CHAPTER 4

PARAMETRIC INSTABILITIES IN A PLASMA

WITH RANDOM FLUCTUATION
4.1 Introduction

The investigation of parametric instability in presence of background turbulence in a plasma is of great relevance to laser fusion problems. Plasma produced by irradiating pellets with a laser or some other agency promotes some fluctuations and is likely to be turbulent in general. The background turbulence will create random density fluctuation and may influence propagation characteristics of the interacting waves of the parametric coupling process. There is a growing interest in the study of the effects of both random and regular density fluctuation upon weak turbulence process involving waves and particles in plasma. Sturrock\(^1\), Jackson and Raether\(^2\), Liu and Rowe\(^3\), Tamolkin and Fahnstin\(^4\), Liu and Rosenbluth\(^5\) and White et al\(^6\), Laval et al\(^7\) considered the effect of finite bandwidth pump wave on the parametric instability using W.K.B. approximation. Coste et al\(^8\) have studied the propagation of unstable plasma waves in presence of random inhomogeneities. Valeo and Oberman\(^9\), Anderson and Bendeson\(^10\) and Brissand and Frish\(^11\) considered the parametric instability of an imperfect pump with random phase.

One model that is often assumed for laser interaction with random turbulent plasma is the stochastic variation of light phase. According to this model the effects of fluctuations can be interpreted as resulting from interference between waves which have acquired different phase shifts during their passage through random inhomogeneities. Valeo and Oberman considered the problem
where the phase changes with many small random jumps and diffuses with a diffusion coefficient which is shown to be equal to band width.

In our findings the basic system of equations for non-linear three wave interactions are reduced to that of the random oscillator. Growth rates are determined for arbitrary excited modes\textsuperscript{12}). In our analysis we shall recall the usual coupled mode equations with a stochastic driver whose phase only stochastically changes (as a Uhlenbeck-Ornstein process)\textsuperscript{13}).

In this work the effect of quasistatic random density fluctuation on the scattering of an intense electromagnetic wave off an electrostatic plasma wave is discussed. The assumption that the density fluctuation is quasistatic is justified if the time period of the density fluctuation is long compared with the time of growth of instability of the waves. We assume that the fluctuations is time independent and the degree of random inhomogeneity is small. One can derive an equation for the ensemble average of the wave amplitudes \( \langle E \rangle \) with its phase as a random function of position in the framework of Bourret's integral equation discussed in Van Kampen\textsuperscript{14}).
4.2 Derivation of the equations which describes the slow varying evolution of complex amplitudes in a plasma medium with random optical index

Let $X(t)$ and $Y(t)$ be two oscillating variables that obey the following equations of damped oscillations.

\[
\mathcal{L}_1 X(t) \equiv \left[ \frac{d^2}{dt^2} + 2 \nu_1 \frac{d}{dt} + (\omega_1^2 + \kappa_1^2) \right] X(t) = 0 \quad (4.2.1)
\]

\[
\mathcal{L}_2 Y(t) \equiv \left[ \frac{d^2}{dt^2} + 2 \nu_2 \frac{d}{dt} + (\omega_2^2 + \kappa_2^2) \right] Y(t) = 0 \quad (4.2.2)
\]

we assume that the frequencies $\omega_s$ and damping rates $\nu_s (s=1,2)$ are constants. Without loss of generality one can assume

\[
\omega_1^2 < \omega_2^2 \quad (4.2.3)
\]

we introduce spatially homogeneous external field of the form

\[
\mathcal{Z}(t) = 2 \zeta_0 \cos \omega_0 t, \quad (\zeta_0 = \text{constant}) \quad (4.2.4)
\]

which produces coupling between $X(t)$ and $Y(t)$. One can think of various type of coupling. Here for simplicity we consider the coupling of the form

\[
\mathcal{L}_1 X(t) = \mu \mathcal{Z}(t) Y(t) \quad (4.2.5)
\]

\[
\mathcal{L}_2 Y(t) = \mu \mathcal{Z}(t) X(t) \quad (4.2.6)
\]
with $\lambda$ and $\mu$ being constant. In this equation the nonlinear modulation of $Y(t) \left[ \omega \times (t) \right]$, due to the external field $\mathbf{E}(t)$, acts as a source to produce a forced oscillation of $X(t) \left[ \omega \gamma (t) \right]$. If this forced oscillation resonates with the natural oscillation, one can expect a resonant energy transfer between the external field and the natural oscillations. Resonance occurs when

$$\omega_0 = \omega_1 + \omega_2$$

(4.2.7)

Taking the Fourier's transform of (4.2.5) and (4.2.6) we obtain

$$D_1(\omega) \times (\omega) = \lambda \mathbf{E}_0 \left[ Y(\omega + \omega_0) + Y(\omega - \omega_0) \right]$$

(4.2.8)

$$D_2(\omega \pm \omega_0) \gamma (\omega \pm \omega_0) = \mu \mathbf{E}_0 \left[ X(\omega) + X(\omega \pm 2\omega_0) \right]$$

(4.2.9)

where $D_2(\omega) = -\omega^2 - 2i\omega \omega_0 + \omega_0^2 + \nu^2$. Equations (4.2.8) and (4.2.9) show that $X(\omega)$ couples with $Y(\omega \pm \omega_0)$, which in turn couples with $X(\omega)$ and $X(\omega \pm 2\omega_0)$. If we are interested in the frequency range of $\Re \omega = \omega_1$, we can neglect $X(\omega \pm 2\omega_0)$ as being off resonant. However, we should retain both $Y(\omega \pm \omega_0)$, since $\omega_1$ (and hence $\omega_0$) can be very small compared with $\omega_0$, so that both $(\omega + \omega_0)$ and $(\omega - \omega_0)$ can stay in the resonant frequency range (near $\pm \omega_2$).

When the pump field has a finite wave number, being of the form $\mathbf{E}(\mathbf{r}, t) = 2 \mathbf{E}_0 \mathbf{e}_\gamma (\mathbf{k}_0 \cdot \mathbf{\gamma} - \omega_0 t)$, we have to
introduce a space time Fourier component of $X$ and $Y$. Equations (4.2.8) and (4.2.9) will then be modified to the form

$$D_1(\omega, \mathbf{k}) \times (\omega, \mathbf{k}) = z_0 \left[ \lambda - \gamma (\omega - \omega_0, \mathbf{k} - \mathbf{k}_0) + \lambda + \gamma (\omega + \omega_0, \mathbf{k} + \mathbf{k}_0) \right]^{(4.2.10)}$$

$$D_2(\omega + \omega_0, \mathbf{k} + \mathbf{k}_0) \gamma (\omega + \omega_0, \mathbf{k} + \mathbf{k}_0) = \mu t z_0 \times (\omega, \mathbf{k})^{(4.2.11)}$$

where

$$D(\omega, \mathbf{k}) = -\omega^2 - 2i \gamma_1 (\mathbf{k}) \omega + \omega_0^2 (\mathbf{k})^2 + \gamma_2 (\mathbf{k})^{(4.2.12)}$$

$\omega_0 (\mathbf{k})$ and $\gamma_1 (\mathbf{k})$ being the frequency and damping of the $\mathbf{k}$th mode of wave number $\mathbf{k}$ in the absence of coupling and we have ignored the off-resonant contribution $\times (\omega \pm 2\omega_0, \mathbf{k} \pm 2\mathbf{k}_0)$. The coupling constant in general depends on the angle between the wave number vectors of the interacting waves, so we used suffix $\pm$ to denote this effect.

