Chapter 5

Waves on a film of conducting liquid flowing down an inclined plane at moderate Reynolds number in presence of electromagnetic field
In the previous chapter we have seen that magnetic field stabilizes the flow whereas electric field stabilizes or destabilizes the flow depending on its orientation with the flow. Further, we have seen that both the subcritical instability and supercritical stability are possible in finite amplitude regime. Two critical Hartmann numbers $M_c$ and $M_c(> M_c)$ are observed for subcritical and supercritical zones respectively. The existence of subcritical unstable zone may be ruled out when $M > M_c$ which is independent of other parameters whereas supercritical and explosive zones are possible till $M < M_c$ which depends on flow parameters. It is to be remembered that the above results are valid so long as the Reynolds number is small. A natural question will arise 'does the above results still valid for large Reynolds number?' or 'will some new results immerse to control the flow field?' To give these answers, we continue in this chapter the study of electrically conducting fluid film down an nonconducting inclined plane in presence of electromagnetic field. In this study we adopt the IBL method to derive the evolution equation. Unlike the LWE method, where relative orders of the film amplitude and the dimensionless parameters, viz. Reynolds, Weber numbers and the angle of inclination, are assigned a priori, the IBL method keeps open on the orders of the amplitude and the dimensionless parameters. Therefore, this method gives us an unique opportunity to analyze the evolution equation in different parameter regions.

5.1 Mathematical formulation of the problem

Consider a layer of an incompressible conducting liquid flows down an inclined nonconducting plane of inclination $\theta$ with the horizon under the action of gravity in presence of electromagnetic field. The co-ordinate system is chosen such that $x-$axis along the flow and $z-$axis normal to the inclined plane. The magnetic field acts parallel to the $z-$axis and the electric field acts normal to the $x-z$ plane. Under the assumption of small magnetic Reynolds number and long-wave approximation the dimensionless governing boundary-layer equations and the corresponding boundary conditions are (see equations (4.10-4.16) of chapter 4)

\[
\frac{\partial u}{\partial x} + \frac{\partial \nu}{\partial z} = 0, \quad (5.1)
\]

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \nu \frac{\partial u}{\partial z} = -\frac{\partial p}{\partial x} + \frac{\sin \theta}{\varepsilon Fr} + \frac{EM^2}{\varepsilon Re} - \frac{M^2}{\varepsilon Re} u + \frac{1}{\varepsilon Re} \frac{\partial^2 u}{\partial z^2}, \quad (5.2)
\]

\[
0 = -\frac{\partial p}{\partial z} - \frac{3 \cot \theta}{Re}. \quad (5.3)
\]

\[
u = 0 = \nu \quad \text{at} \quad z = 0, \quad (5.4)
\]

\[
\frac{\partial u}{\partial z} = 0 \quad \text{at} \quad z = H(x,t), \quad (5.5)
\]

\[
p = p_a - \varepsilon^2 We \frac{\partial^2 H}{\partial x^2} \quad \text{at} \quad z = H(x,t), \quad (5.6)
\]
and
\[ v = \frac{\partial H}{\partial t} + \frac{\partial H}{\partial x} \quad \text{at} \quad z = H(x, t) \] (5.7)
where \( \text{Re} = \frac{\bar{u}_0 H_0}{\nu} \) is the Reynolds number, \( \text{We} = \frac{\sigma_0}{\rho \bar{u}_0^2 H_0} \) is the Weber number, \( M = B_0 H_0 \sqrt{\frac{\sigma}{(\rho \nu)}} \) is the Hartmann number, \( E = \frac{E_0}{(B_0 \bar{u}_0)} \) is the electric parameter and \( \bar{u}_0 = \frac{g H_0^2 \sin \theta}{3 \nu} \) is the Nusselt velocity. \( g, H_0, B_0, E_0, \rho, \nu, \sigma \) and \( \sigma_0 \) denote acceleration due to gravity, mean film thickness, basic magnetic field, basic electric field, density of the fluid, kinematic viscosity, electrical conductivity, and the surface tension respectively. \( u \) and \( v \) are the velocity components along and perpendicular to the flow direction.

We would like to state here that for our future analysis we assume \( E \) and \( M \) are of \( \sim O(1) \) but the orders of Reynolds and Weber numbers will be kept open, since our aim is to study how the wave character changes depending on the different orders of these physical parameters.

**Steady basic flow**

Classical solution for the steady uniform flow satisfying the system obtained as
\[ U_s = \frac{\sigma}{M^2} \left[ 1 - \frac{\cosh(M(1 - z))}{\cosh M} \right], \quad \text{where} \quad \sigma = 3 + EM^2 \] (5.8)
and the corresponding depth averaged velocity becomes
\[ U_{\text{average}} = \int_0^1 U_s(z) \, dz = \frac{\sigma}{M^2} \left[ 1 - \frac{\tanh M}{M} \right]. \]

It is clear from figure 5.1 that the average velocity varies with \( E \) and \( M \). It can be seen that \( U_{\text{average}} \) either decreases or increases with the small or large values of \( E \) respectively for a particular value of \( M \). This is due to the fact that the Lorentz force which is instrumental in driving the \( U_{\text{average}} \) becomes the union of two mechanisms. When the electric field \( E \) is small, the Lorentz force is produced mainly by the combination of velocity and magnetic fields. In this case Lorentz force opposes the basic flow field, on the other hand for strong \( E \), Lorentz force acts favorably to the basic flow field. However, for negative \( E \), which can be obtained by changing the orientation of the electric field, the value of \( \sigma \) will be either zero, positive or negative for a particular value of \( E \). This implies that, for a suitable negative \( E \), the basic flow may cease and even, can turn back when \( \sigma \leq 0 \). In other words, negative \( E \) hinders the liquid flow. It should be pointed out here that we will consider \( E \) to be positive in our future study unless stated otherwise.

