Chapter 7

Multistable behaviour of coupled Lorenz-Stenflo systems

This chapter is based on the article entitled “Multistable behaviour of coupled Lorenz-Stenflo systems” which is published in Physica Scripta 89 (2014) 045202 (10pp).
7.1 Introduction

Mechanisms which support the multistability of a dynamical system are yet incomplete, and remains a fundamental problem for the theory of dynamical systems. A multistable dynamical system is one that possesses a large number of asymptotic stable states for a fixed set of parameters. Therefore a large number of coexisting attractors exist in a multistable system for a fixed set of parameters. Trivial multistability of system can be considered as the coexistence of several stable fixed points i.e. the coexistence of multiple point attractors in the phase space. On the other hand generalized multistability can be considered as the coexistence of many nontrivial attractors i.e. limit cycles, chaotic attractors etc. That are three basic motivations to study multistable systems, that is systems that possess a large number of coexisting attractors for a fixed set of parameters [109]. Firstly, there is ample evidence of multistability in the natural sciences, namely, laser physics [95] [110], neurosciences and neural dynamics [111] [90], biology [112] [113] optics [114] [98] [115] [96], condensed matter [116] and geophysics [117] and chemical reaction [118]. Second motivation is to identify the universal mechanisms that lead to multistability and to prove rigorously under what circumstances the phenomenon may occur. Lastly, the control of multistable systems to obtain desired dynamical behaviour is a challenging problem.

There are many mechanisms for designing multistable systems, the easiest way to construct a multistable system is to take a conservative system and introduce small amounts of dissipation [119]. Coupled dynamical systems with time-delayed [120] and without time delayed [121] interaction show multistability. Sun et al. [104] reported a novel mechanism to obtain multistability in coupled systems and there scheme is the following. Many dynamical systems governed by the the evolution equations

\[ \dot{x}_i = f_i(x_1, x_2, x_3, \ldots, x_n), \quad i = 1, 2, 3, \ldots, n \] (7.1)

in such a way that the long-time evolution of the system can be described by a reduced system

\[ \dot{x}_i = g_i(x_1, x_2, x_3, \ldots, x_m, c_1, c_2, \ldots, c_k), \quad i = 1, 2, 3, \ldots, m, \quad k \leq n - m \] (7.2)

along with a series of algebraic relations

\[ h_j(x_1, x_2, x_3, \ldots, x_n, c_j) = 0, \quad j = 1, 2, \ldots, k \] (7.3)

where \( c_j \) are constants. These constants depends on the initial conditions of the original system. If the reduced system have many qualitatively different dynamical behaviour with the variation of \( c_j \)’s then we say that the original system (7.1) possess generalised multistability. Ngonghala et al. [122] show that the appearance of extreme multistability is associated with the emergence of a conserved
quantity in the long-term limit. They show that the existence of a conserved quantity allows to reduce the dimension of the coupled identical dynamical systems to obtain an uncoupled dynamical system with a constant term. Recently, Hens et al. [105] show that the coexistence of infinitely many attractors in an \( m \)-dimensional coupled system will be possible if \( m - 1 \) of the variables of the two systems are completely synchronized and one of them obeys a constant difference between them.

In this chapter, we formulate generalised multistable system coupling two Lorenz-Stenflo (LS) systems. The LS system [15] is

\[
\begin{align*}
\dot{x}_1 &= \sigma(x_2 - x_1) + sx_4, \\
\dot{x}_2 &= rx_1 - x_2 - x_1x_3, \\
\dot{x}_3 &= x_1x_2 - bx_3, \\
\dot{x}_4 &= -x_1 - \sigma x_4,
\end{align*}
\]  

(7.4)

where the dot denotes the time derivative and \( \sigma > 0, r > 0, b > 0 \) and \( s > 0 \) are the four parameters. We present three coupling schemes to realise extreme multistability in coupled LS systems. In the first and second scheme we mainly follow coupling method of Sun et al. [104]. In the first scheme only one variable synchronize and three other state variables keep constant difference. In the second scheme two variables synchronized and two variables keep a constant difference. In the third scheme we follow the method of Hens et al. [105]. The section-wise split of this chapter is as follows: In the section 7.2, construction of multistable systems suitable coupling LS-systems are discussed. In all these schemes the coupled LS-systems reduced to a single modified LS-system. In section 7.3, the local stability analysis of equilibrium points of the modified LS-system are done and pitchfork bifurcation and Hopf bifurcation conditions are derived. Section 7.4 numerical simulation results (phase diagrams, bifurcation diagrams and maximum Lyapunov exponents) are presented and discussed. Two parameter bifurcation analysis is done with the help of the software package MATCONT ([123]-[125]). In Section 7.5, the conclusion is drawn.