We assume that the medium is uniform and the pump is sinusoidal, but now consider the situation in which the perturbation is initiated at a local point say $\chi = 0$. The perturbation then consists of a wave packet rather than a normal mode, but still satisfies approximately the resonance condition for decay instability. We, therefore, write

$$X(\chi, t) = \tilde{X}(\chi, t) \exp \left[ i (k_1 \chi - \omega_1 t) \right]$$

$$Y(\chi, t) = \tilde{Y}(\chi, t) \exp \left[ i (k_2 \chi - \omega_2 t) \right]$$
with \( \mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_0 \), \( \omega_1 + \omega_2 = \omega_0 \) where \( \tilde{X} \) and \( \tilde{Y} \) are assumed to be slowly varying in space and time. The effect of slow variations of the amplitude can be taken care of by generalizing equations (4.2.10) and (4.2.11) as

\[
\begin{align*}
D_1 \left( \omega_1 + i \frac{\partial}{\partial t}, \mathbf{k}_1 - i \frac{\partial}{\partial x} \right) \tilde{X} &= \lambda \zeta_0 \tilde{Y} \\
D_2 \left( \omega_2 + i \frac{\partial}{\partial t}, \mathbf{k}_2 - i \frac{\partial}{\partial x} \right) \tilde{Y} &= \lambda \zeta_0 \tilde{X}
\end{align*}
\]

where we neglected the anti-Stokes component, \( \gamma \left( \mathbf{k}_1 + \mathbf{k}_0, \omega_1 + \omega_0 \right) \) as off resonant. Expanding the \( D \) -functions in powers of \( \frac{\partial}{\partial t} \) and \( \frac{\partial}{\partial x} \) and linearizing the result with respect to them we get

\[
\begin{align*}
\nu_1 \tilde{X} + \left( \frac{\partial}{\partial t} + \nu_1 \frac{\partial}{\partial x} \right) \tilde{X} &= \frac{i \lambda}{\omega_1} \zeta_0 \tilde{Y} \\
\nu_2 \tilde{Y} + \left( \frac{\partial}{\partial t} + \nu_2 \frac{\partial}{\partial x} \right) \tilde{Y} &= -\frac{i \mu}{\omega_2} \zeta_0 \tilde{X}
\end{align*}
\]

where we neglected \( \gamma \), and used the approximation

\[
\frac{\partial D_i}{\partial \omega_i} = -2 \nu_i
\]

\( \nu_1 \) and \( \nu_2 \) are the group velocity of the waves. \( \nu_i = \frac{\partial \omega_i}{\partial k_i} \).

To simplify the equations (4.2.13) and (4.2.14) we introduce the notation
Equations (4.2.13) and (4.2.14) then become

\[ \gamma_a a_1 + (\frac{\partial}{\partial t} + v_1 \frac{\partial}{\partial x}) a_1 = \gamma_0 a_2 \]  
(4.2.16)

\[ \gamma_a a_2 + (\frac{\partial}{\partial t} + v_2 \frac{\partial}{\partial x}) a_2 = \gamma_0 a_1 \]  
(4.2.17)

where \( \gamma_0 \) is the growth rate in the absence of the damping

\[ \gamma_0 = \left[ \frac{\lambda \mu I_0^2}{4 \omega \omega_2} \right]^{1/2} \]  
(4.2.18)

without damping equations (4.2.16) and (4.2.17) conserve the quantity \( (|a_1|^2 - |a_2|^2) \) in the sense

\[ (\frac{\partial}{\partial t} + v_1 \frac{\partial}{\partial x}) |a_1|^2 = (\frac{\partial}{\partial t} + v_2 \frac{\partial}{\partial x}) |a_2|^2 \]

For simplicity we consider a plasma slab with density gradient in the \( x \) direction and assume that the pump is uniform over the entire space. In the weak pump case, the equations for the action amplitudes of the decay waves in an inhomogeneous medium are a simple extension of equations (4.2.16) and (4.2.17)

\[ \frac{\partial a_1}{\partial t} + v_1 a_1 + v_1 \frac{\partial a_1}{\partial x} = \gamma_0 \gamma a_2 \]  
(4.2.19)
\[ \frac{\partial a_2}{\partial t} + \nu_2 a_2 + \nu_2 \frac{\partial a_2}{\partial x} = \gamma^*_0 a_1 \]  \hspace{1cm} (4.2.20)

when 
\[ \gamma(x) = \exp\left( i \int_0^x \kappa(x') \, dx' \right) \] and the phase
\[ \Phi(x) = \int_0^x \kappa(x') \, dx' \] changes randomly. We ignore its time dependence. \( a_1, a_2 \) are the complex amplitudes, \( v_1 \) and \( v_2 \) are the group velocities and damping rates of the decay waves respectively. \( \gamma^*_0 \) is the growth rate of coherent case. In contrast to the Kubo-Anderson process where \( \Phi(x, t) = S(t - x/\gamma^*_0) \), we assume the static approximation in our problem, i.e., the Uhlenbeck-Ornstein process as recommended by Brissand and Frish. \( S(t) \) describing a stochastic process of the phase only. The intensity of the pump wave is taken as constant but its phase changed randomly. \( S(t - x/\gamma^*_0) \) will be taken as 
\[ S(t - x/\gamma^*_0) = \exp\left( i \int_0^x \kappa(x) \, dx \right) \] .
Hence the process or \( \Phi \) changes in Gaussian and its correlation time is infinitely short \( (\gamma_c \to 0) \). Doub's theorem asserts that the only Gaussian stationary Markov process is the Uhlenbeck-Ornstein process. It has the property of zero mean value and as shown by Uhlenbeck and Ornstein, the autocorrelation function is obtained as
\[ \langle \exp\left( i \int_0^x \kappa(y_1) \, dy_1 \right) \exp\left( -i \int_0^{x'} \kappa(y_2) \, dy_2 \right) \rangle \]
\[ = \exp\left( \frac{-L}{L^2} \right) \left[ 1 - \cos k \left( \frac{x - x'\gamma^*_0}{L} \right) \right] \]
where the generalised Raylon's number \( k \) in this case will have the form \( k = 6L \), \( L \) is the correlation length and
\[ \delta^2 = \langle \varphi^2(0) \rangle \] is the co-variance.

The main advantage of taking the Uhlenbeck-Ornstein process in the present case is that in this process the Bourret approximation turns out to be exact. It will provide a check for different approximation for different generalised Reynold's numbers, as has been discussed by Brissand and Frisch.