**Transient and non-uniform flow**

Integrating (5.3) by using the normal stress condition (5.6) we have pressure \( p(z) \) across the film as
\[ p = p_a - \varepsilon^2 \text{We} \frac{\partial^2 H}{\partial x^2} + \frac{3 \cot \theta}{\text{Re}} (H - z). \] (5.9)
Figure 5.1: Average velocity $U_{\text{average}}$ versus Hartmann number $M$ for different electric parameter $E$. 
5.1. MATHEMATICAL FORMULATION OF THE PROBLEM

Integrating (5.1) and (5.2) with respect to $z$ from 0 to $H$ by Leibnitz rule and using the boundary conditions (5.4)-(5.7), we have

$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial x} \int_0^H u \, dz = 0,$$
(5.10)

$$\frac{\partial}{\partial t} \int_0^H u \, dz + \frac{\partial}{\partial z} \int_0^H u^2 \, dz = \varepsilon^2 W e H \frac{\partial^3 H}{\partial x^3} - \frac{3 \cot \theta}{\text{Re}} \frac{\partial H}{\partial x} + \frac{\sigma}{\varepsilon \text{Re}} H - \frac{M^2}{\varepsilon \text{Re}} \int_0^H u \, dz - \frac{1}{\varepsilon \text{Re}} \frac{\partial u}{\partial z} \bigg|_{z=0}. $$
(5.11)

To deduce the equation (5.11) we have eliminated the pressure term from (5.2) by using the equation (5.9) and then integrated as stated above. A specific velocity profile for transient and non-uniform flow must now be imposed to deduce required information from equations (5.10) and (5.11). Corresponding to the basic uniform flow a velocity profile, which is valid for transient and non-uniform flow, is assumed as

$$u = \frac{1}{\delta_1} \left( \frac{Q}{H} \right) f(Z), \quad f(Z) = \frac{\sigma}{M^2} \left[ 1 - \frac{\cosh M(1-Z)}{\cosh M} \right], \quad Z = \frac{z}{H},$$
(5.12)

where

$$Q = \int_0^H u \, dz, \quad \delta_1 = \int_0^1 f(Z) \, dZ = \frac{\sigma(1-\alpha)}{M^2}, \quad \alpha = \frac{\tanh M}{M}.$$

Further we have,

$$\int_0^H u^2 \, dz = \delta \left( \frac{Q^2}{H} \right), \quad \delta = \frac{\delta_2}{\delta_1}, \quad \delta_2 = \int_0^1 f^2(Z) \, dZ,$$

which gives

$$\delta = \frac{3(1-\alpha) - M^2 \alpha^2}{2(1-\alpha)^2}.$$

and

$$\left. \frac{\partial u}{\partial x} \right|_{z=0} = \frac{M^2 \alpha}{1-\alpha} \left( \frac{Q}{H^2} \right).$$

It is to be noted here that the shape factor $\delta$ characterizing the velocity profile is independent of the electric field $E$ and $\delta$ decreases as Hartmann number $M$ increases. For $M \to 0$, $\delta = 6/5$ and when $M \to \infty$, $\delta \to 1$. Thus the range of $\delta$ is $1 \leq \delta \leq 6/5$. Using above in equations (5.10)-(5.11), we have

$$H_t + Q_z = 0,$$
(5.13)

$$Q_t + \left( \frac{\delta Q^2}{H} + \frac{\gamma H^2}{2} \right) = W e^2 H H_{zzz} - \frac{M^2}{\varepsilon \text{Re}} Q + \frac{\sigma}{\varepsilon \text{Re}} \left[ H - \frac{\alpha}{\delta_1} \left( \frac{Q}{H^2} \right) \right],$$
(5.14)

where

$$\gamma = \frac{3 \cot \theta}{\text{Re}}$$
(5.15)
is used. Here the subscripts denote the derivative of the respective variables with respect to the indicated variables. The system (5.13)-(5.14) has a trivial solution

$$Q = q_0 = \delta_1 = \frac{\alpha(1 - \alpha)}{M^2}, \quad H = 1$$

which is nothing but the Nusselt flat-film solution in presence of electromagnetic field. We like to point out here that the approximate Kármán - Polhausen integral boundary-layer theory, which has been successfully applied in airfoil analysis (Kármán [1]; Schlichting [2]) is being applied to deduce the equations (5.13 - 5.14). Merit of this method is that the effects of inertia are fully incorporated in contrast to the traditional long-wave equations which captures the inertia effects only partially, limiting its validity range to the small vicinity of the critical Reynolds number. Drawback of the integral method consists in the assumption of the conservation of velocity profile even during perturbation. Alekseenko et al. [3] have experimentally established that a parabolic profile is more appropriate. Further Prokopiou et al. [4] and Alekseenko et al. [5] have shown analytically that the integral boundary layer theory is correct to the leading order for $Re \sim 1$ and introduce a difference which amounts to a reduction of 20% in the numerical coefficient in front of the second derivative. Further Alekseenko et al. [5] pointed out that this difference in the numerical coefficient has no prime importance in the wave formation physics. This small difference in the value of the numerical coefficient is attributed to the assumption of self-similar parabolic velocity profile used in the integral method. Ooshida [6] has advocated a regularization method to eliminate the limitation of the traditional long-wave approach. Momentum integral method has also been used by earlier researchers in connection with the stability theory starting from Kapitza [7], Alekseenko et al. [3], Jurman & McCready [8] and others for Newtonian fluid.

