \tilde{Z} \). It is defined by ([19],[20])

\[
\tilde{Z}(X, Y)Z = R(X, Y)Z - \frac{r}{2n(2n + 1)}[g(Y, Z)X - g(X, Z)Y],
\]

where \( X, Y, Z \in TM \). Riemannian manifolds with vanishing concircular curvature tensor are of constant curvature. Thus the concircular curvature tensor is a measure of failure of a Riemannian manifold to be of constant curvature.

Let \( M \) be an almost contact metric manifold equipped with an almost contact metric structure \( (\phi, \xi, \eta, g) \). At each point \( p \in M \), decompose the tangent space \( T_pM \) into direct sum \( T_pM = \phi(T_pM) \oplus \{\xi_p\} \), where \( \{\xi_p\} \) is the 1-dimensional linear subspace of \( T_pM \) generated by \( \{\xi_p\} \). Thus the conformal curvature tensor \( C \) is a map

\[
C : T_pM \times T_pM \times T_pM \rightarrow \phi(T_pM) \oplus \{\xi_p\}, \quad p \in M.
\]

It may be natural to consider the following particular cases:

1. \( C : T_p(M) \times T_p(M) \times T_p(M) \rightarrow L(\xi_p) \), i.e., the projection of the image of \( C \) in \( L(\xi_p) \) is zero.

2. \( C : T_p(M) \times T_p(M) \times T_p(M) \rightarrow \phi(T_p(M)) \), i.e., the projection of the image of \( C \) in \( \phi(T_p(M)) \) is zero.

3. \( C : \phi(T_p(M)) \times \phi(T_p(M)) \times \phi(T_p(M)) \rightarrow L(\xi_p) \), i.e., when \( C \) is restricted to \( \phi(T_p(M)) \times \phi(T_p(M)) \times \phi(T_p(M)) \), the projection of the image of \( C \) in \( \phi(T_p(M)) \) is zero.

Here, cases 2 and 3 are synonymous to \( \xi \)-conformally flat and \( \phi \)-conformally flat respectively.

In [21], it is proved that a \( K \)-contact manifold is \( \xi \)-conformally flat if and only if it is an \( \eta \)-Einstein Sasakian manifold. In [10], the authors studied \( \xi \)-conformally flat \( N(k) \)-contact metric manifold. Moreover, in [3] the author studied \( \phi \)-conformally flat \((k, \mu)\)-contact metric manifold. Again, conharmonic curvature tensor has been studied on \( N(k) \)-contact metric manifold in [11]. Motivated by the above studies, in this paper we study \( \xi \)-conformally flat, \( \xi \)-conharmonically flat and \( \xi \)-concircularly flat \((k, \mu)\)-contact metric manifolds. Analogous to the considerations of conformal curvature tensor, in this paper we define following:
Definition 1.1. A $(2n + 1)$-dimensional $(k, \mu)$-contact metric manifold is said to be $\xi$-conformally flat if
\[ C(X, Y)\xi = 0, \quad \text{where } X, Y \in TM. \] (1.6)

Definition 1.2. A $(2n + 1)$-dimensional $(k, \mu)$-contact metric manifold is said to be $\xi$-conharmonically flat if
\[ \tilde{Z}(X, Y)\xi = 0, \quad \text{where } X, Y \in TM. \] (1.7)

Definition 1.3. A $(2n + 1)$-dimensional $(k, \mu)$-contact metric manifold is said to be $\xi$-concircularly flat if
\[ \hat{Z}(X, Y)\xi = 0, \quad \text{where } X, Y \in TM. \] (1.8)

The paper is organized as follows:
After preliminaries in section 3, we study $\xi$-conformally flat $(k, \mu)$-contact metric manifolds and prove that a $(2n + 1)$-dimensional, $n > 1$, $(k, \mu)$-contact metric manifold is $\xi$-conformally flat if and only if $\mu = 1$. In this section we also prove that a $\xi$-conformally flat $(2n + 1)$-dimensional $(k, \mu)$-contact metric manifold is an $\eta$-Einstein manifold. Section 4 is devoted to study $(2n + 1)$-dimensional $\xi$-conharmonically flat $(k, \mu)$-contact metric manifold and prove that in such a manifold $\mu = 1$ and $k = 2 - n$, provided $n > 1$. Beside this in this section we also prove that a 5-dimensional $\xi$-conharmonically flat $(k, \mu)$-contact metric manifold has constant $\phi$-sectional curvature as a corollary. Section 5 deals with the study of $\xi$-concircularly flat $(k, \mu)$-contact metric manifolds and prove that the manifold is locally isometric to the Example 1, which has been given in section 2. Also an important corollary is given here.

2. Preliminaries

A contact manifold is a $C^\infty$ manifold $M^{2n+1}$ equipped with a global 1-form $\eta$ such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on $M^{2n+1}$. Given a contact form $\eta$, it is well known that there exists a unique vector field $\xi$, called the characteristic vector field of $\eta$, such that $\eta(\xi) = 1$ and $d\eta(X, \xi) = 0$ for any vector field $X$ on $M^{2n+1}$. A Riemannian metric $g$ is said to be associated metric if there exists a tensor field $\phi$ of type $(1,1)$ such that
\[ d\eta(X, Y) = g(X, \phi Y), \quad \eta(X) = g(X, \xi), \quad \phi^2 = -I + \eta \otimes \xi. \] (2.1)

From these equations we have
\[ \phi \xi = 0, \quad \gamma \circ \phi = 0, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y). \] (2.2)

The manifold $M$ equipped with the contact structure $(\phi, \xi, \eta, g)$ is called a contact metric manifold [7].
Given a contact metric manifold $M^{2n+1}(\phi, \xi, \eta, g)$ we define a $(1,1)$ tensor field $h$ by $h = \mathcal{L}\phi$, where $\mathcal{L}$ denotes the Lie differentiation. Then $h$ is symmetric and satisfies $h\phi = -\phi h$. Thus, if $\lambda$ is an eigenvalue of $h$ with eigenvector $X$, $-\lambda$ is also an eigenvalue with eigenvector $\phi X$. Also we have $\text{Tr} h = \text{Tr}\phi h = 0$ and $h\xi = 0$. Moreover, if $\nabla$ denotes the Riemannian connection of $g$, then the following relation holds:

$$\nabla X \xi = -\phi X - \phi h X.$$  \hspace{1cm} (2.3)