4.3 Bourret's Integral Equation

The differential equations for the random oscillator have been studied by a number of authors. The method presented here has been reviewed by Vankampen\textsuperscript{14}). We start with a random harmonic oscillator as

\[ \ddot{x} + \omega^2 x = 0 \quad (4.3.1) \]

where stochastic frequency \[ \omega^2 = [1 + \alpha \xi(t)] \] with \[ \xi(t) \] a stochastic variable with zero mean.

Now writing \[ u = \begin{pmatrix} x \\ \dot{x} \end{pmatrix} \] one gets from equation (4.3.1)

\[ \frac{du}{dt} = Au \quad (4.3.2) \]

with \[ A = A_0 + \alpha \xi(t) B = A_0 + A_1 \]

\[ A_0 = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \quad (4.3.3) \]
Bourret's integral equation takes the form

\[
\frac{d}{dt} \langle u(t) \rangle = A_0 \langle u(t) \rangle + \chi \int_0^t \langle A_1(t') \exp A_0(t-t') A_1(t') \rangle \langle u(t') \rangle dt' (4.3.4)
\]

where \( A_0 \) does not depend on time. \( \langle \rangle \) bracket denotes ensemble average.

4.4 The Bourret equations for averaged amplitudes and average amplitude thresholds

Equations (4.2.19) and (4.2.20) can be written in the form

\[
\frac{d}{d\chi} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} (4.4.1)
\]

with \( A = A_0 + \gamma_0 A_1 \)

\[
A_0 = \begin{pmatrix} (i\omega + \gamma_1)/\nu_1 & 0 \\ 0 & -(i\omega + \gamma_2)/\nu_2 \end{pmatrix} (4.4.2)
\]

\[
A_1 = \begin{pmatrix} 0 & \eta/\nu_1 \\ \eta/\nu_2 & 0 \end{pmatrix} (4.4.3)
\]

Let us now write the Bourret integral equation

\[
\frac{d}{d\chi} \langle u(\chi) \rangle = A_0 \langle u(\chi) \rangle + \gamma_0^2 \int_0^\chi \langle A_1(\chi) \exp [A_0(\chi-x) A_1(x)] \rangle \langle u(x') \rangle dx' (4.4.4)
\]
from which follows a differential equation

\[ \frac{d}{d\chi} \langle U(\chi) \rangle = A_0 \langle U(\chi) \rangle + \left( \begin{array}{c} p_1 \\ 0 \end{array} \right) \langle U(\chi) \rangle \]  

(4.4.4)

where one has \( \langle A_1(\chi) A_1(\chi') \rangle \approx 0 \) when \( \chi - \chi' > \chi_C \). Hence as soon as \( \chi > \chi_C \) no error is made by extending the integral to infinity, where

\[ P_1 = \int_0^\infty \frac{\gamma_0^2}{\gamma_1 \gamma_2} \exp \left[ - \left( \frac{i \omega + \gamma_1}{\gamma_1} \right) (\chi - \chi') \right] \times \langle \exp \left( i \int_0^\chi k(\gamma_1) d\gamma_1 \right) \exp \left( -i \int_0^{\chi'} k(\gamma_2) d\gamma_2 \right) \rangle d\chi' \]

(4.4.5)

\[ P_2 = \int_0^\infty \frac{\gamma_0^2}{\gamma_1 \gamma_2} \exp \left[ - \left( \frac{i \omega + \gamma_1}{\gamma_1} \right) (\chi - \chi') \right] \times \langle \exp \left( -i \int_0^\chi k(\gamma_1) d\gamma_1 \right) \exp \left( i \int_0^{\chi'} k(\gamma_2) d\gamma_2 \right) \rangle d\chi' \]

(4.4.6)

with the autocorrelation function mentioned earlier

\[ \langle \exp \left( i \int_0^\chi k(\gamma_1) d\gamma_1 \right) \exp \left( -i \int_0^{\chi'} k(\gamma_2) d\gamma_2 \right) \rangle \]

\[ = \exp \left( -2 \gamma_1 \right) \left[ 1 - \cosh \frac{\gamma_1 - \gamma_1'}{\gamma_2} \right] \]

(4.4.7)
Substituting equation (4.4.7) in equation (4.4.5) we get

\[ p_1 = \frac{\gamma_0}{\nu_1 \nu_2} \exp \left( \frac{1}{\gamma \nu_2} \right) \int_0^\infty \exp \left( -\gamma x - \frac{1}{\gamma \nu_2} \left[ \cos \omega \frac{x}{L} \right] \right) dx \] (4.4.8)

with \( \gamma = (i \omega + \nu_2)/\nu_2 \)

Then

\[ p_1 = \frac{\gamma_0}{\nu_1 \nu_2} \exp \left( \frac{1}{\gamma \nu_2} \right) \left( \frac{1}{\gamma \nu_2} \right)^{-\gamma \nu_2} \eta \Gamma(\alpha, \gamma \nu_2) \] (4.4.9)

where \( \alpha = \gamma \nu_2 + \frac{1}{\gamma \nu_2} \) and \( \Gamma(\alpha, \gamma \nu_2) \) is the incomplete gamma function defined by (Abramowitz and Stegun)\(^{16}\)

\[ \Gamma(\alpha, \gamma \nu_2) = \int_0^\infty \exp(-t) t^{\alpha-1} dt \]

The mean amplitudes can be written in the form

\[ \langle a_1 \rangle = \exp \left( -\frac{i \omega + \nu_2}{\nu_1} + p_1 \gamma \right) \] (4.4.10)

\[ \langle a_2 \rangle = \exp \left( -\frac{i \omega + \nu_2}{\nu_2} + p_2 \gamma \right) \] (4.4.11)

where \( p_1 \) is given by equation (4.4.9) and \( p_2 \) is obtained from \( p_1 \) by replacing \( \nu_2 \) and \( \nu_1 \) by \( \nu_1 \) and \( \nu_2 \), respectively. For \( \gamma \nu_2 \) small, one can expand \( \Gamma(\alpha, \gamma \nu_2) \) in terms of \( \gamma \nu_2 \)
\( \gamma (\alpha, x) = \chi^\alpha \exp (-x) \sum_{m=0}^{\infty} \frac{\chi_m}{\Gamma(\alpha+m+1)} \)  \hspace{1cm} (4.4.12)

Then we would obtain in first order

\[ \rho_1 = \frac{N^2}{\nu_1 \nu_2} \frac{1}{(i\omega + \sigma)^2} \] \hspace{1cm} (4.4.13)

we obtain a similar result for \( \rho_2 \).