### 5.2 Derivation of the two-wave equation

To study the slightly nonlinear waves, let us assume

$$H = 1 + \zeta \eta(x, t), \quad Q = q_0 + \zeta q(x, t)$$

(5.17)

where $\eta$ and $q$ are dimensionless perturbations of the film thickness and flow rate respectively. Here $\zeta$ is a small parameter defined as $\zeta = h/H_0$, where $h$ is a typical surface elevation above the undisturbed level $H_0$. Substituting (5.17) in (5.13) and (5.14) and retaining the terms up to second order fluctuations, we obtain

$$\eta_t + q_x = 0,$$

(5.18)

$$q_t + \delta(2q_0q_x - q_0^2\eta_x) + \gamma \eta_x + \frac{\sigma}{\epsilon \Re} \left[ (2\beta q_0 - 3)\eta + (\beta + \frac{\alpha}{\delta_1})q \right] - \epsilon^2 \text{Wen}_{xxx}$$

$$= -\zeta [2\gamma q_t + 3\gamma \eta q_x] - \zeta \delta [2(q + q_0)q_t - 2q_0q_\eta x]$$

$$+ \zeta \frac{\sigma}{\epsilon \Re} [(3 - \beta q_0)\eta^2 - 2\beta q_\eta \eta] + 3\epsilon^2 \text{Wen}_{xxx},$$

(5.19)
5.2. DERIVATION OF THE TWO-WAVE EQUATION

where \( \beta = M^2/\sigma \). Equations (5.18) and (5.19) can be expressed into a single equation for the film height disturbance \( \eta \) by differentiating equation (5.19) with respect to \( x \) and eliminating \( q \) and its derivative according to the following procedure described below:

(i) To eliminate the linear derivative of \( q \) use equation (5.18) and (ii) for the nonlinear terms, approximation methods of quasi-stationary process is to be used. Alekseenko et al. [3] and Jurman & McCready [8] have used this method for Newtonian fluid. In this method the basic assumption used in conformation with the experimental observation is that the waves generally evolve in shape rather slowly with the downstream distance. In effect, this procedure limits the ability of the equation to describe the behaviour of very rapidly growing or decaying waves, Jurman & McCready [8]. Following Alekseenko et al. [9], we assume the system of coordinate moving with velocity \( c \), which allows the coordinate transformation \( (t, x) \rightarrow (t, \xi = x - ct) \). It is further assumed that the phase velocity \( c \) is approximately constant for quasi-stationary waves in the interval \( \Delta t \). Under this transformation equation (5.18) gives

\[
\eta_t - c \eta \xi + q \xi = 0. \tag{5.20}
\]

The wave profile in a moving coordinate system is deformed slightly in the quasi-stationary process, under this approximation the equation (5.20) reduces to \( c \eta \xi = q \xi \) from which the following relations are obtained

\[
q = c \eta, \tag{5.21}
\]

\[
\frac{\partial}{\partial t} = -c \frac{\partial}{\partial x}. \tag{5.22}
\]

To derive the two-wave equation from (5.19) first differentiate it with respect to \( x \) and then using the rules stated above, we get

\[
(\partial_t + c \partial_x)\eta + \varepsilon[\alpha_1(\partial_t + c \partial_x)(\partial_t + c \partial_x)\eta + \varepsilon^2 \alpha_2 \eta_{xxxx}] + \varepsilon[\alpha_3 \eta_x + \alpha_4 \eta_t]
= \varepsilon[\alpha_5 (\eta_x)_x + \alpha_6 (\eta_t)_x + \varepsilon^2 \alpha_7 (\eta_{xxxx})_x], \tag{5.23}
\]

where

\[
\alpha_0 = (1 + 2\alpha) \delta_1, \quad c_{1,2} = \delta_1 \pm \sqrt{\delta(\delta - 1) \delta_1^2 + \gamma},
\alpha_1 = \left( \frac{1 - \alpha}{M^2} \right) \Re, \quad \alpha_2 = \Re \alpha_1, \quad \alpha_3 = 2(2 + \alpha) \delta_1, \quad \alpha_4 = 4(1 - \alpha),
\alpha_5 = 3 \gamma \alpha_1, \quad \alpha_6 = 2(\delta - 1) \alpha_1 \quad \& \quad \alpha_7 = -3 \alpha_2. \tag{5.24}
\]

It should be noted here that weakly nonlinear waves are small in curvature, therefore, the contribution from the higher order derivatives of the quadratic terms of (5.23) is very small and hence may be neglected. Therefore equation (5.23) consists of two-wave structure which reveals that two-wave processes occur simultaneously on the thin liquid film. According to Whitham [10] these are (i) Kinematic waves: the lower order waves with
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characteristic velocity \( c_0 \) and (ii) Dynamic waves: the higher order waves with characteristic velocities approximated by \( c_1 \) and \( c_2 \).