A contact metric manifold is said to be Einstein if $S(X,Y) = \alpha g(X,Y)$, where $\alpha$ is a constant and $\eta$-Einstein if $S(X,Y) = \alpha g(X,Y) + \beta\eta(X)\eta(Y)$, where $\alpha$ and $\beta$ are smooth functions. A normal contact metric manifold is a Sasakian manifold. An almost contact metric manifold is Sasakian if and only if

$$(\nabla X \phi) Y = g(X,Y)\xi - \eta(Y)X,$$  \hspace{1cm} (2.4)

$X,Y \in TM$, where $\nabla$ is the Levi-Civita connection of the Riemannian metric $g$. A contact metric manifold $M^{2n+1}(\phi, \xi, \eta, g)$ for which $\xi$ is a Killing vector field is said to be a K-contact metric manifold. A Sasakian manifold is K-contact but not conversely. However, a 3-dimensional K-contact manifold is Sasakian [13]. It is well known that the tangent sphere bundle of a flat Riemannian manifold admits a contact metric structure satisfying $R(X,Y)\xi = 0$ [4]. On the other hand, on a Sasakian manifold the following relation holds:

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y.$$  \hspace{1cm} (2.5)

It is well known that there exists contact metric manifolds for which the curvature tensor $R$ and the direction of the characteristic vector field $\xi$ satisfying $R(X,Y)\xi = 0$ for any vector fields $X$ and $Y$. For example, tangent sphere bundle of a flat Riemannian manifold admits such a structure.

As a generalization of $R(X,Y)\xi = 0$ and the Sasakian case: D. E. Blair, T. Koufogiorgos and B. J. Papantoniou [6] considered the $(k,\mu)$-nullity condition on a contact metric manifold and gave several reasons for studying it. The $(k,\mu)$-nullity distribution $N(k,\mu)$ ([6],[7]) of a contact metric manifold is defined by

$$N(k,\mu) : p \longrightarrow N_p(k,\mu) = [W \in T_pM \mid R(X,Y)W = (kI + \mu h)(g(Y,W)X - g(X,W)Y)]$$

for all $X,Y \in TM$, where $(k,\mu) \in \mathbb{R}^2$. A contact metric manifold $M^{2n+1}$ with $\xi \in N(k,\mu)$ is called a $(k,\mu)$-contact metric manifold. Thus we have

$$...R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY].$$  \hspace{1cm} (2.6)
Applying a D-homothetic deformation to a contact metric manifold with $R(X, Y)\xi = 0$, we obtain a contact metric manifold satisfying (2.6). In [6], it is proved that the standard contact metric structure on the tangent sphere bundle $T_1(M)$ satisfies the condition that $\xi$ belongs to the $(k, \mu)$-nullity distribution if and only if the base manifold is space of constant curvature. There exists examples in all dimensions and the condition that $\xi$ belongs to the $(k, \mu)$-nullity distribution is invariant under D-homothetic deformations; in dimension greater than 5, the condition determines the curvature completely; dimension 3 includes the 3-dimensional unimoduler Lie groups with the left invariant metric.

On a $(k, \mu)$-contact metric manifold, $k \leq 1$. If $k = 1$, the structure is Sasakian ($h = 0$ and $\mu$ is indeterminant) and if $k < 1$, the $(k, \mu)$-nullity condition completely determines the curvature of $M^{2n+1}$ [6]. In fact, for a $(k, \mu)$-contact manifold, the condition of being Sasakian manifold, a K-contact manifold, $k = 1$ and $h = 0$ are all equivalent. Again a $(k, \mu)$-contact manifold reduces to a $N(k)$-contact manifold if and only if $\mu = 0$.

In a $(k, \mu)$ contact metric manifold the following relations hold ([6],[9]):

\[
h^2 = (k - 1)\phi^2, \quad k \leq 1,
\]

\[
S(X, \xi) = 2nk\eta(X),
\]

\[
Q\xi = 2nk\xi,
\]

\[
S(X, Y) = [2(n - 1) - n\mu]g(X, Y) + [2(n - 1) + \mu]g(hX, Y) + [2(1 - n) + n(2k + \mu)]g(X, Y), \quad n \geq 1,
\]

\[
QX = [2(n - 1) - n\mu]X + [2(n - 1) + \mu]hX + [2(1 - n) + n(2k + \mu)]g(X, Y)\xi, \quad n \geq 1,
\]

\[
r = 2n(2n - 2 + k - n\mu).
\]

Given a non-Sasakian $(k, \mu)$-contact manifold $M$, E. Boeckx [9] introduced an invariant

\[
I_M = \frac{1 - k}{\sqrt{1 - k}}
\]

and showed that for two non-Sasakian $(k, \mu)$-manifolds $M_1$ and $M_2$, we have $I_{M_1} = I_{M_2}$ if and only if, up to a D-homothetic deformation, the two manifolds are locally
isometric as contact metric manifolds.

Thus we see that from all non-Sasakian \((k, \mu)\)-manifolds of dimension \((2n + 1)\) and for every possible value of the invariant \(I\), one \((k, \mu)\)-manifold \(M\) can be obtained with \(I_M = 1\). For \(I > -1\) such examples may be found from the standard contact metric structure on the tangent sphere bundle of a manifold of constant curvature \(c\) where we have \(I = \frac{c^2}{(n-c)}\). Boeckx also gives a Lie algebra construction for any odd dimension and value of \(I < -1\).

Using this invariant, D. E. Blair et al. [8] constructed an example of a \((2n + 1)\)-dimensional \(N(1 - \frac{1}{n})\)-contact metric manifold, \(n > 1\). The example is given in the following:

**Example 2.1.** Since the Boeckx invariant for a \((1 - \frac{1}{n}, 0)\)-manifold is \(\sqrt{n} > -1\), we consider the tangent sphere bundle of an \((n + 1)\)-dimensional manifold of constant curvature \(c\) so chosen that the resulting \(D\)-homothetic deformation will be a \((1 - \frac{1}{n}, 0)\)-manifold. That is, for \(k = c(2 - c)\) and \(\mu = -2c\) we solve

\[
1 - \frac{1}{n} = \frac{k + a^2 - 1}{a^2}, \quad 0 = \frac{\mu + 2a - 2}{a}
\]

for \(a\) and \(c\). The result is

\[
c = \frac{\sqrt{n} + 1}{n - 1}, \quad a = 1 + c
\]

and taking \(c\) and \(a\) to be these values we obtain \(N(1 - \frac{1}{n})\)-contact metric manifold.