### 7.2 Construction of multistable systems coupling Lorenz-Stenflo systems

In this section we shall discuss three different schemes for generating multistable systems from couple Lorenz-Stenflo (LS) systems.
Scheme -I We couple two LS systems [15] in the following way:

\[
\begin{align*}
\dot{x}_1 &= \sigma(x_2 - y_1) + sx_4, \\
\dot{x}_2 &= rx_1 - x_2 - x_1x_3, \\
\dot{x}_3 &= x_1x_2 - bx_3, \\
\dot{x}_4 &= -x_1 - \sigma x_4, \\
\dot{y}_1 &= \sigma(y_2 - y_1) + sy_4, \\
\dot{y}_2 &= rx_1 - y_2 - x_1y_3, \\
\dot{y}_3 &= x_1y_2 - by_3, \\
\dot{y}_4 &= -x_1 - \sigma y_4.
\end{align*}
\tag{7.5}
\]

We now show that these eight dimensional dynamical system is a multistable system. Following Sun et al.[104] we construct the dynamical equations for the synchronization errors \(e_1 = y_1 - x_1,\ e_2 = y_2 - x_2, \ e_3 = y_3 - x_3\) and \(e_4 = y_4 - x_4\) as

\[
\begin{align*}
\dot{e}_1 &= \sigma e_2 + s e_4, \\
\dot{e}_2 &= -e_2 - x_1 e_3, \\
\dot{e}_3 &= x_1 e_2 - b e_3, \\
\dot{e}_4 &= -\sigma e_4.
\end{align*}
\tag{7.6}
\]

Clearly the function \(V = (e_2^2 + e_3^2 + e_4^2)/2\) is a lyapunov function for the system \(\dot{e}_2 = -e_2 - x_1 e_3, \dot{e}_3 = x_1 e_2 - b e_3, \dot{e}_4 = -\sigma e_4\) because \(V\) is positive definite and

\[
\begin{align*}
\dot{V} &= e_2 \dot{e}_2 + e_3 \dot{e}_3 + e_4 \dot{e}_4, \\
&= -e_2^2 - b e_3^2 - \sigma e_4^2 < 0, \text{ as } b, \sigma > 0.
\end{align*}
\tag{7.7}
\]

Therefore \(e_2, \ e_3\) and \(e_4\) must tend to zero as \(t \to \infty\) i.e., \(y_2 = x_2, y_3 = x_3\) and \(y_4 = x_4\). Since, \(e_2\) and \(e_4 \to 0\) implies \(e_1 = \text{constant} = c\). Therefore, \(e_1 = y_1 - x_1 \to c\) where \(c\) is some constant (dependent on the initial conditions of the full system). Each new set of initial conditions gives rise to different value of \(c\). Therefore, the dynamics of the system of equations (7.5) is equivalent to the following four dimensional system

\[
\begin{align*}
\dot{x}_1 &= \sigma(x_2 - x_1 - c) + sx_4, \\
\dot{x}_2 &= rx_1 - x_2 - x_1x_3, \\
\dot{x}_3 &= x_1x_2 - bx_3, \\
\dot{x}_4 &= -x_1 - \sigma x_4.
\end{align*}
\tag{7.8}
\]

The system (7.5) is a multistable system if the dynamical behaviour of the system (7.8) varies qualitatively with the variation of the values of \(c\).
We consider the coupled LS systems [15] in the following form:

\[
\begin{align*}
\dot{x}_1 &= \sigma(x_2 - y_1) + sx_4, \\
\dot{x}_2 &= rx_1 - y_2 - x_1x_3, \\
\dot{x}_3 &= x_1x_2 - bx_3, \\
\dot{x}_4 &= -x_1 - \sigma x_4, \\
\dot{y}_1 &= \sigma(x_2 - y_1) + sy_4, \\
\dot{y}_2 &= rx_1 - y_2 - x_1y_3, \\
\dot{y}_3 &= x_1x_2 - by_3, \\
\dot{y}_4 &= -x_1 - \sigma y_4.
\end{align*}
\]  

(7.9)