Inspecting the non-linear wave equation (5.23), it can be seen that the Kinematic waves, associated with the first order terms, dominate the wave field for \( Re \sim 1 \), while for large Reynolds number \( Re \sim 1/e^2 \gg 1 \), the waves of higher order dominate. In ranges where one of these processes controls the type of waves and its behaviour, hence for easier analysis, equation (5.23) is linearized since analytic solutions are rarely available for the full nonlinear system. On the other hand linear results may be used to infer the corresponding nonlinear behaviour of the various waves. To do so, the physical processes of the two distinct wave equations (5.23) will be reduced following the method of Whitham [10]. In this process of reduction, the dominant wave type is first determined from the relative orders of the parameters of the original wave equation.

5.2.1 Case -I: Small flow rate; Kinematic waves, for \( Re \sim 1 \), \( We \sim 1/e^2 \), \( \cot \theta \leq 1 \)

Under this approximation, equation (5.23) reduces to

\[ \eta_t + c_0 \eta_x = 0 \]

which describes the Kinematic wave. At this stage of approximation the kinematic waves are expected to dominate the wave field. To get the reduction equation as outlined above following Whitham [10] the insensitive time derivative, i.e those time that do not correspond to the dominant wave process, are replaced by the relation \( \partial_t = -c_0 \partial_x \), where \( c_0 \) is the approximate speed of the wave and is used to eliminate the time scale by which the overall equation is governed. The equation (5.23) then reduces to

\[
\eta_t + c_0 \eta_x + \epsilon [(c_1 - c_0)(c_2 - c_0) \eta_{xx} + \epsilon^2 c_2 \eta_{xxxx}] + \epsilon (c_3 - c_0 \alpha_4) \eta_{xx} - \epsilon [(c_5 - c_0 \alpha_6) (\eta_{xx})_x + \epsilon^2 \alpha_7 (\eta_{xxxx})_x] = 0
\]

(5.25)

where the nonlinear quadratic terms with higher order derivatives in \( x \) may have been neglected on the basis of weak non-linearity, but have been retained here for the sake of completeness. It would be better to take a note on contribution of different terms of (5.25), although the following conclusions are valid for entire range of Re. In general, odd-order spatial derivatives contribute to the celerity of the waves. Due to the presence of viscosity as reflected in the presence of the second order derivative in (5.25) there exists a mechanism for energy pumping from the mean flow to the perturbations which results in instability but the fourth order derivative containing surface tension term introduces dissipative effects resulting in stability. The nonlinear \( \eta \eta_{xx} \) term causes an asymmetric sharpening of the peak to the steeper front and more shallow back as is observed for solitary waves, while the nonlinear \( \eta^2 \) term contributes to the symmetric growth of the peak. On the other hand both allow for weak interaction between modes. To get a better understanding linear stability analysis is needed.
5.2. DERIVATION OF THE TWO-WAVE EQUATION

5.2.2 Linear stability analysis for the kinematic waves

In this section, we shall examine the linear response for a sinusoidal perturbation of the film by assuming a perturbation of the form

\[ \eta = \Gamma \exp[i(kx - \omega t)], \quad (5.26) \]

where \( \omega(= \omega_r + i\omega_i) \) is the complex wave speed and the amplitude \( \Gamma \) is real. We shall first use the transformation \( x = \xi \) and \( t = \xi t \) in equation (5.25) and then use (5.26) on the linearized part which gives

\[ -i\omega + ikc_0 - \alpha_1(k_0 - c_1)(c_0 - c_2)k^2 + \alpha_2k^4 = 0. \]

Equating the real and imaginary parts, we get

\[ \omega_r = c_0k \]

and

\[ \omega_i = \alpha_1(c_0^2 - 2\delta_1c_0 + \delta \delta_1 - \gamma)k^2 - \alpha_2k^4. \quad (5.27) \]

Hence the phase velocity

\[ c = \frac{\omega_r}{k} = c_0 \]

is obtained. This shows the phase velocity is independent of the wave number \( k \), implying non-dispersive waves. It is to be noted here that for \( \lim M \to 0, c = 3 \) which was obtained by Yih [11], Benjamin [12] for Newtonian fluid. But \( \omega_i \) is different from zero containing two summands which appeared due to second and fourth derivatives present in the equation (5.25). It is clear that the second term on the right hand side of equation (5.27) which is related to surface tension is always negative and results in the attenuation of perturbations implying dissipation. On the other hand, the first term may have any sign depending on the values of \( E, M \) and \( \cot \theta \). If \( \gamma > (c_0^2 - 2\delta_1c_0 + \delta \delta_1) \), then this term also contributes to dissipation. But for \( k \neq 0 \), if

\[ \gamma < (c_0^2 - 2\delta_1c_0 + \delta \delta_1) - \text{We}k^2 \]

then the flow becomes unstable. This shows the second derivative yields the energy pumping into the perturbation causing the instability while the fourth derivative term describes the stabilizing effects. Using the relation (5.15) for \( \gamma \) in (5.22), one arrive at the stability criterion

\[ \text{Re} < \text{Re}_{\text{linear}} = 3 \cot \theta \delta_1^2 \left[ 1 + 2\alpha \right] - \delta (1 + 4\alpha) \text{We}k^2 \] \quad (5.28)