Here we state some Lemmas which will be needed to prove main results:

**Lemma 2.2.** [4] A contact metric manifold \(M^{2n+1}\) satisfying the condition \(R(X, Y)\xi = 0\) for all \(X, Y\) is locally isometric to the Riemannian product of a flat \((n + 1)\)-dimensional manifold and an \(n\)-dimensional manifold of positive curvature \(4\), i.e., \(E^{n+1}(0) \times S^n(4)\) for \(n > 1\) and flat for \(n = 1\).

**Lemma 2.3.** [14] A \((2n + 1)\)-dimensional \((n > 1)\) non-Sasakian \((k, \mu)\)-contact Riemannian manifold \(M\) has constant \(\phi\)-sectional curvature if and only if \(\mu = k + 1\).

3. \(\xi\)-Conformally flat \((k, \mu)\)-contact metric manifolds

In this section we consider a \(\xi\)-conformally flat \((k, \mu)\)-contact metric manifold. Let \(M\) be a \((2n + 1)\)-dimensional \((k, \mu)\)-contact metric manifold which is \(\xi\)-conformally flat. Then putting \(Z = \xi\) in (1.1) and using (1.6), (2.1) and (2.8), we obtain

\[
R(X, Y)\xi = \frac{1}{2n - 1}[\eta(Y)QX - \eta(X)QY + 2nk[\eta(Y)X - \eta(X)Y]] (3.1)
\]

\[
-\frac{r}{2n(2n - 1)}[\eta(Y)X - \eta(X)Y].
\]
Using (2.6) in (3.1), we have
\[ (k - \frac{2nk}{2n - 1} \frac{r}{2n(2n - 1)}) [\eta(Y)X - \eta(X)Y] \]
\[ + \mu [\eta(Y)hX - \eta(X)hY] = \frac{1}{2n - 1} [\eta(Y)QX - \eta(X)QY]. \]
Putting \( Y = \xi \) in (3.2) and using \( h\xi = 0 \), (2.1) and (2.9), we get
\[ (k - \frac{2nk}{2n - 1} \frac{r}{2n(2n - 1)}) [X - \eta(X)] + \mu hX \]
\[ = \frac{1}{2n - 1} [QX - 2n\kappa\eta(X)\xi]. \]
Using (2.11) in (3.3) yields
\[ (k - \frac{2nk}{2n - 1} \frac{r}{2n(2n - 1)}) [X - \eta(X)] + \mu hX \]
\[ = \frac{1}{2n - 1} [(2(n - 1) - n\mu)X + (2(n - 1) + \mu)hX \]
\[ + 2(1 - n) + n\mu] \eta(X)\xi]. \]
Applying \( h \) in (3.4) and using \( h\xi = 0 \), we get
\[ (k - \frac{2nk}{2n - 1} \frac{r}{2n(2n - 1)}) hX + \mu h^2 X \]
\[ = \frac{1}{2n - 1} [(2(n - 1) - n\mu)hX + (2(n - 1) + \mu)h^2 X] \]
We take the trace on both sides of (3.5) and use \( tr.h = 0 \) yields
\[ 2(n - 1)(\mu - 1)tr.h^2 = 0. \]
Since \( tr.h^2 \neq 0 \), therefore from (3.6), we have \( \mu = 1 \) for \( n > 1 \).
Conversely, it is easy to show that for \( \mu = 1 \).
\[ C(X, Y)\xi = 0. \]
Therefore we state the following:

**Theorem 3.1.** A \((2n + 1)-\text{dimensional} \ (n > 1) \ (k, \mu)\)-contact metric manifold is \( \xi \)-conformally flat if and only if \( \mu = 1 \).

By using Lemma 2.2., we have the following:

**Corollary 3.2.** Let \( M \) be a \((2n + 1)-\text{dimensional} \ (n > 1) \ (0, \mu)\)-contact metric manifold. If \( M \) is \( \xi \)-conformally flat manifold, then \( M \) has constant \( \phi \)-sectional curvature.
Proof. Let \( M \) be a \( \xi \)-conformally flat \((0, \mu)\)-contact metric manifold of dimension \((2n + 1), (n > 1)\). Then by Theorem 3.1 \( \mu = 1 \). Hence by virtue of Lemma 2.2, \( M \) must have a constant \( \phi \)-sectional curvature. \( \square \)

In the remaining part of this section we consider \( \mu \neq 1 \). Then putting the value of \( hX \) from (2.11) in (3.4) yields

\[
QX = \frac{1}{2n-1} \left[ 2n - 2 - n\mu \right] X - \frac{\mu(2(1-n) + n(2k + \mu))}{2(n-1) + \mu} - \frac{2nk}{2n-1} \eta(X)\xi,
\]

where \( a = \frac{1}{2n-1} \). Therefore we have from (3.8)

\[
S(X,Y) = \alpha g(X,Y) + \beta \eta(X)\eta(Y),
\]

where \( \alpha \) and \( \beta \) are given by

\[
\alpha = \frac{1}{2n-1} \left[ 2n - 2 - n\mu \right] - \frac{\mu(2(1-n) + n(2k + \mu))}{2(n-1) + \mu} - \frac{2nk}{2n-1},
\]

and

\[
\beta = \frac{1}{2n-1} \left[ 2n - 2 - n\mu \right] + \frac{\mu(2(1-n) + n(2k + \mu))}{2(n-1) + \mu} - \frac{2nk}{2n-1}.
\]

In view of (3.9), we state the following:

**Theorem 3.3.** A \( \xi \)-conformally flat \((2n + 1)\)-dimensional \((k, \mu)\)-contact metric manifold is an \( \eta \)-Einstein manifold, provided \( \mu \neq 1 \).

