Chapter 2

Synchronization of coupled chaotic systems using active control

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2.1 Introduction

Since 1990 chaos synchronization has been a topic of great attention. Chaos synchronization become very important topics in the non-linear science over the last two decades, due to its potential applications in many areas such as secure communication, information processing, biological system, chemical reaction, neural networks and in engineering. Usually two dynamical systems are called synchronized if the distance between their corresponding states converges to zero as time goes to infinity. This type of synchronization is known as identical synchronizations, Caroll and Pecora, 1990 [18]. Since 1990 various synchronization method are developed by many researchers. Synchronization between two different noise perturbed chaotic system with fully unknown parameters was proposed by Sun Y.Cao [26]. Synchronization of a modified Chua’s circuit system using adaptive control was studied by Yassen [27]. In 2005 Park [28] have observed adaptive synchronization of hyper-chaotic Chen system with uncertain parameters. Xian et al. [29] have investigated synchronization of two hyper-chaotic systems via adaptive control. Based upon adaptive control Shihua et al. [30] have studied parameters identification and synchronization of chaotic systems. Lü et al. [31] have observed adaptive feed back synchronization of unified chaotic system. Huang [32] proposed chaos synchronization between two different novel hyper-chaotic systems with unknown parameters. Controlling uncertain Lü system using linear feedback was proposed by Lü [33]. Chen [34] introduce chaos synchronization of new chaotic system via non linear control. Using linear and non linear feed back control chaos synchronization have been studied in various chaotic systems [[35]-[41]]. In 2005 Bio et al. [42] presented synchronization of two Lorenz systems using active control. Recently, Al-Sanalha et al.[43] investigated anti synchronization between two different novel hyper-chaotic systems.

In this chapter, we propose synchronization scheme for two identical Nuclear Spin Generating systems using active control. We have also introduced the synchronization scheme for coupled Lorenz [1] and Rossler[13] systems using active control. Numerical simulation results are presented to show the effectiveness of our scheme.

2.2 Nuclear Spin Generator system

The Nuclear Spin Generator(NSG) is a high-frequency oscillator which generates and controls the oscillation of the motion of a nuclear magnetisation vector in a magnetic field. The purpose of NSG is to overcome this damping and maintain the precisions of the nuclear magnetisation vector about the magnetic field.
The system consists of a suitable sample of matter containing proper nuclei in a relatively strong magnetic field, the equilibrium field, defining z-direction, an exciting coil with axis in the x-direction, perpendicular to z, a pick up coil with axis in y-direction perpendicular to both x and z, and a high-gain amplifier feeding the voltage induced in the pick-up coil back to the exciting coil, we take the normalised NSG system [12] in the following form

\begin{align}
\dot{x}_1 &= -\beta x_1 + y_1, \\
\dot{y}_1 &= -x_1 - \beta y_1(1 - kz_1), \\
\dot{z}_1 &= \beta(\alpha(1 - z_1) - k y_1^2), \tag{2.1}
\end{align}

where \(x_1, y_1, z_1\) are the components of the nuclear magnetisation vector in the x,y,z directions respectively. \(\alpha, \beta\) and \(k\) are parameters where \(\alpha \beta \geq 0\) and \(\beta \geq 0\) are linear damping terms, while the non-linear parameter \(\beta k\) is proportional to the amplifier gain in voltage feedback. Physical considerations limit the parameter \(\alpha\) to the range \(0 < \alpha < 1\). The mathematical formulation of the problem was done by Sherman[12]. The NSG problem was by Sachdev and Sarathy [44] and the Hegizi et al. [45]. They showed that for some values of the parameters this NSG system displays very rich and typical bifurcation and chaotic phenomena.

### 2.3 Synchronization of two coupled identical Nuclear Spin Generator systems

We assume that the Nuclear Spin Generator system with subscript 1 is driving and the subscript 2 is driven system. The systems are

\begin{align}
\dot{x}_1 &= -\beta x_1 + y_1, \\
\dot{y}_1 &= -x_1 - \beta y_1(1 - kz_1), \\
\dot{z}_1 &= \beta(\alpha(1 - z_1) - k y_1^2), \tag{2.2}
\end{align}

and

\begin{align}
\dot{x}_2 &= -\beta x_2 + y_2 + \mu_1, \\
\dot{y}_2 &= -x_2 - \beta y_2(1 - kz_2) + \mu_2, \\
\dot{z}_2 &= \beta(\alpha(1 - z_2) - k y_2^2) + \mu_3. \tag{2.3}
\end{align}