Hence the threshold for \( \langle a_1 \rangle \) would read (for \( \nu_1 \nu_2 > 0 \))

\[ \gamma_0^2 = \nu_1 (\nu_2 + \nu_2 \sigma^2 \gamma_2) \] \hspace{1cm} (4.4.14)

and similar for \( \langle a_2 \rangle \)

\[ \gamma_0^2 = \nu_2 (\nu_1 + \nu_1 \sigma^2 \gamma_1) \] \hspace{1cm} (4.4.15)

This results agree with those obtained by Laval et al \(^7\) for the case \( \nu_0 = 0 \). In particular the coherent threshold is

\[ \gamma_0^2 = \nu_1 \nu_2 \] \hspace{1cm} (4.4.16)

Integrals in equation (4.4.8) will exist irrespective of the signs of \( \nu_1 \) and \( \nu_2 \). Thus for \( |\sigma| \ll 1 \)

\[ \rho_1 = \frac{N^2}{\nu_1 \nu_2} \frac{1}{\alpha} \left[ \exp \left( \sigma^2 \gamma_2 \right) \right] \left( \sigma^2 \gamma_2 \right)^{\sigma^2 \gamma_2} \] \hspace{1cm} (4.4.17)
The threshold for $\langle 0_1 \rangle$ is given by

$$y_0^L = \nu_1 (\nu_2 + \nu_2 s^L) \left[ \exp \left( s^L (s^L)^2 \right) \right]^{-1}$$

(4.4.18)

It is clear that the threshold given by the Uhlenbeck-Ornstein process of equation (4.4.18) is greater than that given by equation (4.4.14). The instability threshold conditions can be written as (for $\nu_1 \nu_2 < 0$)

$$y_0^L = \nu_1 \nu_2 (2 s^2) \min \left( \frac{\nu_1^2}{\nu_1^2}, \frac{\nu_2^2}{\nu_2^2} \right)$$

(4.4.19)

provided $\left( \sqrt{2} s^2 \right) > \max \left( \left\| \nu_1 \right\|, \left\| \nu_2 \right\| \right)$

(4.4.20)

for $s^2 L^2 << 1$ the lossless modified growth rate is obtained from (4.4.13)

$$\Lambda = y_0^L \left\| \nu_1 \nu_2 \right\| (s^L L)$$

(4.4.21)

with $\left| s^L L \right| = \left| (\delta k^2) \right| L$

This agrees with Bondeson. It may be interpreted as the effect of turbulence which introduces random mismatch in the three interacting waves.

Assuming $s L = 0$

$$\frac{\Lambda}{\Lambda_0} = \left( \frac{\nu_1}{\nu_1} \right) \left( 1 - \frac{\nu_1 \nu_2}{y_0^2} \right)$$

(4.4.22)
It is interesting to note that the degree of inhomogeneous fluctuation increases the threshold values but decreases the growth rate.

4.5 Power Stability Condition

The Bourret equation for the average intensity can be written in the following way

\[
\frac{\partial}{\partial x} \left\langle \begin{pmatrix} a_{1,1}^2 \\ a_{1,2} a_{2,1} \\ a_{2,2}^2 \end{pmatrix} \right\rangle = \begin{pmatrix} -(i\omega + \nu_1) + p_{11} & 0 & p_{13} \\ 0 & -[i\omega (\frac{\nu_1}{v_1} - \frac{\nu_2}{v_2}) + \frac{\nu_1}{v_1} + \frac{\nu_2}{v_2}] + p_{22} & 0 \\ p_{31} & 0 & -(i\omega + \nu_2) + p_{33} \end{pmatrix} \left\langle \begin{pmatrix} a_{1,1} \\ a_{1,2} a_{2,1} \\ a_{2,2} \end{pmatrix} \right\rangle
\]

where

\[
p_{11} = \frac{\chi_0}{\nu_1 v_2} \int_0^\infty \left\langle \eta(x) \eta^*(x') \right\rangle \exp \left[- \frac{i\omega (\frac{\nu_1}{v_1} - \frac{\nu_2}{v_2})}{\nu_1} + \frac{\nu_1}{v_1} + \frac{\nu_2}{v_2} \right] (x-x') \, dx'
\]

\[
p_{22} = \frac{\chi_0}{\nu_1 v_2} \int_0^\infty \left\langle \eta(x) \eta^*(x') \right\rangle \left\{ \exp \left[- \frac{(i\omega + \nu_1)}{v_1} \right] (x-x') \right\} \exp \left[- \frac{(i\omega + \nu_2)}{v_2} \right] (x-x') \, dx'
\]

\[
p_{13} = \frac{\chi_0}{\nu_1 v_2} \int_0^\infty \left\langle \eta(x) \eta(x') \right\rangle \exp \left[- \frac{(i\omega + \nu_1)}{v_1} + \frac{i\omega + \nu_2}{v_2} \right] (x-x') \, dx'
\]
\[ P_{31} = \frac{\gamma_{0}^2}{\nu_1 \nu_2} \int_0^\infty\langle \eta^*(x) \eta^*(x') \rangle \exp \left[-\left(\frac{i\omega+\nu_1}{\nu_1} + \frac{i\omega+\nu_2}{\nu_2}\right)\right] (x-x') \, dx' \]

\[ P_{11} = P_{33} \tag{4.5.5} \]

Which after integration becomes

\[ P_{11} = \frac{\gamma_{0}^2}{\nu_1 \nu_2} \int_0^\infty \exp \left(6^2 L^2\right) \gamma (\alpha, 6^2 L^2) \left(6^4 L^4\right)^{-\gamma_{11} L} \tag{4.5.6} \]

\[ P_{22} = \frac{\gamma_{0}^2}{\nu_1 \nu_2} \int_0^\infty \exp \left(6^2 L^2\right) \gamma (\tilde{\alpha}, 6^2 L^2) \left(6^4 L^4\right)^{-\gamma_{22} L} \tag{4.5.7} \]

\[ P_{13} = P_{31} = 0 \]

with \( \alpha = \gamma_{11} L + 6^2 L^2 \), \( \tilde{\alpha} = \gamma_{22} L + 6^2 L^2 \) \tag{4.5.8}

\[ \gamma_{11} = \left[i\omega \left(\frac{1}{\nu_1} - \frac{1}{\nu_1}\right) + \frac{\nu_1}{\nu_1} + \frac{\nu_2}{\nu_2}\right] \tag{4.5.9} \]

\[ \gamma_{22} = \left[i\omega \left(\frac{1}{\nu_1} - \frac{1}{\nu_2}\right) + \frac{\nu_1}{\nu_1} + \frac{\nu_2}{\nu_2}\right] \tag{4.5.10} \]

However, the integrants for \( P_{13} \) can not be expressed in terms of \( (\chi - \chi') \). In the limit \( \chi \to \infty \), \( P_{13} = P_{31} = 0 \)

The final result is
The threshold condition for small $\delta^2 L^2$ can be expressed in a simple form

$$\gamma_1 \gamma_2 \exp(-\delta^2 L^2) = \gamma_0^2 \left[ \frac{1}{\gamma_2} + \delta^2 L^2 / (\gamma_2 - 1) \right]$$ (4.5.12)

$$\gamma_2 = \gamma_1 / \gamma_1 + \gamma_2 / \gamma_2$$

For $\gamma_1 \gamma_2 < 0$ the absolute instability threshold can be obtained as

$$\gamma_0^2 = \gamma_1 \gamma_2 \frac{2 \delta^2 L^2}{\gamma_1 / \gamma_1 + \gamma_2 / \gamma_2} \min\left(\frac{|\gamma_1|}{\gamma_1}, \frac{|\gamma_2|}{\gamma_2}\right)$$ (4.5.13)

with the validity conditions

$$\sqrt{2} \theta > \gamma_1 / |\gamma_1| + \gamma_2 / |\gamma_2|$$

It is interesting to note that eigen vectors of the unperturbed intensities have undergone an equal shift because of the presence of fluctuation. The effect of fluctuation in case of the amplitudes is to shift the damping rates from $\gamma_1, \gamma_2$ to $\gamma_1 + \gamma_1$ and $\gamma_2 + \gamma_2$. Furthermore, it is evident from
equation (4.5.11) that while average amplitudes shift to \( \bar{P}_1 \) and \( \bar{P}_2 \), the energy shift are completely different and are \( \bar{P}_{11} \) and \( \bar{P}_{33} \) respectively. From equations (4.4.19) and (4.5.13) it follows that to first order, the absolute instability threshold for intensity \( \langle \alpha, \alpha^* \rangle \) is lower than that for amplitude \( \langle \alpha \rangle \).