**4. \( \xi \)-Conharmonically Flat \((k, \mu)\)-Contact Metric Manifolds**

This section is devoted to study of \( \xi \)-conharmonically flat \((k, \mu)\)-contact metric manifolds. Let a \((2n+1)\)-dimensional \((k, \mu)\)-contact metric manifold is \( \xi \)-conharmonically flat. From (1.4), we have the \((0, 3)\)-type conharmonic curvature tensor as

\[
\tilde{C}(X,Y,Z) = R(X,Y)Z - \frac{1}{2n-1} g(Y,Z)QX - g(X,Z)QY + \frac{S(Y,Z)X - S(X,Z)Y}{2n-1};
\]

Putting \( Z = \xi \) in (4.1) and using (1.7), (2.1), (2.6), (2.8) and (2.9), we get

\[
-k[\eta(Y)X - \eta(X)Y] + \mu(2n-1)[\eta(Y)hX - \eta(X)hY] = [\eta(Y)QX - \eta(X)QY].
\]

Putting \( Y = \xi \) in (4.2) and using (2.1), (2.9) and \( h\xi = 0 \) yields...
Comparing (4.3) with (2.10) we obtain

\begin{align}
2(n - 1) - \eta \mu &= -k, \quad \mu(2n - 1) = 2(n - 1) + \mu, \\
(2n + 1)k &= 2(1 - n) + 2nk + n\mu.
\end{align}

(4.4)

Solving the equations of (4.4), we get \( \mu = 1 \) and \( k = 2 - n \) for \( n > 1 \).

In view of above discussions we state the following:

**Theorem 4.1.** In a \((2n + 1)\)-dimensional, \( n > 1 \), \( \xi \)-conharmonically flat \((k, \mu)\)-contact metric manifold, \( \mu = 1 \) and \( k = 2 - n \).

Here we see that, for a 5-dimensional manifold i.e., for \( n = 2 \), \( k = 0 \) and in that case \( \mu = k + 1 \). Therefore in view of Lemma 2.2, we can state the following:

**Corollary 4.2.** A 5-dimensional \( \xi \)-conharmonically flat \((k, \mu)\)-contact metric manifold has constant \( \phi \)-sectional curvature tensor.

5. \( \xi \)-Concircularly Flat \((k, \mu)\)-Contact Metric Manifolds

This section deals with the study of \( \xi \)-concircularly flat \((k, \mu)\)-contact metric manifolds. Let a \((2n + 1)\)-dimensional \((k, \mu)\)-contact metric manifold is \( \xi \)-concircularly flat. Putting \( Z = \xi \) in (1.5) and using (2.1), we obtain

\[
Z(X, Y)\xi = R(X, Y)\xi - \frac{r}{2n(2n + 1)}[\eta(Y)X - \eta(X)Y].
\]

(5.1)

Using (1.8) in (5.1) yields

\[
R(X, Y)\xi = \frac{r}{2n(2n + 1)}[\eta(Y)X - \eta(X)Y].
\]

(5.2)

Using (2.6) in (5.2) gives

\[
[k - \frac{r}{2n(2n + 1)}][\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY] = 0.
\]

(5.3)

Putting \( Y = \xi \) in (5.3) and using (2.1) and \( h\xi = 0 \), we obtain

\[
[k - \frac{r}{2n(2n + 1)}][X - \eta(X)\xi] + \mu hX = 0.
\]

(5.4)

Applying \( h \) on (5.4) and using \( h\xi = 0 \), we get

\[
[k - \frac{r}{2n(2n + 1)}]hX + \mu h^2X = 0.
\]

(5.5)

Taking trace on both sides of (5.5) and using \( tr.h = 0 \), we have \( \mu = 0 \), since \( tr.h^2 \neq 0 \). Using \( \mu = 0 \) in (5.5), we obtain
Using the value of \( r \) from (2.12) in (5.6) gives \( k = 1 - \frac{1}{n} \). Therefore we state the following:

**Theorem 5.1.** A \((2n + 1)\)-dimensional, \( n > 1 \), \( \xi \)-concircularly flat \((k, \mu)\)-contact metric manifold is isometric to the Example 1.

For a 3-dimensional \( \xi \)-concircularly flat \((k, \mu)\)-contact metric manifold, \( n = 1 \) and so \( k = 0 \). Hence from (2.6) we have \( \mathcal{R}(X,Y)\xi = 0 \) and in view of Lemma 2.1., the manifold is flat. Therefore we state the following:

**Corollary 5.2.** A 3-dimensional \( \xi \)-concircularly flat \((k, \mu)\)-contact metric manifold is flat.

**References**


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On a class of generalized \((k, \mu)\)-contact metric manifolds

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ON A CLASS OF GENERALIZED \((k, \mu)\)-CONTACT METRIC MANIFOLDS

SUJIT GHOSH AND UDAY CHAND DE

ABSTRACT. The object of the present paper is to study 3-dimensional generalized \((k, \mu)\)-contact metric manifolds with harmonic curvature tensor. Also locally \(\phi\)-Ricci symmetric and locally \(\phi\)-conharmonically symmetric 3-dimensional generalized \((k, \mu)\)-contact manifolds have been considered. An example of a locally \(\phi\)-conharmonically symmetric three dimensional generalized \((k, \mu)\)-contact manifold has been given.

Key words and phrases: generalized \((k, \mu)\)-contact manifold, harmonic curvature tensor, locally \(\phi\)-Ricci symmetric, locally \(\phi\)-conharmonically symmetric, Einstein manifold, Sasakian manifold.

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1. INTRODUCTION

In 1995 Blair, Koufogiorgos and Papantoniou [1] introduced the notion of \((k, \mu)\)-contact metric manifolds, where \(k\) and \(\mu\) are real constants and a full classification of such manifolds was given by E. Boeckx [4]. Assuming \(k\), \(\mu\) as smooth functions, T. Koufogiorgos and C. Tsichlias [11] introduced the notion of generalized \((k, \mu)\)-contact manifolds and gave several examples. Again they also show that such manifolds does not exist in dimension greater than three. Three dimensional generalized \((k, \mu)\)-contact manifolds have been studied by few authors such as K. K. Baishya, S. Eyasmin and A. A. Shaikh [5] and F. Gouli-Andreou and P. J. Xenos [9].