The minimum \( \text{Re} \), at which instability sets in may be denoted as critical Reynolds number \( \text{Re}_c \) for wave formation and obtained from (5.28) as

\[ \text{Re}_c = \frac{3M^4 \cot \theta}{\omega^2 (1 - \alpha)^2} \left[ 1 + 2\alpha \right] - \delta (1 + 4\alpha)^{-1}. \quad (5.29) \]
Figure 5.2: Critical Reynolds number $Re_c$ versus Hartmann number $M$ for different electric parameter $E$ at $\theta = 75^\circ$. 
5.2. DERIVATION OF THE TWO-WAVE EQUATION

It is to be noted here that the equation (5.29) is same as equation (48) of Korsunsky [13] if \( \omega(1 - \alpha)/M^2 \) is replaced by \( \varphi_0 \) obtained in (5.16). He has pointed out that the magnetic and electric field stabilize and destabilize the flow respectively. It is clear from equation (5.29) that \( \text{Re}_c \) has a point of discontinuity at \( M = M_c \) (say) = 4.629 (approx). A close scrutiny shows that the factor \( [(1 + 2\alpha)^2 - \delta(1 + 4\alpha)] \rightarrow 0 \) as \( M \rightarrow M_c \). The relation (5.29) is valid so long \( M \leq M_c \); \( \text{Re}_c \) becomes negative for \( M > M_c \), which is meaningless. It is to be remembered here that the result (5.29) is obtained for higher Reynolds number by using the momentum integral method. It has the built-in error of the velocity problem not really being the Nusselt solution in presence of perturbation. In the present problem, the magnetic field effects the averaged velocity, as can be seen in figure 5.1 and may contain an unknown error in perturbations of \( \delta \) which probably effects the stability results. However it is expected that this unknown error may not effect the results qualitatively, only that they are certainly quantitatively off by some percent, as yet unknown. Further it is clear from figure 5.2 that as \( E \) increases for any fixed value of \( M < M_c \), \( \text{Re}_c \) decreases implying the destabilizing role of \( E \), while \( \text{Re}_c \) increases with the increase of \( M < M_c \), exhibiting the stabilizing role of \( M \). The destabilizing and stabilizing role of \( E \) and \( M \) can be explained in the following way: The basic flow has only one downstream component. In the perturbed state the downstream velocity component is larger in order of magnitude than the transverse velocity component. One part of the Lorentz force connected with the electric field, helps to accelerate the flow in the downstream direction while the other part of the Lorentz force, due to the interaction of the velocity and the magnetic fields, is directed upstream to oppose the downstream flow. The magnetic lines of force act like elastic string which tend to resist any deviation from the mean flow due to perturbation. When Hartmann number \( M \) increases, the field strength increases to provide more restoring force in suppressing disturbances while increase of \( E \) supplies more energy to the growth of the disturbance. We conclude that an electrical field destabilizes the film flow whereas magnetic field stabilizes it for not too large \( E \). At large \( E \) and for small \( M \), the effect of electrical field prevails over the magnetic field. However, by changing the orientation of the electric field one may have \( E \) to be negative. In this case one should change \( E \) to \(-E\) where \( E > 0 \) and this technical substitution gives rise to new result. By inspection from equations (5.29) and (5.8) we can see that the value of \( \omega \) depends on \( M \) for a particular value of \( E \). This implies that for a particular value of \( E = E_c \) (say), \( \omega \) vanishes implying complete stability of the flow at \( E_c \). Consequence of this complete stability can be clear if we look at the basic flow \( U_b \), given in equation (5.8), which becomes zero at \( E_c \). Now for fixed \( M < M_c \), as \( E \) increases, \( \text{Re}_c \) increases so long \( E < E_c \) implying stabilizing role of \( E \). However, for \( E > E_c \), \( \text{Re}_c \) decreases with the increase of \( E \) implying destabilizing role of \( E \). Further we would like to mention here that one can obtain a mirror image of the curves about the \( \text{Re}_c \) axis in figure 5.2 if the orientation of the magnetic field be changed with the result remaining same. This is due to the fact that \( \text{Re}_c \) is an even function of \( M \). One can obtain from (5.29) that as \( M \rightarrow 0 \), \( \text{Re}_c = \cot \theta \). This shows a discrepancy between this result and that by Yih [11] for linear analysis at low Reynolds number. This discrepancy with the exact result from the long wave length expansion is a well-known deficiency of the integral boundary
layer method, due to the lack of flexibility of the velocity profile used in the computation. But the present result agrees well with that of Lee & Mei [14] and Prokopiou et al. [4].

For neutral perturbations \( \omega_i = 0 \) gives two relations

\[
\begin{align*}
    k &= 0, \\
    k_N &= \left\{ \delta^2 \left[ (1 + 2 \alpha)^2 - \delta (1 + 4 \alpha) \right] - \gamma \right\} \frac{1}{We}^{1/2},
\end{align*}
\]

which correspond to two branches of the neutral curves and the flow instability takes place between them.

Therefore at \( Re = Re_c \) two neutral curves bifurcate. The wave number of the waves with maximum growth is obtained from the relation \( dw_i/dk = 0 \), gives

\[
k_m = \left\{ \delta^2 \left[ (1 + 2 \alpha)^2 - \delta (1 + 4 \alpha) \right] - \gamma \right\} \frac{1}{2 We}^{1/2} = \frac{k_N}{\sqrt{2}},
\]

where \( k_N \) is given by formula (5.30-b). It should be emphasized here that the above analysis is meaningful so long \( M < M_c \).