We now show that the eight dimensional dynamical system is a multistable system. Following Sun et al. [104] we construct the governing equations for the synchronization errors \(e_1 = y_1 - x_1, \ e_2 = y_2 - x_2, \ e_3 = y_3 - x_3\) and \(e_4 = y_4 - x_4\) as

\[
\begin{align*}
\dot{e}_1 &= se_4, \\
\dot{e}_2 &= -x_1e_3, \\
\dot{e}_3 &= -be_3, \\
\dot{e}_4 &= -\sigma e_4.
\end{align*}
\]  

(7.10)

Since \(b > 0, \ \sigma > 0\), it is clear that the function that \(e_3\) and \(e_4\) must tend to zero with time i.e., \(y_3 = x_3\) and \(y_4 = x_4\). Since \(x_1\) is a bounded physical quantity therefore

\[
\begin{align*}
\dot{e}_1 &= 0, \\
\dot{e}_2 &= 0,
\end{align*}
\]

which implies \(e_1 = \text{constant}=c_1\) and \(e_2 = \text{constant}=c_2\). Hence, \(y_1 = x_1 + c_1\) and \(y_2 = x_2 + c_2\), where \(c_1, c_2\) are some constants (dependent on the initial condition of the full system). Each new set of initial conditions gives rise to different value of \(c_1\) and \(c_2\). Therefore, the dynamics of the system of equations (7.9) is equivalent to following four dimensional system

\[
\begin{align*}
\dot{x}_1 &= \sigma(x_2 - x_1 - c_1) + sx_4, \\
\dot{x}_2 &= rx_1 - x_2 - c_2 - x_1x_3, \\
\dot{x}_3 &= x_1x_2 - bx_3, \\
\dot{x}_4 &= -x_1 - \sigma x_4.
\end{align*}
\]  

(7.11)

The system (7.9) is a multistable system if the dynamical behaviour of the system (7.11) varies qualitatively with the variation of \(c_1\) and \(c_2\). In this scheme two
variables of the coupled systems synchronized and two other variables keep a constant difference between them. In general coupling two \( m \)-dimensional systems we may obtain multistable nature if \( i \) number of variables of the two systems are completely synchronized and \( j \) number of variables keep a constant difference between them, where \( i + j = m \) and \( 1 \leq i, j \leq m - 1 \). In this way we can easily find \( j \) \((1 \leq j \leq m - 1)\) number of conserved quantities of the coupled system.

**Scheme -III** We consider the coupled LS in the following form:

\[
\begin{align*}
\dot{x}_1 &= \sigma(x_2 - x_1) + sx_4 + u_{11}, \\
\dot{x}_2 &= r x_1 - x_2 - x_1 x_3 + u_{12}, \\
\dot{x}_3 &= x_1 x_2 - bx_3 + u_{13}, \\
\dot{x}_4 &= -x_1 - \sigma x_4 + u_{14}, \\
\dot{y}_1 &= \sigma(y_2 - y_1) + sy_4 + u_{21}, \\
\dot{y}_2 &= ry_1 - y_2 - y_1 y_3 + u_{22}, \\
\dot{y}_3 &= y_1 y_2 - by_3 + u_{23}, \\
\dot{y}_4 &= -y_1 - \sigma y_4 + u_{24}.
\end{align*}
\]

(7.12)

Following Hens et al. [105] we choose controllers \( u_{11}, u_{12}, u_{13}, u_{14}, u_{21}, u_{22}, u_{23} \) and \( u_{24} \) such that the above system become multistable. Controllers \( u_{11}, u_{12}, u_{13}, u_{14}, u_{21}, u_{22}, u_{23} \) and \( u_{24} \) are selected as \( u_{21} - u_{11} = \sigma(y_1 - x_1), u_{22} - u_{12} = y_1 y_3 - x_1 x_3 - x_1 (y_3 - x_3) - r(y_1 - x_1), u_{23} - u_{13} = y_2 (x_1 - y_1) \) and \( u_{24} - u_{14} = y_1 - x_1 \).

We choose \( u_{12} = u_{13} = u_{14} = u_{21} = 0 \). We now show that the eight dimensional dynamical system is a multistable system.

We construct the governing equations for the synchronization errors \( e_1 = x_1 - y_1, \ e_2 = x_2 - y_2, \ e_3 = x_3 - y_3 \) and \( e_4 = x_4 - y_4 \) as

\[
\begin{align*}
\dot{e}_1 &= se_4, \\
\dot{e}_2 &= -e_2 - x_1 e_3, \\
\dot{e}_3 &= x_1 e_2 - b e_3, \\
\dot{e}_4 &= -\sigma e_4. 
\end{align*}
\]

(7.13)

It follows that the function that \( e_2, e_3 \) and \( e_4 \) must tend to zero with time since \( V = (e_2^2 + e_3^2 + e_4^2)/2 \) is a lyapunov function for the system \( \dot{e}_2 = -e_2 - x_1 e_3, \dot{e}_3 = x_1 e_2 - b e_3, e_4 = -\sigma e_4 \). Therefore, \( y_2 = x_2, y_3 = x_3 \) and \( y_4 = x_4 \). Since \( x_1 \) is a bounded physical quantity therefore

\[
\dot{e}_1 = 0
\]

(7.14)

which implies \( e_1 = \text{constant} = c \). Hence, \( y_1 = x_1 + c \) where \( c \) is some constants (dependent on the initial condition of the full system). Each new set of initial
conditions gives rise to different value of \( c \). Therefore, the dynamics of the system of equations (7.12) is equivalent to following four dimensional system

\[
\begin{align*}
\dot{x}_1 &= \sigma(x_2 - x_1 - c) + sx_4, \\
\dot{x}_2 &= r x_1 - x_2 - x_1 x_3, \\
\dot{x}_3 &= x_1 x_2 - bx_3, \\
\dot{x}_4 &= -x_1 - \sigma x_4.
\end{align*}
\]

(7.15)

The system (7.12) is a multistable system if the dynamical behaviour of the system (7.15) changes qualitatively with the variation of the value of \( c \).

### 7.3 Theoretical results

The equilibrium points of the modified LS-system (7.8) are the origin \( O \equiv (0,0,0,0) \) and the point \( C^\pm \equiv (x^*_1, x^*_2, x^*_3, x^*_4) \) where \( x^*_1 = 2b\sigma^2 x_3/(c\sigma^2 + \phi(x_3)) \), \( x^*_2 = (c\sigma^2 + \phi(x_3))/2\sigma^2 \), \( x^*_3 = r - 1 - s/\sigma^2 - (c(c\sigma^2 + \phi(x_3)))/2b\sigma^2 x_3 \) and \( x^*_4 = -x_1/\sigma \), with \( \phi(x_3) = \pm \sqrt{(c^2\sigma^4 + 4\sigma^2(s + \sigma^2)bx_3)} \). Clearly, the origin is a trivial stationary point for all parameter values, but \( C^\pm \) exist only when \( (c^2\sigma^4 + 4\sigma^2(s + \sigma^2)bx_3) > 0 \). Therefore, existence of nontrivial equilibrium points depend on the value of \( c \).

We now investigate the stability property of the equilibrium points and bifurcation nature following Zhou et al.[126]. The Jacobian matrix of the modified LS-system at the equilibrium point \( C^\pm \equiv (x^*_1, x^*_2, x^*_3, x^*_4) \). The Jacobian matrix of the system (7.8) at the equilibrium point \( C^\pm \) is given by

\[
J(C^\pm) = \begin{pmatrix}
-\sigma & \sigma & 0 & s \\
 r - x^*_3 & -1 & -x^*_1 & 0 \\
x^*_2 & x^*_1 & -b & 0 \\
-1 & 0 & 0 & -\sigma
\end{pmatrix}.
\]

The eigenvalues of the Jacobian matrix are roots of the following equation

\[
a_4\lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0,
\]

(7.16)

where

\[
\begin{align*}
a_0 &= bs + (1 - r)b\sigma^2 + b\sigma^2 x^*_3 + (s + \sigma^2)x^*_1 x^*_2, \\
a_1 &= (1 + b)s + (2 - r)b\sigma + (1 + b - r)\sigma^2 + (b\sigma + \sigma^2)x^*_3 + 2\sigma x^*_1 + \sigma x^*_1 x^*_2, \\
a_2 &= b + s + (2 - 2b - r)\sigma + \sigma^2 + x^*_1^2 + \sigma x^*_3, \\
a_3 &= b + 2\sigma + 1, \text{ and } a_4 = 1.
\end{align*}
\]

(7.17)

For the fixed point \( O \), the characteristc equation (7.16) becomes

\[
(b + \lambda)[\lambda^3 + (1 + 2\sigma)\lambda^2 + (s + 2\sigma - r\sigma + \sigma^2)\lambda + s + (1 - r)\sigma^2] = 0
\]

(7.18)
which indicates that for $0 < r < \min\{1 + s/\sigma^2, 2 + \sigma + s/\sigma, 2[1 + \sigma + s/(1 + \sigma)]\}$ the origin $O$ is stable and attracting [126]. However, the sign of one root changes from negative to positive as $r$ increases through $1 + s/\sigma^2$. In fact, there is a pitchfork bifurcation at $r = 1 + s/\sigma^2$. The origin then loses its stability and $C^\pm$ appear.

We now consider the equilibrium point $C^\pm$. The latter exist for $r > 1 + s/\sigma^2$ which we note is domain of instability of the fixed point $O$. Equation (7.18) can be rewritten as

$$ (\lambda^2 + a_3 \lambda + a_2 - a_1/a_3)(\lambda^2 + a_1/a_3) = (a_2 - a_1/a_3)a_1/a_3 - a_0, \quad (7.19) $$

from which the conditions for the roots to lie on the left half of the complex plane can be shown to be (i) $a_3 > 0$, (ii) $a_2 > a_1/a_3$, (iii) $a_1 > a_3^2a_0/(2a_2a_3 - a_1)$, and (iv) $a_0 > 0$. They are the necessary and sufficient conditions for the non-trivial fixed points $C^\pm$ to be stable. In the following, we shall obtain the critical condition for instability to occur. Clearly the condition $a_0 > 0$ is satisfied when $r > 1 + s/\sigma^2$. It follows that the real root never vanishes. Thus, when the equality

$$ a_0 = (a_1/a_3)(a_2 - a_1/a_3), \quad (7.20) $$

is satisfied, Hopf bifurcation can occur. Notice that the pitchfork bifurcation and Hopf bifurcation condition of the modified LS-system are same as the original LS system because the Jacobian of the two systems are equal.

### 7.4 Numerical simulation results

We have simulated dynamical behaviours of the system (7.8) for $\sigma = 10$, $r = 340$, $s = 30$ and $b = 8/3$ and $\sigma = 1$, $r = 28$, $s = 1.5$ and $b = 0.7$. We have varied the vital parameter $c$ throughout the numerical simulations. Firstly, we discuss simulation results of scheme-I. Phase diagram of the system (7.8) in the $x_1x_3$ plane for different values of $c$ are plotted in figure 7.1 taking the other parameters $\sigma = 10$, $r = 340$, $s = 30$ and $b = 8/3$. Taking $c = 0.5, 1, 5$ and 8, the phase diagrams are presented in figure 7.1(a), 7.1(b), 7.1(c) and 7.1(d) respectively. Existence of qualitatively different behaviour is obvious from the figure. Therefore, the coupled LS system have multistable behaviour. For same set of parameter values and for different values of $c$ phase diagrams are depicted in figure 7.2. Figure 7.2(a), 7.2(b), 7.2(c) and 7.2(d) respectively represent the phase diagram for $c = 12$, 16, 21 and 29. The multistable dynamics is clear from figure 7.2. The phase diagram of the system (7.8) in the $x_1x_3$ plane for different $c$ are plotted in figure 7.3 taking $\sigma = 1$, $r = 28$, $b = 0.7$ and $s = 1.5$.

In order to characterise the periodic orbits, we represent their projections on the $x_1x_3$ plane in figures 7.1-7.3 and in terms of the symbol $P_+$ and $P_-$ in
Figure 7.1: Phase diagram in $x_1 - x_3$ plane for different $c$ with $\sigma = 10$, $r = 340$, $s = 30$ and $b = 8/3$.

Figure 7.2: Phase diagram in $x_1 - x_3$ plane for different $c$ with $\sigma = 10$, $r = 340$, $s = 30$ and $b = 8/3$.

Table 7.1 and Table 7.2. Here $P_+$ and $P_-$ stands for a trajectory encircling the equilibrium points $C^+$ and $C^-$ respectively. When a trajectory makes $n$ revolutions around some fixed point then we label it with superscript $n$. From figures 7.1-7.3, it is clear that the system (7.8) has non-symmetric periodic orbits as well as chaotic behaviour for various values of the parameter $c$.

7.4.1 Bifurcation analysis

The bifurcation diagram with respect to $c$ of system (7.8) is plotted in figure 7.4 for different range of $c$. The bifurcation diagram in the intervals $0 \leq c \leq 6$, $6 \leq c \leq 15$, $15 \leq c \leq 30$ and $30 \leq c \leq 60$ are plotted in figure 7.4(a), 7.4(b), 7.4(c) and 7.4(d) respectively. The multistable nature of the system is established from these bifurcation diagrams. The bifurcation diagrams of the system (7.8) with respect to $c$ for $\sigma = 1$, $r = 28$, $b = 0.7$ and $s = 1.5$ are presented in figure
Figure 7.3: Phase diagram in $x_1 - x_3$ plane for different $c$ with $\sigma = 1.0, r = 28, b = 0.7$ and $s = 1.5$.

Figure 7.4: Bifurcation of $x_3$ with respect to $c$ with, $\sigma = 10.0, r = 340, b = 8/3$ and $s = 30$ (a) $0 \leq c \leq 6$ (b) $6 \leq c \leq 15$ (c) $15 \leq c \leq 30$ (d) $30 \leq c \leq 60$.

7.4.2 Hopf point continuation

The aim of this section is to investigate the bifurcation pattern with respect to $c$. This is done by studying the change in the eigenvalues of the Jacobian matrix and following the continuation algorithm. We choose initial points $x_{10} = 2.6736143, x_{20} = 19.408265, x_{30} = 271.07421$ and $x_{40} = 1.0903714$ for parameters $\sigma = 10, r = 340, s = 30$ and $b = 8/3$. In case of parameter set $\sigma = 1, r = 28, s = 1.5$ and $b = 0.7$ we choose $x_{10} = -3.8657198, x_{20} = -5.2784958, x_{30} = 26.592774$ and $x_{40} = 3.3470576$. The characteristics of Hopf point, limit cycle and the general bifurcation nature are explored using the software package MATCONT2.5.1. In this package we use prediction-correction continuation algorithm based on the Moore-Penrose matrix pseudo inverse for computing the curves of equilibria, limit
Parameter | Orbits |
--- | --- |
0.5 | \((P_1^-, P_1^+)\) |
1.0 | \((P_2^-, P_2^+)\) |
5.0 | \((P_3^-, P_2^+)\) |
8.0 | \((P_2^-, P_1^+)\) |
12.0 | \((P_3^-, P_1^+)\) |
16.0 | \((P_\infty^-, P_2^+)\) |
21.0 | \((P_\infty^-, P_1^+)\) |
29.0 | \((P_\infty^-, P_1^+)\) |

The continuation curves from the equilibrium point with respect to \(r\) for \(\sigma = 10, b = 8/3, s = 30\) and \(c = 3\) is depicted in figure 7.6(a). Existence of one Hopf point (\(H\)), one limit point (LP) and two neutral saddle \(\{H_1, H_2\}\) are observed. In figure 7.6(b) the continuation curve from the equilibrium point with respect to \(r\) for \(\sigma = 1, b = 0.7, s = 1.5\) and \(c = 3\) is plotted. Only existence of one Hopf (H) point of \(x_3\) with the variation of \(r\) is clear form the figure 7.6(b). The eigenvalues corresponding to all equilibrium points of the two sets of parameters are listed in Table 7.3 and Table 7.4 respectively. From Table 7.3, The limit point (LP) occurs at \((x_1, x_2, x_3, x_4, r)\equiv (1.154414, 4.500739, 1.948881, -0.115441, 5.847601)\) with normal form of coefficient \(a = -2.346875\). As the parameter is increasing, we observe Hopf point
Figure 7.5: Bifurcation of $x_3$ with respect to $c$ with $\sigma = 1.0$, $r = 28$, $b = 0.7$ and $s = 1.5$.

Table 7.3: Parameter values of $r$ and $c$ at the bifurcation points in figure 7.6(a), together with First Lyapunov coefficients/normal form coefficients and eigenvalues (scaled, see [127]) for Lorenz parameters $\sigma = 10$, $b = 8/3$, $s = 30$. $H$-Hopf point; $H_1$ and $H_2$-Neutral saddle; LP-limit point; BP-branch point.

$(H)$ which is located at $(x_1, x_2, x_3, x_4, r) \equiv (5.792844, 10.530697, 22.881729, -0.579284, 24.699609)$ with first Lyapunov coefficient is to be $0.002982796$, indicating a sub critical Hopf bifurcation. Indeed, there are two complex eigenvalues of the equilibrium with real $\lambda_{3,4} \approx 0$ at this parameter. First Lyapunov coefficient is positive implies that an unstable limit cycle appears from the equilibrium. The other points $\{H_1, H_2\}$ indicate the neutral saddle equilibrium. From the Table 7.4, we observe that Hopf point occurs at $(x_1, x_2, x_3, x_4, r) \equiv (-3.874012, -6.685029, 36.996971, 3.874012, 38.722580)$ with first lyapunov coefficient $l_1 = 0.0005576716$ for $\sigma = 1$, $b = 0.7$, $s = 1.5$ and $c = 3$ indicating that this is also a sub-critical Hopf bifurcation.

Now, we analyse co-dimension-2 bifurcation with respect to $r$ and $c$ as bifurcation parameters. We now consider our starting point to be the Hopf point $(H)$ occurring at $r = 24.699609$, as in figure 7.6(a). Applying forward and backward continuation techniques, we observe Bogdanov-Taken points $(BT_1)$ at...
Table 7.4: Parameter values of $r$ and $c$ at the bifurcation points in figure 7.6(b), together with First Lyapunov coefficients and eigenvalues for Lorenz-stenflo parameters $\sigma = 1$, $b = 0.7$, $s = 1.5$. $H$-Hopf point.

\[
(x_1, x_2, x_3, x_4, r, c) \equiv (1.255628, 6.356972, 2.993996, -0.125563, 8.056779, 4.724656) \quad \text{and} \quad BT_2 \text{ at } (x_1, x_2, x_3, x_4, r, c) \equiv (-1.255628, -6.356972, 2.993996, 0.125563, 8.056779, -4.724656).
\]

This phenomena are presented in figure 7.7. The Bogdanov-Taken points are common points for the limit point curves and curves corresponding to equilibria with eigenvalues $\lambda_1 + \lambda_2 = 0$, $\lambda_3 \neq 0$. Actually, at each BT point, the Hopf bifurcation curve (with $\lambda_{1,2} = \pm i\omega, \omega > 0$) turn into the neutral saddle curve (with real $\lambda_1 = -\lambda_2$). Now, we start LP point continuation from a Bogdanov-Taken (BT) point. If we choose $r$ and $c$ as free parameters and start from the BT points, the continuation curve shows cusp point ($CP$) and two zero-Hopf ($ZH$) points which are shown in figure 7.7. The eigenvalues corresponding these bifurcation points are listed in Table 7.5. Again, starting from Hopf point at $r = 38.722580$ occurring in figure 7.6(b), the continuation of Hopf curve is shown in figure 7.8 and we observe Bogdanov-Taken ($BT$), zero-Hopf ($ZH$) point and also cusp point ($CP$) for the variation of $r$ and $c$. We list the co-dimension-2 bifurcation points in Table 7.6.

Table 7.5: Hopf and limit curve: Parameter values of $r$ and $c$ at the bifurcation points in figure 7.7, together with normal form coefficients and eigenvalues (scaled, \([127]\)) for parameters $\sigma = 10$, $b = 8/3$, $s = 30$. BT- Bogdanov-Takens; ZH-zero-Hopf; CP-Cusp Point.
Figure 7.6: Continuation curves of equilibrium with the variation of the parameter \( r \) for \( c = 3 \) and \( \sigma = 10.0, s = 30, b = 8/3 \) in (a) and for \( \sigma = 1.0, s = 1.5, b = 0.7 \) in (b) \( H_1, H_2 \)- neutral saddle, \( LP \)- limit point, \( H \)- Hopf point.

Figure 7.7: Hopf point continuation of the system (7.8) in the plane \((r, c)\) for \( \sigma = 10.0, s = 30, b = 8/3 \): \( LP \)-limit point, \( ZH_1, ZH_2 \)-Zero Hopf, \( CP \)-cusp point, \( BT_1, BT_2 \) - Bogdanov-Takens.
<table>
<thead>
<tr>
<th>$c$</th>
<th>$r$</th>
<th>Label</th>
<th>Normal form coefficient</th>
<th>Eigenvalues</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.50825089</td>
<td>4.629008</td>
<td>$BT_1$</td>
<td>$a = -0.2574046$</td>
<td>$-2.55875, -1.14125, 0.00, 0.00$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$b = 0.09692810$</td>
<td></td>
</tr>
<tr>
<td>-0.50825089</td>
<td>4.629008</td>
<td>$BT_2$</td>
<td>$a = -0.2574046$</td>
<td>$-2.55875, -1.14125, 0.00, 0.00$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$b = 0.09692810$</td>
<td></td>
</tr>
<tr>
<td>0.0</td>
<td>2.5</td>
<td>$CP$</td>
<td>$c = -0.2164502$</td>
<td>$-2, -1, -0.7, 0.0$</td>
</tr>
<tr>
<td>3.3649552</td>
<td>12.528644</td>
<td>$ZH_1$</td>
<td>$-$</td>
<td>$-3.7, -1.09855, 0.00, 1.09855$</td>
</tr>
<tr>
<td>-3.464952</td>
<td>12.528644</td>
<td>$ZH_2$</td>
<td>$-$</td>
<td>$-3.7, -1.09855, 0.00, 1.09855$</td>
</tr>
</tbody>
</table>

Table 7.6: Hopf and limit curve: Parameter values of $r$ and $c$ at the bifurcation points in Figure 7.8, together with normal form coefficients and eigenvalues (scaled, [127]) for Lorenz-Stenflo parameters $\sigma = 1$, $b = 0.7$, $s = 1.5$. BT-Bogdanov-Takens; ZH-zero-Hopf; CP-Cusp Point.

### 7.4.3 Maximum Lyapunov exponent calculation

The variation of maximum Lyapunov exponent of the modified LS system (7.8) with respect to $c$ are presented for $\sigma = 10$, $r = 340$, $b = 8/3$, $s = 30$ in figure 7.9(a) and for $\sigma = 1$, $r = 28$, $b = 0.7$, and $s = 1.5$ in figure 7.9(b). The multistable behaviour of the coupled LS-system is prominent from these figures. Order to chaos or chaos to order transition is possible by changing only the initial conditions of the full coupled system since maximum lyapunov exponent is positive for some $c$ and it is negative for some other $c$.

Using the scheme-II we obtain the modified LS system (7.11). The bifurcation of the system (7.11) with respect to $c_2$ for different $c_1$ are shown in figure 7.10 for $\sigma = 1$, $r = 28$, $b = 0.7$, and $s = 1.5$. The bifurcation diagrams for $c_1 = 2.4$ and $c_1 = 2.5$ are plotted in figures 7.10(a) and 7.10(b) respectively. The multistable nature of the system (7.9) is confirmed from these diagrams. Using the scheme-III we obtain the modified LS system same as scheme-I.

### 7.5 Conclusion

We introduce three schemes for designing multistable systems coupling LS systems. In the first scheme only one state variable of the coupled systems synchronize and three other state variables keep constant difference. In the second scheme two variables of the coupled systems synchronized and two other variables keep a constant difference. In the third scheme only one state variable of the coupled systems synchronize and three other state variables keep constant difference. In all these three schemes the coupled LS-systems reduced to a single
Figure 7.8: Hopf point continuation of the system (7.8) in the plane \((r,c)\) for \(\sigma = 1.0, s = 1.5, b = 0.7\): \(ZH_1, ZH_2\)-Zero Hopf, \(CP\)-cusp point, \(BT_1, BT_2\)-Bogdanov-Taken.

Figure 7.9: Maximum Lyapunov exponent of \(x_3\) with respect to \(c\) for parameters (a) \(\sigma = 10, b = 8/3, r = 340, s = 30\) and (b) \(\sigma = 1, r = 28, b = 0.7,\) and \(s = 1.5\).

Figure 7.10: Bifurcation with respect to \(c_2\) of the system (7.11) with \(\sigma = 1, s = 1.5, b = 0.7, r = 28\) for \(c_1 = 2.4\) in (a) and \(c_1 = 2.5\) in (b).
modified LS-system. We generalize the method for finding constant of motion of the coupled system to obtain multistable behaviour. We find that coupling two $m$-dimensional systems we may obtain multistable nature if $i$ number of variables of the two systems are completely synchronized and $j$ number of variables keep a constant difference between them, where $i + j = m$ and $1 \leq i, j \leq m - 1$. In this way we can easily find $j(1 \leq j \leq m - 1)$ number of conserved quantities of the coupled system and obtain desired multistable nature of the system. Equilibrium points are determined and there local stability criteria is derived. Conditions for pitchfork bifurcation and Hopf bifurcation of the modified LS-system are obtained. Multistable nature of the coupled LS systems are presented by phase diagrams for different initial conditions. One parameter bifurcation analysis is done with respect to difference in initial conditions of the two systems. Two parameter bifurcation analysis are done using MATCONT software. Variation of largest Lyapunov exponent with respect to initial conditions are calculated which show that chaos to order transition is possible in the coupled system changing only the initial conditions of the system. Our observation may be very useful for designing multistable systems of biological, physical and engineering importance.