In this paper we have considered three dimensional generalized \((k, \mu)\)-contact manifolds with some curvature properties. In section 2 some preliminary results of \((k, \mu)\)-contact and three dimensional generalized \((k, \mu)\)-contact manifolds are given. In section 3 we characterize 3-dimensional generalized \((k, \mu)\)-contact manifolds with harmonic curvature tensor and prove that the manifold is either an Einstein Sasakian manifold or a flat manifold. In section 4 we study locally \(\phi\)-Ricci symmetric generalized \((k, \mu)\)-contact manifolds and prove that the scalar curvature of this manifold is constant. Section 5 of our paper deals with a locally \(\phi\)-conharmonically symmetric generalized \((k, \mu)\)-contact manifold which is not a \((k, \mu)\)-contact manifold. In this section we prove that a 3-dimension generalized \((k, \mu)\)-contact manifold which is not a \((k, \mu)\)-contact manifold is locally \(\phi\)-conharmonically symmetric if and only if the scalar curvature \(r\) vanishes. Finally in the last section we construct an example of a three dimensional locally \(\phi\)-conharmonically symmetric generalized \((k, \mu)\)-contact manifold.
2. Preliminaries

A contact manifold is a $C^0$ manifold $M^{2n+1}$ equipped with a global 1-form $\eta$ such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on $M^{2n+1}$. Given a contact form $\eta$ it is well known that there exists a unique vector field $\xi$, called the characteristic vector field of $\eta$, such that $\eta(\xi) = 1$ and $d\eta(X, \xi) = 0$ for any vector field $X$ on $M^{2n+1}$. A Riemannian metric $g$ is said to be associated metric if there exists a tensor field $\phi$ of type $(1,1)$ such that

\begin{equation}
(2.1) \quad d\eta(X, Y) = g(X, \phi Y), \quad \eta(X) = g(X, \xi), \quad \phi^2 = -I + \eta \otimes \xi.
\end{equation}

From these equations we have

\begin{equation}
(2.2) \quad \phi \xi = 0, \quad \eta \circ \phi = 0, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y).
\end{equation}

The manifold $M$ equipped with the contact structure $(\phi, \xi, \eta, g)$ is called a contact metric manifold [2].

Given a contact metric manifold $M^{2n+1}(\phi, \xi, \eta, g)$ we define a $(1,1)$ tensor field $h$ by $h = L_\phi \phi$, where $L$ denotes the Lie differentiation. Then $h$ is symmetric and satisfies $h\phi = -\phi h$. Thus, if $\lambda$ is an eigenvalue of $h$ with eigenvector $X$, $-\lambda$ is also an eigenvalue with eigenvector $\phi X$. Also we have $\text{Tr} h = \text{Tr} \phi h = 0$ and $h\xi = 0$. Moreover, if $\nabla$ denotes the Riemannian connection of $g$, then the following relation holds:

\begin{equation}
(2.3) \quad \nabla_X \xi = -\phi X - \phi h X.
\end{equation}

A contact metric manifold is said to be Einstein if $S(X, Y) = a g(X, Y)$, where $a$ is a constant and $\eta$-Einstein if $S(X, Y) = ag(X, Y) + \eta(X)\eta(Y)$, where $a$ and $b$ are smooth functions. A normal contact metric manifold is a Sasakian manifold. An almost contact metric manifold is Sasakian if and only if

\begin{equation}
(2.4) \quad (\nabla_X \phi) Y = g(X, Y)\xi - \eta(Y)X,
\end{equation}

$X, Y \in TM$, where $\nabla$ is the Levi-Civita connection of the Riemannian metric $g$.

A contact metric manifold $M^{2n+1}(\phi, \xi, \eta, g)$ for which $\xi$ is a Killing vector field is said to be a K-contact metric manifold. A Sasakian manifold is K-contact but not conversely. However a 3-dimensional K-contact manifold is Sasakian[10]. It is well known that the tangent sphere bundle of a flat Riemannian manifold admits a contact metric structure satisfying $R(X, Y)\xi = 0$ [3]. On the other hand, on a Sasakian manifold the following relation holds:

\begin{equation}
(2.5) \quad R(X, Y)\xi = \eta(Y)X - \eta(X)Y.
\end{equation}

It is well known that there exists contact metric manifolds for which the curvature tensor $R$ and the direction of the characteristic vector field $\xi$ satisfying $R(X, Y)\xi = 0$ for any vector fields $X$ and $Y$. For example, tangent sphere bundle of a flat Riemannian manifold admits such a structure.

As a generalisation of $R(X, Y)\xi = 0$ and the Sasakian case : D. E. Blair, T. Koufogiorgos and B. J. Papantoniou [1] considered the $(k, \mu)$-nullity condition on a
contact metric manifold and gave several reasons for studying it. The \((k, \mu)\)-nullity distribution \(N(k, \mu)\) of a contact metric manifold is defined by

\[
N(k, \mu) : p \rightarrow N_p(k, \mu) = \{W \in T_pM \mid R(X, Y)W = (kI + \mu h)(g(Y, W)X - g(X, W)Y)\}
\]

for all \(X, Y \in TM\), where \((k, \mu) \in \mathbb{R}^2\). A contact metric manifold \(M^{2n+1}\) with \(\xi \in N(k, \mu)\) is called a \((k, \mu)\)-contact metric manifold. Thus we have

\[
R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY].
\]

Applying a D-homothetic deformation to a contact metric manifold with \(R(X, Y)\xi = 0\), we obtain a contact metric manifold satisfying (2.6). In [1], it is proved that the standard contact metric structure on the tangent sphere bundle \(T_1(M)\) satisfies the condition that \(\xi\) belongs to the \((k, \mu)\)-nullity distribution if and only if the base manifold is the space of constant curvature. There exist examples in all dimensions and the condition that \(\xi\) belongs to the \((k, \mu)\)-nullity distribution is invariant under D-homothetic deformations; in dimension greater than 5, the condition determines the curvature completely; dimension 3 includes the 3-dimensional unimodular Lie groups with the left invariant metric.

On \((k, \mu)\)-contact metric manifold, \(k \leq 1\). If \(k = 1\), the structure is Sasakian \((h = 0 \text{ and } \mu \text{ is indeterminant})\) and if \(k < 1\), the \((k, \mu)\)-nullity condition completely determines the curvature of \(M^{2n+1}\) [1]. In fact, for a \((k, \mu)\)-contact manifold, the condition of being Sasakian manifold, a K-contact manifold, \(k = 1\) and \(h = 0\) are all equivalent. Again a \((k, \mu)\)-contact manifold reduces to an \(N(k)\)-contact manifold if and only if \(\mu = 0\).