4.6 Conclusion

In this work parametric coupling processes have been investigated in presence of background density fluctuation. The results are applicable to stimulated Raman scattering and can be extended to stimulated Brillouin scattering and other laser fusion decay instabilities. The density fluctuations have enhanced the thresholds but quenched the amplification growth rate. It is found that for absolute instabilities the threshold obtained from the averaged intensities is lower than the one obtained from the averaged amplitudes. It was pointed out by Laval et al that in the Kubo-Anderson process, the case \( \nu_1 \nu_2 > 0 \) can be solved exactly. In the present case we have taken a more general case and obtained the growth rate as a transcendental function of the frequencies and velocities.
References


ABSTRACT. This study is concerned with the theory of parametric coupling of waves in a hot inhomogeneous magnetized plasma in which the temperature gradient has been taken into account. The general dispersion relation and the polarization of the ordinary and the extra-ordinary wave modes are discussed. The eigen-mode solutions of the coupled differential equations for the wave amplitudes are obtained in the terms of the so called three wave interaction matrix elements. The theory of nonlinear wave-wave interactions, which has been extended to the case of an inhomogeneous magnetized plasma, is used to determine the threshold value of the electric field and the frequency shift. The results of this paper are also compared...
with the other known results. It is shown that the findings of this study are in excellent agreement with the results of earlier investigators.

KEY WORDS AND PHRASES. Parametric instability, Waves in inhomogeneous plasma, wave-wave interactions, Threshold electric field and frequency shift.

1980 MATHEMATICS SUBJECT CLASSIFICATION CODES. 76X05, 76E30.

1. INTRODUCTION.

In recent years there has been considerable interest in the theory of parametric instability in an inhomogeneous plasma [1-8] because of its fundamental role in the study of weak plasma turbulence, and of many important physical applications. In their recent research-expository and survey article on parametric phenomena in a plasma, Galeev and Sagdeev [9] reviewed the theory of parametric instabilities in an inhomogeneous plasma using the generalized Mathieu equation as a model equation. They have also presented the latest advances of the nonlinear theories of parametric instabilities based on the ideas of weakly turbulent plasma. Perkins and Flick [10] have made an interesting study of parametric instabilities in an inhomogeneous plasma, and then calculated the threshold electric field. It is shown that the value of the threshold electric field increases in an inhomogeneous plasma because energy propagates away from the unstable region by electron plasma waves. Thus an additional energy loss occurs and is solely responsible for the increase of the threshold electric field not observed in a homogeneous plasma. Eubank [11] experimentally confirmed the theoretical prediction of Perkins and Flick. In a recent paper, Kroll, Ron and Rostoker [12] have suggested that the nonlinear resonance of two transverse electromagnetic waves whose frequencies differ slightly by the electron plasma frequency can be applied to excite longitudinal electron plasma oscillations. Montgomery [13] observed certain mathematical inaccuracy and physical limitations of the work of Kroll, Ron and Rostoker, and then analyzed the problem of nonlinear wave interactions in plasma with laser beams.
Using the perturbation method of Krylov-Bogliubov-Mitropolskii, Montgomery obtained the amplitude-dependent frequency shift and wave number shift with physical significance. It was shown that the resonant excitation of longitudinal plasma oscillations is possible by the transverse electromagnetic waves. Etievant, Ossakow, Ozizmir and Su [14] have investigated the nonlinear wave-wave interactions of electromagnetic waves in an infinite homogeneous plasma. It is interesting and important to take into account the effects of density and temperature gradients on the above problem.

The present study deals with the theory of parametric coupling of waves in a hot inhomogeneous magnetized plasma in which the temperature gradient has been taken into consideration. The general dispersion relation and the polarization of two different wave modes are investigated. The eigen-mode solutions of the coupled equations for the wave amplitudes are obtained in terms of the three wave interaction matrix elements. The theory of nonlinear wave-wave interactions is used to determine the threshold electric field and the frequency shift. The results of this analysis are found to be in excellent agreement with those of earlier workers.

2. BASIC EQUATIONS FOR TWO PLASMA MODEL.

In two plasma model, the equations of motion and the continuity equation for each kind of component are in the usual notations given by

$$\frac{D \mathbf{v}}{D t} = \frac{\mathbf{a}}{m_a} \mathbf{F} - \frac{k}{m_a} \mathbf{G} - \mathbf{Z} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{v},$$

$$\frac{D \rho_a}{D t} + \rho_a \mathbf{v} \cdot \mathbf{v} = 0,$$

where $\Omega_a = \frac{e B_0}{cm_a}$ is the cyclotron frequency, $\nu_a$ is the collision frequency of each
component and \( B = B_0 z \) is the external magnetic field. The subscript \( a \) stands for 
\( e \) or \( i \) corresponding to electrons and ions respectively, \( Z_e \) is the charge of an.
electron or ion and \( k \) is the Boltzmann Constant.

The Maxwell equations are

\[
\text{curl } E = -\frac{1}{c} \frac{\partial B}{\partial t}, \quad \text{div } E = -4\pi \epsilon (n-n_0), \quad (2.3ab)
\]

\[
\text{curl } B = \frac{1}{c} \frac{\partial E}{\partial t} - \frac{4\pi}{c} ne \nu_a, \quad \text{div } B = 0, \quad (2.4ab)
\]

We consider a plasma model whose density distribution varies as \( \rho_a = \rho_{a_0} (1-x\delta) \).
with the density gradient \( \delta = \frac{1}{\rho_{a_0}} \frac{d\rho}{dx} \) and neglect the time dependence of \( \rho_{a_0} \)
due to any external electric field. The zeroth order solution of (2.1) gives

\[
\left< v_{x} (t) \right>_e = \frac{kT_e}{m_e} \frac{\delta v_x}{\delta} \left( \frac{2}{m_e} - v_e^2 \right), \quad (2.5)
\]

\[
\left< v_{y} (t) \right>_e = \frac{kT_e}{m_e} \frac{\delta v_y}{\delta} \left( \frac{2}{m_e} - v_e^2 \right), \quad (2.6)
\]

with

\[
\delta = \delta \times \left( \frac{\nu_{0_0}}{P_{0_0}} + \frac{\nu_{0}}{P_{0}} \right). \quad (2.7)
\]