### 5.2.3 Case-II: High flow rate; Dynamic or inertial waves

This section will deal with the waves of higher order that dominate in the range of large Reynolds numbers \( Re \sim 1/\varepsilon^2 \gg 1 \). In this range dynamic or inertial waves have a controlling position over the kinematic waves. Different limiting cases are considered depending on the relative orders of magnitude of the parameter \( We \).

**Case-(i): \( Re \sim 1/\varepsilon^2, We \sim 1/\varepsilon, \cot \theta \leq 1 \).**

Under this limit equation (5.23) will be controlled by the dynamic or inertial wave field. Keeping the linear terms of the order of \( 1/\varepsilon \) as the first basic approximation, equation (5.23) will be reduced to

\[
\left( \frac{\partial}{\partial t} + c_1 \frac{\partial}{\partial x} \right) \left( \frac{\partial}{\partial t} + c_2 \frac{\partial}{\partial x} \right) \eta = 0.
\]

The equivalent forms of the above equation are

\[
\left( \frac{\partial}{\partial t} + c_1 \frac{\partial}{\partial x} \right) \eta = 0,
\]

\[
\left( \frac{\partial}{\partial t} + c_2 \frac{\partial}{\partial x} \right) \eta = 0
\]
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describing the propagation of the travelling waves in the mean flow direction with velocities $c_1$ and $c_2$, given in (5.24) above. It is clear from the equations (5.33), (5.34) and (5.24-b) that the first and second wave propagates faster and slower than the mean phase speed $(c_1 + c_2)/2$. If the system (5.33)-(5.34) is transformed through the system of moving coordinate $\xi = x - (c_1 + c_2)t/2$, then we have

$$\frac{\partial \eta}{\partial t} + \left(\frac{c_1 - c_2}{2}\right) \frac{\partial \eta}{\partial \xi} = 0,$$

$$\frac{\partial \eta}{\partial t} - \left(\frac{c_1 - c_2}{2}\right) \frac{\partial \eta}{\partial \xi} = 0.$$

The above set of equations show that the two waves move in opposite direction with same velocity $(c_1 - c_2)/2$. Equation (5.32) describes the waves of higher order and these waves are classified by Whitham [10] as dynamic waves. These waves are also called inertial waves, since equation (5.32) has appeared due to the inertial terms of the Navier-Stokes equation.

Alekseenko et al. [5] reported their observation in experiment on periodic waves that increase in Re reduces the characteristic phase velocity. This suggests the wave with phase speed $c_1$, along the direction of shear, should be the basis of the wave process at high Reynolds numbers since $c_1 < c_2$. To deduce the reduction equation we follow the procedure described before, the time derivatives in equation (5.23) are replaced by the relation $\partial_t = -c_l \partial_x$, except for, naturally, the operator $\partial_t + c_l \partial_x$. The linear equation after integrating once with respect to $x$ yields

$$\eta_t + c_1 \eta_x - \frac{1}{\varepsilon \alpha_1} \frac{c_2 - c_1}{c_1 - c_2} \eta - \frac{\varepsilon^2 \alpha_2}{\alpha_1 (c_1 - c_2)} \eta_{xxx} = 0. \quad (5.35)$$

Linear stability analysis for the dynamic/inertial waves

In this section, we are interested to understand the contribution of separate terms into the formation of the wave process (5.35). To achieve this goal, linear stability analysis is being performed by assuming the perturbation of the form (5.26) and proceeding as before, we get

$$-i\omega + ik c_1 + \frac{1}{\alpha_1 (c_1 - c_2)} \frac{\alpha_2}{\alpha_1 (c_1 - c_2)} k^3 = 0. \quad (5.36)$$

Equating the real and imaginary parts give,

$$\omega_r = c_1 k + \frac{\alpha_2}{\alpha_1 (c_1 - c_2)} k^3 = c_1 k + \frac{We}{(c_1 - c_2)} k^3 \quad (5.37)$$

$$\omega_i = \frac{1}{\alpha_1 (c_1 - c_2)} \quad (5.38)$$

and the phase speed

$$c = c_1 + \frac{We}{(c_1 - c_2)} k^2. \quad (5.39)$$
It is clear from (5.39) that the surface tension yields dispersion in this case. Under this approximation we have

\[ c_1 = \delta_1 \left( \delta + \sqrt{\delta(\delta - 1)} \right) + O(\varepsilon^2) \quad c_2 = \delta_1 \left( \delta - \sqrt{\delta(\delta - 1)} \right) + O(\varepsilon^2). \]

Third term in equation (5.35) comes from the contribution of the kinematic waves and it leads to \( \omega_i \) (equation 5.38) positive for the values of \( M < M_c \), implying instability to the film flow. Thus it can be concluded that this term whose appearance is due to the kinematic waves is responsible for a low frequency energy pumping resulting in instability to film flow at high Reynolds number and this energy transfer is always possible while kinematic waves interact with the dynamic waves.

A general comment on the wave process described by equation (5.23) can be noted as follows. The lower order waves (kinematic waves) obtain energy from the mean flow and control the process at small Reynolds number. On the other hand higher order waves (dynamic waves) dominate the mechanism at high Reynolds number and obtain energy from the kinematic wave process. The above energy transfer is possible so long \( M < M_c \). For \( E > 0 \), the electrical field destabilizes the flow while for \( E < 0 \), electrical field either stabilizes or destabilizes the flow depending on its strength less or greater than the critical value \( E_c \). The surface tension plays a double role. For the kinematic wave process, it exerts stabilizing effect, so that a finite-amplitude case may be established, but for the dynamic wave process it yields dispersion.