In a \((k, \mu)\) contact metric manifold, the following relations hold [1], [4]:

\[
(2.7) \quad h^2 = (k-1)\phi^2, \quad k \leq 1,
\]

\[
(2.8) \quad (\nabla_X \phi)Y = g(X + hX, Y)\xi - \eta(Y)(X + hX),
\]

\[
(2.9) \quad R(\xi, X)Y = k[g(Y, X)\xi - \eta(Y)X] + \mu[g(hX, Y)\xi - \eta(Y)hX],
\]

\[
(2.10) \quad S(X, \xi) = 2\kappa n\eta(X),
\]

\[
(2.11) \quad S(X, Y) = [2(n - 1) - n]\eta[g(X, Y) + [2(n - 1) + \mu]g(hX, Y)] + [2(1 - n) + n(2k + \mu)]\eta(X)\eta(Y), \quad n \geq 1,
\]

\[
(2.12) \quad r = 2n(2n - 2 + k - n\mu),
\]

\[
(2.13) \quad S(\phi X, \phi Y) = S(X, Y) - 2n\kappa n\eta(X)\eta(Y) - 2(2n - 2 + \mu)g(hX, Y),
\]

where \(S\) is the Ricci tensor of type \((0, 2)\) and \(r\) is the scalar curvature of the manifold. From (2.3) it follows that

\[
(2.14) \quad (\nabla_X \eta)Y = g(X + hX, \phi Y).
\]
A generalized \((k, \mu)\)-contact metric manifold \(M^3(\phi, \xi, \eta, g)\) is a \((k, \mu)\)-contact metric manifold in which \(k\) and \(\mu\) are smooth functions on \(M\). In \([11]\) the authors prove that a generalized \((k, \mu)\)-contact metric manifold does not exist for dimension greater than three. Hence the generalized \((k, \mu)\)-contact metric manifold exists for dimension three and several examples are given in \([11]\). In a generalized \((k, \mu)\)-contact metric manifold \(M^3(\phi, \xi, \eta, g)\), beside the relations (2.1) – (2.15) the following relations also hold \([11]\): \([12]\):

\[
\begin{align*}
(2.16) \quad \xi \kappa &= 0, \\
(2.17) \quad \xi \tau &= 0, \\
(2.18) \quad h \text{ grad } \mu &= \text{ grad } k.
\end{align*}
\]

3. THREE DIMENSIONAL GENERALIZED \((k, \mu)\)-CONTACT MANIFOLDS WITH HARMONIC CURVATURE TENSOR

The curvature of a Riemannian manifold is said to be harmonic if the divergence of its curvature tensor is zero. It is well known that a Riemannian manifold has harmonic curvature tensor if and only if

\[
(3.1) \quad (\nabla_X S)(Y, Z) = (\nabla_Y S)(X, Z),
\]

for any vector fields \(X\), \(Y\) and \(Z\). One reason for the interest in such manifolds is that a Riemannian manifold has harmonic curvature tensor if and only if the Riemannian connection is a solution of the Yang-Mills equations on the tangent bundle. If \(S\) is parallel, then (3.1) is valid, but not conversely. So it is meaningful to study a three dimensional generalized \((k, \mu)\)-contact manifolds by replacing the hypothesis that \(S\) is parallel with the weaker hypothesis (3.1).

In a 3-dimensional generalized \((k, \mu)\)-contact manifold the Ricci tensor \(S\) is given by

\[
(3.2) \quad S(Y, Z) = -\mu g(Y, Z) + \mu g(hY, Z) + (2k + \mu)\eta(Y)\eta(Z).
\]

Differentiating (3.2) covariantly with respect to \(X\) we have

\[
(3.3) \quad (\nabla_X S)(Y, Z) = -(X\mu)g(Y, Z) + (X\mu)g(hY, Z)
+ \mu g((\nabla_X h)Y, Z) + (2Xk + X\mu)\eta(Y)\eta(Z)
+ (2k + \mu)((\nabla_X \eta)(Y)\eta(Z) + \eta(Y)(\nabla_X \eta)(Z)).
\]

Using (2.14) and (2.15) in (3.3) we obtain
Generalized \((k, \mu)\)-contact metric manifolds

\[(\nabla_X S)(Y, Z) = - (X \mu)g(Y, Z) + (X \mu)g(hY, Z)
+ \mu(1 - k)g(X, \phi Y)\eta(Z) + \mu g(X, h\phi Y)\eta(Z)
+ \mu \eta(Y)g(h\phi X, Z) + \mu \eta(Y)g(h\phi hX, Z)
- \mu^2 \eta(Y)g(h\phi hY, Z) + (2k + \mu)\eta(Y)g(hX, \phi Y)\eta(Z)
+ \eta(Y)g(X, \phi Z) + \eta(Y)g(hX, \phi Z).\]

Again using (3.4) in (3.1) and then using (2.7) we have
\[(3.5) - (X \mu)g(Y, Z) + (X \mu)g(hY, Z) + \mu(1 - k)g(X, \phi Y)\eta(Z)
+ \mu g(X, h\phi Y)\eta(Z) + \mu(1 - k)\eta(Y)g(hX, \phi Y)\eta(Z)
+ \mu^2 \eta(Y)g(h\phi hY, Z) + (2k + \mu)\eta(Y)g(hX, \phi Y)\eta(Z)
+ \mu^2 \eta(Y)g(h\phi hX, Z) + (2k + \mu)\eta(Y)g(hX, \phi Y)\eta(Z)
+ \mu^2 \eta(Y)g(h\phi hX, Z) + (2k + \mu)\eta(Y)g(hX, \phi Y)\eta(Z)
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+ \mu^2 \eta(Y)g(h\phi hX, Z) + (2k + \mu)\eta(Y)g(hX, \phi Y)\eta(Z)
+ \mu^2 \eta(Y)g(h\phi hX, Z) + (2k + \mu)\eta(Y)g(hX, \phi Y)\eta(Z).
\]

Putting \(X = Z = \xi\) in (3.5) and using (2.7) we obtain
\[(3.6) (Y \kappa) = 0.\]

The equation (3.6) implies that \(k\) is a constant.

Then by (2.18) we have

\[h \text{ grad } \mu = 0.\]

i.e., either \(h = 0\) or \(\mu\) is constant.

Using \(h = 0\) in (2.7) we have \(k = 1\).

In view of the above discussions we state the following:

**Proposition 3.1.** A three dimensional generalized \((k, \mu)\)-contact manifold with harmonic curvature tensor is either Sasakian or a \((k, \mu)\)-contact manifold.

In Lemma (3.7) of [6] we see that a Sasakian manifold with harmonic curvature tensor is Einstein. Again in [13] the author prove that a three dimensional \((k, \mu)\)-contact manifold with harmonic curvature tensor is a flat manifold. Hence we state the following:
Theorem 3.1. A three dimensional generalized \((k, \mu)\)-contact manifold with harmonic curvature tensor is either an Einstein Sasakian manifold or a flat manifold.