3. THE FIRST ORDER EQUATIONS.

In the first order approximation, equations (2.1) - (2.2) can be written as,
dropping the subscript \( a \),

\[
\frac{\partial \nu_{i}}{\partial t} + \nu_{i} \cdot \nabla \nu_{i} + \nu_{i} \cdot \nu_{v} = -\frac{Z_e}{m} \frac{\rho_v}{\rho_0} - \frac{Z_0}{m} \times \nu_{i} - \frac{\nu_{i} \cdot \nabla (\rho_0)}{\rho_0} - \frac{\nu_{v} \times \nu_{v}}{\rho}, \quad (3.1)
\]

where \( \left( \frac{\nu_{v}}{\rho} \right)_l \) represents the first order terms in \( \left( \frac{\nu_{v}}{\rho} \right) \).
Assuming the density gradient is small, we take the space and time dependent
of the first order quantities as \( \exp \left[ \frac{1}{2} (k \cdot \mathbf{r} - \omega t) \right] \). The equation (3.1) can
then readily be solved to obtain

\[
\nabla \times \mathbf{V}_1 = \mu \cdot \mathbf{E}_1 ,
\]

(3.2)

where \( \mu \) is the mobility tensor given by

\[

\mu_a = \frac{Z_{\text{m}}}{m} \mu_c \left[ (1 - \frac{k \cdot v_o}{\omega}) \mathbf{I} + \frac{k \cdot v}{\omega} \right] ,
\]

(3.3)

when \( \mathbf{I} \) is the unit dyadic and \( \mu_c \) is

\[

\mu_c = \frac{1}{\Delta} \left[ \left( \bar{\omega} - k \cdot v_o \right) \dddot{\mathbf{G}} + \dddot{\mathbf{G}} \right] + \frac{\Delta \dddot{\mathbf{G}}}{\bar{\omega} - k \cdot v_o} + \frac{\Delta \dddot{\mathbf{G}}}{\bar{\omega} - k \cdot v_o} + \frac{\Delta \dddot{\mathbf{G}}}{\bar{\omega} - k \cdot v_o} ,
\]

(3.4)

with

\[

\dddot{\mathbf{G}} = \frac{\bar{\omega} T}{m(\bar{\omega} - k \cdot v_o)} \left\{ \gamma k^2 + 2i(1-\gamma)k \delta \right\} ,
\]

(3.5)

\[

\Delta = (\bar{\omega} - k \cdot v_o)^2 - (\bar{\omega} - k \cdot v_o)\dddot{\mathbf{G}} - \Omega_c^2 ,
\]

(3.6)

and \( \gamma \) is the ratio of the two specific heats.

Using equations (3.2) and taking the Maxwell equations (2.3ab) - (2.4ab),
it turns out that

\[

D \cdot \mathbf{E}_1 = 0 ,
\]

(3.7)

with

\[

D = (k^2 - \frac{\omega^2}{c^2}) \mathbf{I} - \frac{\dddot{k}}{c^2} \sum \frac{\omega_a^2 \left( \frac{\omega_a k}{c} \right) \cdot \mu_c} ,
\]

(3.8)

where

\[

\frac{\omega_a^2 \left( \frac{\omega_a k}{c} \right) \cdot \mu_c}{\omega_a} ,
\]

(3.9)

In the case of wave propagation along with \( x \)-direction (\( k = k_x \)), we obtain
the dispersion relation

$$\det D(k, \omega) = 0,$$

There are two independent solutions of equation (3.10) in the form

$$D_{zz} = 0,$$

which corresponds to the ordinary wave, and

$$D_{xx} D_{yy} - D_{xy} D_{yx} = 0,$$

which corresponds to the extra-ordinary wave, where $D_{xx}, D_{yy}, \ldots$ are the elements of the matrix $D$.

The first order fields for the different modes (ordinary and extra-ordinary) are obtained as

$$E^O = \frac{a_0}{a_0} A_0 \exp[i(k_0 \cdot x - \omega_0 t)],$$

$$E^E = \frac{a_e}{a_e} A_e \exp[i(k_e \cdot x - \omega_e t)],$$

with

$$a_0 = \frac{\gamma}{\sqrt{2}}, \quad a_e = \left(1 + \frac{b^2}{1 + b^2}\right)^{\frac{1}{2}},$$

and

$$b = -\frac{E_x}{E_y} = \frac{D_{yy}}{D_{yx}}.$$

where

$$D_{yy} = \left(\omega - \frac{\omega_0^2}{c^2}\right) \sum_{\alpha} \omega_\alpha^2 \frac{\omega}{\omega_\alpha^2} \Delta \left[ (\omega - k \cdot v_\alpha - \omega_0) + \frac{\xi_0 \omega_\alpha v_\alpha \omega}{\omega} \right],$$

and

$$D_{yx} = \sum_{\alpha} \frac{\omega_\alpha^2}{\omega_\alpha^2} \left[ \frac{(\omega - k \cdot v_\alpha) k \cdot v_\alpha \omega}{\omega_\alpha} - \xi_0 \right].$$

4. THE SECOND ORDER EQUATIONS.

In the second order approximation, we obtain from equations (2.1) - (2.2)
\[
\frac{\partial v_i}{\partial t} + v_j \cdot \nabla v_i + v_i \cdot \nabla v_j + v_j \cdot \nabla v_i = \frac{\rho}{m} \frac{\partial E_j}{\partial t} - v_j \cdot \nabla v_i - z \frac{\partial \alpha_2}{\partial t} \varepsilon_{i2} - \frac{z \alpha_2}{\rho_0} \cdot \nabla v_i - z \alpha_1 \cdot \nabla v_i,
\]

with

\[
\kappa \left( \frac{\nabla \phi}{\rho} \right) = \frac{i \kappa T}{m} \left[ \frac{\gamma_k - 1(y - 2) \delta}{\rho_0} + \frac{y - 1}{2} \left( \frac{\gamma_k - 1(y - 2) \delta}{\rho_0} \right) \frac{\rho_1}{2} + \frac{1}{y} \frac{\rho_1}{2} \right],
\]

\[
\frac{\rho_2}{\rho_0} = \left( k \cdot v_1 \right) + (k - i \delta) \cdot v_2 + \left( (v - k) \cdot v_0 \right),
\]

Using the space and time dependence of all second order quantities in the form exp\((i(k \cdot r - \omega t))\), we find

\[
\Sigma_2 = \frac{2e}{m} \mu_c \left[ \left( 1 - \frac{k \cdot v_0}{\omega} \right) I + \frac{k \cdot v_0}{\omega} \right] \cdot \frac{E_2}{\rho} + \mu_c \cdot \frac{H}{\rho},
\]

with

\[
H = -v_1 \cdot \nabla v_1 - \frac{\partial}{\partial t} \varepsilon_{i1} \cdot \varepsilon_{i1} - \frac{\partial}{\partial t} - Q,
\]

\[
Q = \frac{\kappa T}{m} \left[ \frac{\rho_0 \left( k \cdot v_1 \right)}{\rho_0 (v - k) \cdot v_0} + \frac{y - 1}{2} \left( \frac{\gamma_k - 1(y - 2) \delta}{\rho_0} \right) \frac{\rho_1}{2} + \frac{1}{y} \frac{\rho_1}{2} \right]
\]