**Case-(ii) \( Re \sim 1/\varepsilon^2, \ We \sim 1/\varepsilon^2, \ cot \theta \leq 1 \)**

At this order of approximations equation (5.23) will reduce to the form

\[
(\partial_t + c_1 \partial_x)(\partial_t + c_2 \partial_x)\eta + \varepsilon^2 \left( \frac{\partial_2}{\partial_1} \right) \eta_{xxxx} = 0. \tag{5.40}
\]

To get the dispersion relation, thickness perturbation is assumed as before defined in (5.26). The dispersion relation becomes

\[-\omega^2 + (c_1 + c_2)k\omega - c_1c_2k^2 + Wek^4 = 0.\]

Equating real and imaginary parts, we get

\[ \omega_i = 0, \]

\[ \omega_r = \delta_1 k \pm k\sqrt{\delta(\delta - 1)\delta_1^2 + Wek^2} \approx \delta_1 k \pm k^2\sqrt{We}. \]

To obtain the above approximation one should remember that \( 1 < \delta < 1.2 \) and \( \delta_1 \) lies between 1 and \( E \), for all values of \( M \). Further one should remember \( E \sim 1 \) and \( We \sim O(1/\varepsilon^2) \) so that \( Wek^2 >> 1 \). The phase speed becomes

\[ c = \omega_r/k \approx \delta_1 k \pm k\sqrt{We}. \tag{5.41}\]
5.2. DERIVATION OF THE TWO-WAVE EQUATION

It is thus at this order of approximations of the parameters that the exchange of stability takes place and the wave is dispersive. It is interesting to note that the phase speed will depend on both $E$ and $M$ for long waves. As $k$ increases (close to 1), $c$ approaches an asymptotic value $\sqrt{E/We}$. Comparing equations (5.39) and (5.41) one can find that the phase speed is more when the surface tension is less.

5.2.4 Case-III: Moderate flow rate: $Re \sim 1/\varepsilon, We \sim 1/\varepsilon^2$, $\cot \theta \leq 1$

At this approximation the linearized form of the equation (5.23) reduces to

$$(\xi_t + c_0 \xi_x)\eta + \varepsilon[\alpha_1(\xi_t + c_1 \xi_x)(\xi_t + c_2 \xi_x)\eta + \alpha_2 \varepsilon^2 \eta_{xxxx}] = 0. \quad (5.42)$$

To study the stability of this film flow on the basis of the two wave-equation (5.42), introduce the time varying perturbations of the film height

$$\eta = \Gamma \exp[ik(\tilde{x} - c\tilde{t}) + \lambda \tilde{t}]. \quad (5.43)$$

Here $\tilde{x} = x/\varepsilon$, $\tilde{t} = t/\varepsilon$, $k$ is the real wave number, $c$ is the real part of the phase velocity and $\lambda$ is an increment (the imaginary part of the frequency). Using (5.43) in (5.42), we get, after equating real and imaginary parts of the dispersion relation,

$$\lambda \alpha_1 = -\left[\frac{c - c_0}{2c - (c_1 + c_2)}\right]. \quad (5.44)$$

$$\lambda + \alpha_1 \lambda^2 - \alpha_1[c^2 - (c_1 + c_2)c + c_1c_2]k^2 + \alpha_2 k^4 = 0. \quad (5.45)$$

Elimination of $\lambda$ from (5.44) by using (5.45) gives a quadratic relation for $(k\alpha_1)^2$ as

$$(k\alpha_1)^4 - \frac{\alpha_1^2}{\alpha_2}[(c-c_1)(c-c_2)](k\alpha_1)^2 - \frac{\alpha_1^2}{\alpha_2} \left(\frac{c - c_0}{2c - (c_1 + c_2)}\right) \left(\frac{c + c_0 - (c_1 + c_2)}{2c - (c_1 + c_2)}\right) = 0. \quad (5.46)$$

The solution of equation (5.46) reduces to

$$(k\alpha_1)^2 = \frac{\alpha_1^2}{\alpha_2} (c - c_1)(c - c_2) \left[1 \pm \sqrt{1 + 4 \frac{\alpha_1^2}{\alpha_2^2} \frac{(c - c_0)(c + c_0 - (c_1 + c_2))}{(c - c_1)^2(c - c_2)^2(2c - (c_1 + c_2))^2}}\right]. \quad (5.47)$$

Equation (5.47) has two pairs of roots representing two nonintersecting branches of the dispersion curves, which are symmetric to each other about the axis $c = (c_1 + c_2)/2 = \delta \delta_1$, corresponding to them. The lower branch represents the strongly damped modes and hence are of no interest in this study.