4. Locally \(\phi\)-Ricci Symmetric Three Dimensional Generalized
\((k, \mu)\)-Contact Manifolds

The notion of locally \(\phi\)-symmetry first introduced by T. Takahashi [14] on a Sasakian manifold. Again in a recent paper [7] U. C. De and Avijit Sarkar introduced the notion of locally \(\phi\)-Ricci symmetric Sasakian manifolds. In this paper we introduced the notion of locally \(\phi\)-Ricci symmetric three dimensional generalized \((k, \mu)\)-contact manifolds.

Definition 4.1. A three dimensional generalized \((k, \mu)\)-contact manifold is said to be locally \(\phi\)-Ricci symmetric if the Ricci operator \(Q\) defined by \(g(QX, Y) = S(X, Y)\) satisfies

\[
\phi^2(\nabla Q)(Y) = 0,
\]

where \(X, Y\) are horizontal vector fields i.e., \(X, Y\) are orthogonal to \(\xi\).

In a three dimensional generalized \((k, \mu)\)-contact manifold the Ricci operator \(Q\) is given by

\[
QX = -\mu X + \mu hX + (2k + \mu)\eta(X)\xi.
\]

Differentiating (4.2) covariantly with respect to \(Z\) we get

\[
(\nabla Z Q)(X) = -(Z\mu)X + (Z\mu)hX + \mu((1-k)g(Z, \phi X)\xi + g(Z, hX)\xi + \mu(1-k)g(Z, \phi Z)\xi + \mu g(Z, hZ, \phi X)\xi + (2k + \mu)\eta(X)\eta(X)\xi + (2k + \mu)\eta(X)(-\phi Z - \phi hZ).
\]

Taking \(\phi^2\) on both sides of (4.4) and applying (2.1) and (2.2) we obtain

\[
\phi^2(\nabla Z Q)(X) = -(Z\mu)(-X + \eta(X)\xi + \eta(X)\xi + \mu(1-k)g(Z, \phi Z + \phi hZ)\xi + \mu g(Z, hZ, \phi X)\xi + (2k + \mu)\eta(X)(-\phi Z - \phi hZ).
\]

Taking \(\phi^2\) on both sides of (4.4) and applying (2.1) and (2.2) we obtain

\[
(2k + \mu)\eta(X)(-\phi Z - \phi hZ).
\]

Since the manifold of our consideration is locally \(\phi\)-Ricci symmetric, then we obtain from (4.5)

\[
(2k + \mu)\eta(X)(-\phi Z - \phi hZ).
\]

Therefore from (4.6) we have either \(Z\mu = 0\) or \(-X + \eta(X)\xi + hX = 0\). Now putting \(\phi X\) instead of \(X\) in the relation \(-X + \eta(X)\xi + hX = 0\) we obtain
Since $\phi X$ is non-zero then the relation (4.7) gives $h = 1$.

Using $h = 1$ in (2.7) and then using (2.1) we obtain

\[(4.8)\quad kX = (k - 1)\eta(X)\xi.\]

Putting $\phi X$ instead of $X$ in (4.8) and using (2.2) we have

\[(4.9)\quad k(\phi X) = 0.\]

Since $\phi X$ is non-zero we have $k = 0$. Again using $k = 0$ in (2.18) we have

\[(4.10)\quad h\ \text{grad}\ \mu = 0.\]

Since $h = 1$ therefore $\mu$ must be constant. Therefore the relation (2.18) gives $k$ is also constant. Hence in this case the manifold is a $(k, \mu)$-contact manifold.

Again $Z\mu = 0$ gives $\mu$ is constant. Therefore from (2.18) we have $k$ is also constant.

Hence in either case the manifold is a $(k, \mu)$-contact manifold. Therefore we state the following:

**Proposition 4.1.** A locally $\phi$-Ricci symmetric three dimensional generalized $(k, \mu)$-contact manifold is a $(k, \mu)$-contact manifold.

Again in a recent paper [8] we prove that a three dimensional $(k, \mu)$-contact manifold is locally $\phi$-Ricci symmetric if and only if the scalar curvature is constant. Hence we state the following:

**Theorem 4.1.** A three dimensional generalized $(k, \mu)$-contact manifold is locally $\phi$-Ricci symmetric if and only if the scalar curvature is constant.
Using (2.11) and (4.2) in (5.1) we have

\[
C(X, Y)Z = R(X, Y)Z + \mu g(Y, Z)X - \mu g(Y, Z)hX \\
-(2k + \mu)g(Y, Z)\eta(X)\xi - \mu g(X, Z)Y \\
+ \mu g(X, Z)hY + (2k + \mu)g(X, Z)\eta(Y)\xi \\
+ \mu g(Y, Z)X - \mu g(hY, Z)X \\
-(2k + \mu)\eta(Y)\eta(Z)X \\
- \mu g(X, Z)Y + \mu g(hX, Z)Y \\
+(2k + \mu)\eta(X)\eta(Z)Y.
\]

Putting \( Z = \xi \) in (5.2) and using \( h\xi = 0, \eta(\xi) = 1, (2.1) \) and (2.6) we obtain

\[
C(X, Y)\xi = (k - \mu)\eta(X)Y - \eta(Y)X.
\]

Differentiating (5.3) covariantly with respect to \( W \) and using (2.14) we have

\[
(\nabla_W C)(X, Y)\xi = \left[(2k - Z\mu)\eta(X)\phi^2 Y - \eta(Y)\phi^2 X \right] \\
+ (k - \mu)[g(W, \phi X)Y + g(hW, \phi X)Y \\
- g(W, \phi Y)X - g(hW, \phi Y)X].
\]

Taking \( \phi^2 \) on both sides of (5.4) we have

\[
\phi^2(\nabla_W C)(X, Y)\xi = \left[(Wk - W\mu)\eta(X)\phi^2 Y - \eta(Y)\phi^2 X \right] \\
+ (k - \mu)[g(W, \phi X)\phi^2 Y \\
+ g(hW, \phi X)\phi^2 Y - g(W, \phi Y)\phi^2 X \\
- g(hW, \phi Y)\phi^2 X].
\]

Since the manifold is locally \( \phi \)-conharmonically symmetric therefore (5.5) yields

\[
\phi^2(W - W\mu)[\eta(X)\phi^2 Y - \eta(Y)\phi^2 X] \\
+ (k - \mu)[g(W, \phi X)\phi^2 Y \\
- g(W, \phi Y)\phi^2 X - g(hW, \phi Y)\phi^2 X] = 0.
\]

Hence either

\[
k - \mu = 0
\]

or

\[
[g(W, \phi X)\phi^2 Y + g(hW, \phi X)\phi^2 Y \\
- g(W, \phi Y)\phi^2 X - g(hW, \phi Y)\phi^2 X] = 0.
\]

Since in a three dimensional generalized \((k, \mu)\)-contact manifold the scalar curvature \( r = 2(k - \mu) \), therefore for the first case the scalar curvature \( r = 0 \). Again if in a three dimensional generalized \((k, \mu)\)-contact manifold the scalar curvature \( r \) vanishes, then \( k - \mu = 0 \) and in view of (5.5) we have the manifold is locally \( \phi \)-conharmonically symmetric.

Now for the second case using (2.1) in (5.8) we have

\[
g(W, \phi Y)X - g(W, \phi X)Y + g(hW, \phi Y)X - g(hW, \phi X)Y = 0.
\]
Replacing \( W \) by \( hW \) in (5.9) and using (2.7) and (2.2) we have
\[
(5.10) \quad g(hW, \phi Y)X - g(hW, \phi X)Y = (k - 1)[g(W, \phi Y)X - g(W, \phi X)Y].
\]
Using (5.10) in (5.9) we get
\[
(5.11) \quad k[g(W, fY)X - g(W, fX)Y] = 0.
\]
Now \( g(W, fY)X - g(W, fX)Y \neq 0 \), since if \( g(W, fY)X - g(W, fX)Y = 0 \), then putting \( X = \xi \) and using (2.2) we have \( g(W, fY)\xi = 0 \), which can not happen. Therefore in this case \( k = 0 \). Hence by the similar argument as in section 4 we have the manifold is a \((k, \mu)\)-contact manifold.

Since we consider the manifold is not a \((k, \mu)\)-contact manifold therefore the second case can not happen. Hence we can state the following:

**Theorem 5.1.** A 3-dimensional generalized \((k, \mu)\)-contact manifold which is not a \((k, \mu)\)-contact manifold is locally \( \phi \)-conharmonically symmetric if and only if the scalar curvature \( r \) vanishes.

6. **Example**

In this section we construct an example of a locally \( \phi \)-conharmonically symmetric three dimensional generalized \((k, \mu)\)-contact manifold. We consider 3-dimensional manifold \( M = \{(x, y, z) \in \mathbb{R}^3 \mid x \neq 0 \} \), where \( (x, y, z) \) are the standard coordinate in \( \mathbb{R}^3 \). Let \( \{e_1, e_2, e_3\} \) be linearly independent global frame on \( M \) given by
\[
e_1 = \frac{2}{x} \frac{\partial}{\partial y}, \quad e_2 = \frac{2}{x} \frac{\partial}{\partial z} - \frac{4z}{x} \frac{\partial}{\partial y} + \frac{xy}{z} \frac{\partial}{\partial z}, \quad e_3 = \frac{\partial}{\partial z}.
\]
Let \( g \) be the Riemannian metric defined by
\[
g(e_1, e_2) = g(e_2, e_3) = g(e_1, e_3) = 0, \quad g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1.
\]
Let \( \eta \) be the 1-form defined by
\[
\eta(U) = g(U, e_3)
\]
for any \( U \in \chi(M) \). Let \( \phi \) be the \((1, 1)\)-tensor field defined by
\[
\phi e_1 = e_2, \quad \phi e_2 = -e_1, \quad \phi e_3 = 0.
\]
Using the linearity of \( \phi \) and \( g \) we have
\[
\eta(e_3) = 1, \quad \phi^2(U) = -U + \eta(U)e_3
\]
and
\[
g(\phi U, \phi W) = g(U, W) - \eta(U)\eta(W)
\]
for any \( U, W \in \chi(M) \). Moreover
\[
h e_1 = -e_1, \quad h e_2 = e_2 \quad \text{and} \quad h e_3 = 0.
\]
Thus for \( e_3 = \xi, \phi, \xi, \eta, g \) defines a contact metric structure on \( M \).

Let \( \nabla \) be the Levi-Civita connection with respect to the Riemannian metric \( g \). Then we have
\[
[e_1, e_2] = 2e_3 + \frac{2}{x} e_1, \quad [e_1, e_3] = 0, \quad [e_2, e_3] = 2e_1.
\]
The Riemannian connection $\nabla$ of the metric tensor $g$ is given by
\[ 2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]). \]

Taking $e_3 = \xi$ and using the above formula for the Riemannian metric $g$, we can easily calculate
\[ \nabla_{e_1} e_1 = -\frac{2}{z} e_2, \quad \nabla_{e_1} e_2 = \frac{2}{z} e_1, \quad \nabla_{e_1} e_3 = 0, \]
\[ \nabla_{e_2} e_1 = -2e_3, \quad \nabla_{e_2} e_2 = 0, \quad \nabla_{e_2} e_3 = 2e_1, \]
\[ \nabla_{e_3} e_1 = 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_3 = 0. \]

From the above it can be easily seen that $(\phi, \xi, \eta, g)$ is a generalized $(k, \mu)$-contact structure on $M$. Consequently $M^3(\phi, \xi, \eta, g)$ is a generalized $(k, \mu)$-contact metric manifold with $k = -\frac{2}{z} \neq 0$ and $\mu = -\frac{2}{z} \neq 0$.

Using the above relations, we can easily calculate the non-vanishing components of the curvature tensor as follows:
\[ R(e_1, e_2) e_3 = -\frac{4}{z} e_2, \quad R(e_2, e_3) e_1 = \frac{4}{z} e_2, \quad R(e_2, e_3) e_2 = \frac{4}{z} e_1. \]

Using the relation $S(X, Y) = \sum_{i=1}^{3} g(R(e_i, X)Y, e_i)$ we get the following:
\[ S(e_1, e_1) = 0, \quad S(e_2, e_2) = 0, \quad S(e_3, e_3) = 0. \]

Therefore the scalar curvature
\[ r = S(e_1, e_1) + S(e_2, e_2) + S(e_3, e_3) = 0. \]

Therefore in view of Theorem 5.1 we can say that this manifold is a locally $\phi$-conharmonically symmetric generalized $(k, \mu)$-contact manifold.

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