Using the Maxwell equations and eliminating the second order magnetic field \( B_2 \), it follows that

\[
\mathbf{v} \times \left( \mathbf{v} \times \mathbf{E}_2 \right) = \frac{1}{c^2} \frac{\partial^2 \mathbf{E}_2}{\partial t^2} = -\frac{4 \pi}{c^2} \frac{\partial \mathbf{J}_2}{\partial t},
\]

whence

\[
\mathbf{J}_2 = \rho_0 \mathbf{v}_2 + \rho_2 \mathbf{v}_2 + \rho_1 \mathbf{v}_1,
\]

Substituting the values of \( \mathbf{v}_2 \) in equation (4.4), we obtain

\[
\mathbf{v} \times \left( \mathbf{v} \times \mathbf{E}_2 \right) + \frac{1}{c^2} \frac{\partial^2 \mathbf{E}_2}{\partial t^2} + \frac{4 \pi}{c^2} \sum_{a} \left( \rho_{a0} \mathbf{v} + \frac{\rho_{a0} \mathbf{v} \cdot \mathbf{k}_a}{\mathbf{w} - k \cdot \mathbf{v}_{a0}} \right) \frac{\delta \mathbf{E}_2}{\delta t}
\]

\[
= \frac{4 \pi \mathbf{v}}{c^2} \sum_{a} \rho_{a0} \left( \mathbf{I} + \frac{\mathbf{v} \cdot \mathbf{k}_a}{\mathbf{w}} \right) \cdot \left( \mu_c \cdot \mathbf{H}_a + \frac{\nu_{a1}}{\rho_{a0}} \right),
\]
5. **EQUATION FOR THE INTERACTING MODES.**

We assume a plane wave solution with an amplitude varying in the direction of propagation and write

\[ E_2 = a \, A_w (x, t) \, \exp[i (k \cdot x - \omega t)] , \tag{5.1} \]

where \( A_w \) is a slowly varying function of \( x \) and \( t \) because the nonlinearity is assumed to be weak.

We next drop the second derivatives of \( A \) in comparison to \( k^2 A \) and \( \omega^2 A \), and then use the linear dispersion relation \( \mathbf{D} \cdot \mathbf{E}_2 = 0 \) to obtain the final result

\[ (u_1 \frac{\partial}{\partial x} + \frac{\partial}{\partial t}) A_1 = A_2 A_3 \, V_{123} (\omega_2 |\omega_3|) \exp[i (x \Delta k - t \Delta \omega)] , \tag{5.2} \]
\[ (u_2 \frac{\partial}{\partial x} + \frac{\partial}{\partial t}) A_2 = A_1 A_3 \, V_{213} (-\omega_2 |\omega_3|) \exp[i (-x \Delta k + t \Delta \omega)] , \tag{5.3} \]
\[ (u_3 \frac{\partial}{\partial x} + \frac{\partial}{\partial t}) A_3 = A_1 A_2 \, V_{321} (-\omega_3 |\omega_2|) \exp[i (-x \Delta k + t \Delta \omega)] , \tag{5.4} \]

where \( \Delta k = k_3 - (k_2 + k_1) \) and \( \Delta \omega = \omega_3 - (\omega_2 + \omega_1) \) \( \tag{5.5ab} \)

We write the matrix elements for \( \hat{a}_1 = \hat{a}_3 = a_0 \) and \( \hat{a}_2 = a_2 \), where \( a_0 \) and \( a_2 \) are defined in (3.15ab).

\[ V_{123} = 2\pi \sum_{\omega} \rho_{0\omega} \left\{ -(a_1 \cdot \mu_c \cdot a_3 \cdot \hat{a}_2) (k_3 \cdot \mu_2 \cdot \hat{a}_2) \frac{ze}{m_3 \omega_3} + \right\} \]

\[ + \left( a_1 \cdot \mu_c \cdot a_3 \cdot \hat{a}_2 \right) \left( \hat{a}_1 \cdot \mu_c \cdot \hat{a}_2 \right) \left( k_2 \cdot \mu_a \cdot \hat{a}_2 \right) \frac{2}{\omega_2 - k_2 \cdot \nu_{a0}} \]

\[ = \frac{1}{2} \sum_{\omega} \frac{ze}{m_3 (\omega_3 - k_3 \cdot \nu_{ox}) (\omega_2 - k_2 \cdot \nu_{ox})} \left[ \left( 1 + \frac{k_2 \nu_{ox}}{\omega_2}\right) \left( \omega_2 - k_2 \nu_{ox} \right) \frac{b_2}{\sqrt{1 + b_2^2}} \right] \]

\[ + ( Z \, \Omega_0 + \frac{\omega_2 - k_2 \nu_{ox}}{\omega_2} \frac{k_2 \nu_{oy}}{1 + b_2^2} ) \frac{1}{\sqrt{1 + b_2^2}} , \tag{5.6} \]
$V_{321}$ is obtained by interchanging $u_1$ and $u_3$ in $V_{123}$ and

$$V_{231} = 2\pi i \left[ \frac{\rho a Tw_{2}}{a} \left[ (a_2 + \frac{1}{u_2} \frac{w_{3}k_{3}}{u_{3}} a_3) \cdot \frac{1}{u_{3}} \frac{w_{3}k_{3}}{u_{3}} (a_{3}'w_{1}'a_{1}') \right] + \frac{k_{1}}{u_{1}} (a_{3}'w_{1}'a_{1}') \right]$$

$$= \frac{1}{2} \left[ \frac{w_{2}^{2}/a}{\Delta_{2}^{2}/a} \left( \sqrt{(\omega_{2} - k_{2} - \omega_{0x})b_{2}^{-2} + 2\omega_{1}k_{3}w_{1}k_{1}(\omega_{2} - k_{2} - \omega_{0x})} \right) + \frac{w_{3}k_{3}}{u_{3}} \right] \left( \omega_{1} - k_{1} - \omega_{0x} \right) + \frac{k_{1}}{u_{1}} (a_{3}'w_{1}'a_{1}') \right]$$

(5.7)

We consider $\omega$, $k$, $\Delta k$ real and $A_3$ is the pump of fixed amplitude. The small amplitudes are described by equations (5.2) - (5.4). We take $A_1$'s to be space independent so that

$$A_1 = [\exp \left( \int_{0}^{t} \frac{1}{2} \omega dt \right)] \psi^* ,$$

(5.8)

and then we find

$$\frac{\partial^2 \psi}{\partial t^2} + \Delta \omega \frac{\partial \psi}{\partial t} + \frac{1}{4} \Delta \omega^2 \psi = \nu \nu_a \psi ,$$

(5.9)

where

$$\nu_a^2 = (V_{123} V_{213} A_{3}^2)$$

(5.10)