From relations (5.44) and (5.45) we can find that at the neutral state ($\lambda = 0$), the waves propagates with a phase velocity $c = c_0$ and same set of neutral curves, obtained earlier in connection with the kinematic waves (5.30-a,b). It is further clear from equation (5.44) that the perturbations will decay so long either

$$c > c_0 > \delta \delta_1 \quad \text{or} \quad c < \delta \delta_1 < c_0.$$
Figure 5.3: Upper branch of the dispersion curve about the symmetric axis $\delta \delta_1$ at $\theta = 75^\circ$, $We = 400$ (a) $M = 1$, $E = 1$ (b) $M = 1$, $E = 0$, $M = 1.5$, $E = 0$

But for the growth of perturbations the phase velocity $c$ must lie in

$$\delta \delta_1 < c < c_0,$$

as $c_0$ is always greater than $\delta \delta_1$. It should be remembered that $M$ should be less than $M_c$ for both the cases of growth and decay of perturbations. Figure 5.3 depicts the upper branches of the dispersion curve about the symmetric axis $c = \delta \delta_1$, for various values of $M$ and $E$. It is clear from the figure that the phase speed $c$ increases with $k$ for both $M$ and $E$ but for a particular $k$, when either $M$ or $E$ increases, $c$ decreases or increases respectively. Figure 5.4 shows the variation of the growth rate $\lambda$ with the wave number $k$, for various values of $Re$, $M$ and $E$. It is clear from the figure that as $M$ increases the growth rate $\lambda$ decreases implying stabilizing influence of magnetic field. It is also evident from the figure that $\lambda$ increases with $E$ implying the destabilizing role of the electric field. Further, one can see that as $Re$ increases the growth rate $\lambda$ increases with $k$, attains a maximum corresponding to the wave of maximum growth. It can also be seen that as $Re$ increases the region of instability increases with the wave number $k$. 

\[
\delta \delta_1 < c < c_0,
\]
Figure 5.4: Variation of the growth rate $\lambda$ with the wave number $k$ at $\theta = 75^\circ$, $We = 400$
(a) $Re = 30$, $M = 1$ & $E = 1$  (b) $Re = 10$, $M = 1 \& E = 1$, $Re = 30$, $M = 1 \& E = 0$  (d) $Re = 10$, $M = 1.5 \& E = 1$
5.3 Weakly non-linear waves

In this section we are interested in studying those small-amplitude waves which develop immediately after the breakdown of the flat film solution and travel at a constant speed \( c = c_0 \) of kinematic waves. We define the new time scale as

\[
\tau = \varepsilon t. \tag{5.48}
\]

Further we introduce a transformation which defines as a new co-ordinate moving with a constant speed \( c_0 \), given by

\[
\xi = x - c_0 t. \tag{5.49}
\]

Introducing equations (5.48), (5.49) into the two wave equation (5.23) and retaining only \( O(1) \) terms for \( \text{Re} \sim O(1) \), \( \text{We} \sim O(1/\varepsilon^2) \) and \( \cot \theta \sim 1 \), we get

\[
\eta_t + A \eta_{\xi \xi} + B \eta_{\xi \xi \xi} + C \eta_{\xi \xi \xi \xi} = 0. \tag{5.50}
\]

where \( A = \alpha_3 - c_0 \alpha_4, B = (c_0 - c_1)(c_0 - c_2)\alpha_1 \) and \( C = \varepsilon^2 \alpha_2 \).

Introducing a re-scaled film height \( \zeta = \varepsilon \eta \) in (5.50), we get

\[
\zeta_t + \zeta_{\xi \xi} + B \zeta_{\xi \xi \xi} + C \zeta_{\xi \xi \xi \xi} = 0, \tag{5.51}
\]

which reduces to the Kuramoto-Sivashinsky (K-S) equation. It is to be noted here that K-S equation takes into account of non-linearity and dissipation but lacks the dispersion. Even though more general equation suitable for non-linear waves on viscous film have a behaviour similar to K-S equation (Chang et al. [15]) and the wave structure is in good agreement with experimental data of Kapitza and Kapitza [16] and Nakoryakov et al. [17]. K-S equation also deduced by Korsunsky [13] for conducting fluid flowing down an inclined plane in presence of electromagnetic field through long-wave approximation. As expected, only the coefficient \( B \) in equation (5.51) differs a small amount from \( \gamma_2 \) (equation (20) of Korsunsky) by 20% as \( M \to 0 \) while this difference further reduces as \( M \) increases. This discrepancy is again due to the use of integral method as discussed earlier. This equation represents an infinite number of stationary wave families which ultimately leads to a solitary wave with different number of humps. A detail discussion on K-S equation can be found in Chang [15].

5.4 Conclusion

In this section, we shall summarize some of the results obtained in this study. We have analyzed the waves that occur at the surface of a thin conducting fluid flowing down an inclined plane in presence of electromagnetic field. Using the method of integral relations an evolution equation representing two-wave equation is derived under the assumption of small magnetic Reynolds number and long wave approximations. Considering different ranges of the physical parameters, it is shown that different types of waves are possible on the surface of the thin film. At small flow rate, kinematic waves dominate the flow
field and it acquires energy from the mean flow, while, for high flow rate, inertial waves dominate and the energy comes from the kinematic waves. Further it is shown that the wave process stabilizes as the Hartmann number $M < M_c$ increases for a small electrical parameter $E$. It is also found that the flow field destabilizes for $E > 0$. For negative $E$, electric field stabilizes the wave process so long $\varpi > 0$. However when $\varpi < 0$, the electric field destabilizes the flow. Surface tension plays a double role, for the kinematic wave process, it exerts stabilizing influence, but for the inertial wave process it yields dispersion. At a small flow rate, Kuramoto-Sivashinsky equation can be deduced from the evolution equation when the Reynolds number $Re$ crosses $Re_c$. 
Bibliography