In (2.3) we have introduce three control functions \(\mu_1, \mu_2, \mu_3\). These unknown functions are to be determined in such a way that the systems (2.2) and (2.3) will synchronize. Now, we define three new variables in the following way

\[ x_3 = x_2 - x_1; y_3 = y_2 - y_1; z_3 = z_2 - z_1. \tag{2.4} \]
Using the set of transformation we obtain the system
\[
\begin{align*}
\dot{x}_3 &= -\beta x_3 + y_3 + \mu_1, \\
\dot{y}_3 &= -x_3 - \beta y_3 + k\beta(y_2z_2 - y_1z_1) + \mu_2, \\
\dot{z}_3 &= \beta\alpha z_3 - \beta ky_3(y_1 + y_2) + \mu_3.
\end{align*}
\]

We choose the active control functions \(\mu_1, \mu_2\) and \(\mu_3\) as
\[
\begin{align*}
\mu_1 &= v_1, \\
\mu_2 &= -k\beta(y_2z_2 - y_1z_1) + v_2, \\
\mu_3 &= \beta ky_3(y_1 + y_2) + v_3.
\end{align*}
\]

With this choice the system (2.5) transformed to
\[
\begin{align*}
\dot{x}_3 &= -\beta x_3 + y_3 + v_1, \\
\dot{y}_3 &= -x_3 - \beta y_3 + v_2, \\
\dot{z}_3 &= -\beta\alpha z_3 + v_3.
\end{align*}
\]

Equation (2.7) describes the error dynamics and can be considered in terms of a control problem where the system to be controlled is a linear system with control input \(v_1, v_2\) and \(v_3\) as functions of \(x_3, y_3\) and \(z_3\). When \(x_3, y_3\) and \(z_3\) goes to zero as time \(t\) goes to infinity then system (2.7) stabilizes at zero. This implies that two Nuclear Spin Generator systems are synchronized with feedback control \(v_1, v_2\) and \(v_3\). We choose
\[
\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},
\]

where \(A\) is an \(3 \times 3\) constant matrix. For our propose we choose elements of the matrix \(A\), in such a way that the system (2.7) must have all the eigenvalues with negative real parts. Let us choose a particular form of the matrix \(A\) that is given by
\[
A = \begin{pmatrix} \beta - 1 & -1 & 0 \\ 1 & \beta - 1 & 0 \\ 0 & 0 & \beta\alpha - 1 \end{pmatrix}.
\]

For this particular choice, the closed loop system has negative eigenvalues for \(\alpha = 0.15, \beta = 0.75\). This choice will lead the system (2.7) to a stable system which stabilizes at origin.
Figure 2.1: The time evolution of the error systems (2.7) is plotted.

Figure 2.2: Display the trajectories of $x_1$ and $x_2$ when the control functions are activated.
2.4 Lorenz and Rossler system

The Lorenz system of ordinary differential equations has been used to model the dynamics movement of an atmospheric fluid by E.N.Lorenz 1963 [1]

\[
\begin{align*}
\dot{x} &= \sigma(y - x), \\
\dot{y} &= rx - y - xz, \\
\dot{z} &= xy - \beta z. \\
\end{align*}
\]

(2.10)

Where \(\sigma, \beta, r > 0\) are parameters. It is known that for parameter values \(\sigma = 10, \beta=8/3, r = 28\) the system have chaotic behaviour.

Rossler (1976) [13] found a system of three differential equations with a simple strange attractor, which is known as Rossler system

\[
\begin{align*}
\dot{x} &= -y - z, \\
\dot{y} &= x + ay, \\
\dot{z} &= b + z(x - c). \\
\end{align*}
\]

(2.11)

Here \(a, b, c > 0\) are three parameters, the Rossler system has chaotic behavior for \(a = 0.02, b = 0.02\) and \(c = 5.0\).

2.5 Synchronization scheme for two coupled non-identical chaotic systems

In order to achieve the synchronization between coupled Lorenz system and Rossler system by using the method of active control, we assume that Lorenz system is the drive system whose variables are denoted by subscript 1 and Rossler system is response system whose variables are denoted by subscript 2. The drive and response systems are described, respectively by the following equations

\[
\begin{align*}
\dot{x}_1 &= \sigma(y_1 - x_1), \\
\dot{y}_1 &= rx_1 - y_1 - x_1z_1, \\
\dot{z}_1 &= x_1y_1 - \beta z_1. \\
\end{align*}
\]

(2.12)

and

\[
\begin{align*}
\dot{x}_2 &= y_2 - z_2 + \mu_{r1}, \\
\dot{y}_2 &= x_2 + ay_2 + \mu_{r2}, \\
\dot{z}_2 &= b + x_2z_2 - cz_2 + \mu_{r3}. \\
\end{align*}
\]

(2.13)

Where \(\mu_{r1}, \mu_{r2}\) and \(\mu_{r3}\) are three control functions which we have to determine.
Now using $x_3 = x_2 - x_1; y_3 = y_2 - y_1$ and $z_3 = z_2 - z_1$ we obtain

\[
\begin{align*}
\dot{x}_3 &= -y_2 - z_2 + \sigma(x_1 - y_1) + \mu_{r1}, \\
\dot{y}_3 &= x_2 + ay_2 - rx_1 + y_1 + x_1z_1 + \mu_{r2}, \\
\dot{z}_3 &= b - cz_2 + \beta z_1 + x_2z_2 - x_1y_1 + \mu_{r3}.
\end{align*}
\] (2.14)

We define the active control functions $\mu_{r1}, \mu_{r2}$ and $\mu_{r3}$ as

\[
\begin{align*}
\mu_{r1} &= z_1 + y_2 + \sigma(y_1 - x_2) - v_{r1}, \\
\mu_{r2} &= (r - 1)x_2 - (a + 1)y_2 - x_1z_1 - v_{r2}, \\
\mu_{r3} &= (c - \beta)z_2 - b - x_2z_2 + x_1y_1 - v_{r3}.
\end{align*}
\] (2.15)

With this suitable choice the system (2.14) transformed to

\[
\begin{align*}
\dot{x}_3 &= -\sigma x_3 - z_3 - v_{r1}, \\
\dot{y}_3 &= r x_3 - y_3 - v_{r2}, \\
\dot{z}_3 &= -\beta z_3 - v_{r3}.
\end{align*}
\] (2.16)

Equation (2.16) describes the error dynamics and can be considered in terms of a control problem where the control function $v_{r1}, v_{r2}$ and $v_{r3}$ are the functions of $x_3, y_3$ and $z_3$. As long as these feedbacks stabilise the system at origin, $x_3, y_3$ and $z_3$ converges to zero as time goes to infinity. This implies that two different systems (Lorenz and Rossler system) are synchronized with active control. We choose

\[
\begin{pmatrix}
v_{r1} \\
v_{r2} \\
v_{r3}
\end{pmatrix} = \tilde{A}
\begin{pmatrix}
x_3 \\
y_3 \\
z_3
\end{pmatrix}.
\] (2.17)

Where $\tilde{A}$ is an $3 \times 3$ constant matrix. Let us choose

\[
\tilde{A} = \begin{pmatrix}
-\sigma & 0 & -1 \\
 r & -1 & 0 \\
 0 & 0 & -\beta
\end{pmatrix}.
\] (2.18)

For this particular choice, the closed loop system has eigenvalues $-\sigma, -1$ and $-\beta$. This choice will lead to a stable system and as we will observe in a numerical investigation, that the two different chaotic systems will synchronize.

### 2.6 Results and discussion

We present here simulation results for the identical Nuclear Spin Generator systems. The parameters are taken as $\alpha = 0.15, \beta = 0.75$ and $k=10.5$. The Nuclear
Spin Generator system behave chaotically under these values of $\alpha$, $\beta$ and $k$. The fourth order Runge-Kutta method with step size 0.002 is used for solving the error system. Figure 2.1 shows the synchronization errors between two identical Nuclear Spin Generator systems. We observe that the synchronization error converges to zero and therefore coupled Nuclear Spin Generator systems are synchronized. Figure 2.2 displays the trajectories of $x_1$ and $x_2$ of two coupled Nuclear Spin Generator systems. These figures shows the synchronization behaviour of the $x$ components of the drive and response systems. Similar results hold for $y$ and $z$ components also. We also present here simulation results for synchronization scheme for two coupled Lorenz and Rossler systems. The parameters of the Lorenz system are taken as $\sigma=10$, $\beta=(8/3)$ and $r=28$ and parameters of Rossler systems are $a=0.2$, $b=0.2$ and $c=5.0$. The synchronization errors of two coupled Lorenz and Rossler systems are presented in figure (2.3). We observe that the synchronization error converges to zero and hence coupled chaotic Lorenz and Rossler systems are synchronized. Figure 2.4 displays the trajectories of $x_1$ and $x_2$ of two different Lorenz and Rossler systems. This figure confirms the synchronization behaviour of $x$ components of the coupled drive and response systems. Similarly behaviour are observed for $y$ and $z$ components of the coupled systems.
2.7 Conclusion

We have successfully design scheme for synchronization between two coupled identical chaotic N.S.G systems using active control. We have also applied the method of active control for synchronization between the coupled chaotic Lorenz and Rossler systems. These synchronization schemes may be useful for secure communication purpose and for designing different types of electrical circuits.