Neglecting $\Delta \omega \frac{\partial \psi}{\partial t}$, it follows that the solution of (5.9) by W.K.B. method assumes the form

$$\psi = \frac{1}{\sqrt{\nu}} \left[ A_1 \exp \left( \int_{0}^{t} g(t) dt \right) + B_2 \exp \left( -\int_{0}^{t} g(t) dt \right) \right] ,$$

(5.11)

with

$$g = \left\{ V_{123} V_{213} A_{3}^2 - \frac{1}{4} \Delta \omega^2 \right\}^{1/2}$$

(5.12)

When the wave packet drifts along $x$ the time increment $dt$ can be written as $dt = (dx) \left[ \frac{3\omega}{2k} \right]$ to obtain

$$A_1 \sim \left[ \exp \left( \int_{0}^{x} g(x) dx \right) \right] \left[ \frac{3\omega}{2k} \right] ,$$

(5.13)

where the integration is limited to the instability zone. It is interesting to note that the threshold value of the wave can be calculated when $g < 0$.
Thus the value of the frequency shift can be obtained from (5.14) as

$$
\Delta \omega = 2A_3 \sqrt{\frac{V_2}{V_{123}V_{213}}} \approx \frac{A_3 \omega_e^2 e}{\pi A_2} \left\{ \frac{1}{(\omega_3 - k_3 \omega_0)(\omega_2 - k_2 \omega_0)} \right\}^{\frac{1}{2}} \left[ \frac{1}{(\omega_2 - k_2 \omega_0)(\omega_1 - k_2 \omega_0)} \right]^{\frac{1}{2}}
$$

where the assumption $\omega_e^2 \gg \omega_x^2$ is invoked.

With the following numerical values

$$
\delta = 0 \quad \theta_0 = 0
$$

$$
\omega_1 = \omega_p = 2.7 \times 10^{15} \text{ sec}^{-1}
$$

$$
\omega_2 = \omega_e = 5.6 \times 10^{11} \text{ sec}^{-1}
$$

$$
k_3 - k_2 = k_1 = 19 \text{ cm}^{-1}
$$

$$
a_1, a_3 = \frac{10^6}{3} \text{ e. s. u/cm}
$$

it turns out that $\Delta \omega \approx 10^8 \text{ sec}^{-1}$ which is in good agreement with that of Montgomery [13].

Similarly, considering the space dependence only, one can calculate the wave number shift from equations (5.2) - (5.4).

6. DISCUSSION.

The general features of density gradient and magnetic field in the nonlinear interactions of plasma oscillations have been investigated. This is important in connection with its use as an optical density probe suggested by Kroll, Ron and
Rostoker [13] or as a controlled source of plasma oscillations conceived by Montgomery [13] the frequency shift will deviate with the increased value of the density gradient. However, strong magnetic field will have only influence on the frequency shift.

ACKNOWLEDGMENT. This work was partially supported by a summer grant from the East Carolina University Research Committee. Authors express their sincere thanks to Mrs. Lela Skinner for typing the final manuscript.

REFERENCES


PARAMETRIC INSTABILITIES IN RANDOM PLASMA

T.P. KHAN, R.K. CHOWDHURY, M. DAS, B. GHOSH and T. ROY
Jadavpur University, Calcutta-32.

In this paper the basic system of equations for nonlinear three wave interactions are reduced to that of the random oscillator (1). Growth rates are determined for arbitrary excited nodes. A comparison of results predicted by random phase theory show good agreement with those of the others.

We consider the equations which describe the slow varying evolution of complex amplitudes which can be written as

\[ \begin{align*}
\frac{\dot{a}_1}{\dot{a}_1} + \gamma_1 \frac{a_1}{\dot{a}_1} + \gamma_2 \frac{a_2}{\dot{a}_1} &= Y_0 \gamma_2 a_2 \\
\frac{\dot{a}_2}{\dot{a}_2} + \gamma_2 \frac{a_2}{\dot{a}_2} + \gamma_2 \frac{a_2}{\dot{a}_2} &= Y_0 \gamma_2 a_2
\end{align*} \] ...

(4)

We follow the same notations used by Lavel et.al (2). Following VanKampen equation (1) can be written in the form

\[ \frac{\dot{A}}{A} = \gamma_1 \left( \frac{a_1}{\dot{a}_1} \right) \] ...

(5)

We now write the Bourret's integral equation

\[ \frac{1}{\dot{a}_1} = A_0 \left( \frac{a_1}{\dot{a}_1} \right) + A_1 \left( \frac{a_1}{\dot{a}_1} \right) \]

(6)

where

\[ A_0 = \left( \begin{array}{c} \alpha_0 \\
\beta_0 \end{array} \right) \]

\[ A_1 = \left( \begin{array}{c} \alpha_1 \\
\beta_1 \end{array} \right) \]

\[ \alpha_0 = \left( \begin{array}{c} \gamma_1 \\
\gamma_2 \end{array} \right) \]

(7)

Let us now write the Bourret's integral equation

\[ \frac{1}{\dot{a}_1} = A_0 \left( \frac{a_1}{\dot{a}_1} \right) + A_1 \left( \frac{a_1}{\dot{a}_1} \right) \]

(8)

\[ \alpha_0 = \left( \begin{array}{c} \alpha_0 \\
\beta_0 \end{array} \right) \]

(9)

\[ \alpha_1 = \left( \begin{array}{c} \alpha_1 \\
\beta_1 \end{array} \right) \]

(10)

\[ \beta_0 = \left( \begin{array}{c} \gamma_1 \\
\gamma_2 \end{array} \right) \]

(11)

\[ \beta_1 = \left( \begin{array}{c} \gamma_2 \\
\gamma_2 \end{array} \right) \]

(12)

\[ \alpha_0 = \left( \begin{array}{c} \alpha_0 \\
\beta_0 \end{array} \right) \]

(13)

\[ \alpha_1 = \left( \begin{array}{c} \alpha_1 \\
\beta_1 \end{array} \right) \]

(14)

\[ \beta_0 = \left( \begin{array}{c} \gamma_1 \\
\gamma_2 \end{array} \right) \]

(15)

\[ \beta_1 = \left( \begin{array}{c} \gamma_2 \\
\gamma_2 \end{array} \right) \]

(16)
It is interesting to write the exact mean amplitude as a series of partial waves.

\[ \langle \psi \rangle = \frac{1}{n} \sum_{j=1}^{n} \langle \psi_j \rangle \]

The mean amplitude \( \langle \psi \rangle \) can be studied in an exactly similar way. The corresponding equation for the average intensity can be written in the following way.

\[ \begin{align*}
\left( \begin{array}{c}
\alpha_1 \\
\alpha_2 \\
\end{array} \right) &= \left( \begin{array}{c}
\beta_1 \\
\beta_2 \\
\end{array} \right) \\
\left( \begin{array}{c}
\gamma_1 \\
\gamma_2 \\
\end{array} \right) &= \left( \begin{array}{c}
\delta_1 \\
\delta_2 \\
\end{array} \right)
\end{align*} \]

References: