Chapter 6

Uncertain destination dynamics of delay coupled systems

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6.1 Introduction

The phenomenon of multistability has been found in almost all areas of science and nature, e.g., in biological systems [83], in hydrodynamics [85], in chemical reactions [106], in ecosystems [88], in neuron dynamics [90], in climate dynamics [91], in social systems [93], in optical systems [95], in semiconductor materials [98], in visual perceptions [99]. Multistability was first observed by Atteneave [99] in visual perception in 1971. A multistable dynamical system is one that possesses a large number of asymptotic stable states for a fixed set of parameters depending on initial conditions. Trivial multistable system can be considered as the coexistence of several stable fixed points i.e. the coexistence of multiple point attractors in the phase space. On the other hand, generalized multistability can be considered for the coexistence of many non-trivial attractors such as limit cycles, chaotic attractors etc [95]. Generalized multistability was first discovered in an experimental nonlinear dynamical system by Arecchi et al. [95] in 1982 and in a computational set up by Arecchi et al. [101] in 1985. The phenomenon of multistability has been identified in different classes of systems, such as weakly dissipative systems, coupled systems, delayed feedback systems, parametrically excited systems, and stochastic systems ([102],[107]). The appearance of a multitude of attractors depends in general on the most important parameters characterising a particular system class, such as, the strength of dissipation, the kind and strength of coupling, the value of the time delay, amplitude and frequency of the parameter perturbation and the noise intensity [103]. Here our basic motivation to study multistable systems is to identify the mechanisms that lead to multistability and to prove rigorously under what circumstances the phenomenon may occur. In extreme multistability the number of coexisting attractors is infinite and this kind of system has been reported in a coupled system by Sun et al. [104]. It has been shown that the reason for the emergence of infinitely many attractors lies in the appearance of a conserved quantity in the long-term limit. Recently, a systematic scheme of the coupling to obtain multistable systems has been developed by Hens et al. [105]. The coupling scheme is developed in a systematic way by using the principle of partial synchronization. Hens et al. [108] formulate one precondition for the emergence of extreme multistability and reported that the coexistence of infinitely many attractors in two m-dimensional coupled systems will be possible if \(m-1\) of the variables of the two systems are completely synchronized and one of them obeys a constant difference between them. We generalise this condition and observe that the coexistence of infinitely many attractors in two m-dimensional coupled systems will be possible if \(m-i\) of the variables of the two systems are completely synchronized and \(i\) (where \(1 \leq i \leq n-1\)) of them obeys a constant difference between them.

Due to signal propagation delays in natural systems it is useful to study delay
coupled systems. Previously Kim et al. PRL1997 had shown multistability in delay coupled systems. They have considered coupled phase oscillators to show multistability but in this chapter we have considered coupled dynamical systems which are not necessarily phase oscillators. Our technique is based on partial synchronization between the coupled systems.

6.2 Designing of multistable delay coupled Lorenz-Stenflo (LS) systems

Lorenz-Stenflo [15] introduced the following system of equations

\[
\begin{align*}
\dot{x} &= \sigma(y - x) + sw, \\
\dot{y} &= rx - y - xz, \\
\dot{z} &= xy - bz, \\
\dot{w} &= -x - \sigma w,
\end{align*}
\]  

where \(\sigma\), \(r\), \(b\) and \(s\) are the four positive parameters.

**Scheme-I:** We coupled two LS systems in the following way:

\[
\begin{align*}
\dot{x}_1 &= \sigma(x_2 - y_1) + sx_4, \\
\dot{x}_2 &= rx_1(t - \tau) - x_2 - x_1(t - \tau)x_3, \\
\dot{x}_3 &= x_1(t - \tau)x_2 - bx_3, \\
\dot{x}_4 &= -x_1(t - \tau) - \sigma x_4, \\
\dot{y}_1 &= \sigma(y_2 - y_1) + sy_4, \\
\dot{y}_2 &= rx_1(t - \tau) - y_2 - x_1(t - \tau)y_3, \\
\dot{y}_3 &= x_1(t - \tau)y_2 - by_3, \\
\dot{y}_4 &= -x_1(t - \tau) - \sigma y_4.
\end{align*}
\]  

(6.2)

Where \(\tau\) is amount of time delay.

We now show that these eight dimensional dynamical system is a multistable system. We construct the dynamical equations for the synchronization errors \(e_1 = x_1 - y_1\), \(e_2 = x_2 - y_2\), \(e_3 = x_3 - y_3\) and \(e_4 = x_4 - y_4\) as

\[
\begin{align*}
\dot{e}_1 &= \sigma e_2 + se_4, \\
\dot{e}_2 &= -e_2 - x_1(t - \tau)e_3, \\
\dot{e}_3 &= x_1(t - \tau)e_2 - be_3, \\
\dot{e}_4 &= -\sigma e_4.
\end{align*}
\]  

(6.3)
Clearly the function $V = (e_2^2 + e_3^2 + e_4^2)/2$ is a Lyapunov function for the system
\[
\dot{e}_2 = -e_2 - x_1(t-\tau)e_3, \quad \dot{e}_3 = x_1(t-\tau)e_2 - be_3 \quad \text{and} \quad \dot{e}_4 = -\sigma e_4 \quad \text{because} \quad V \text{ is positive definite and}
\]
\[
\dot{V} = e_2\dot{e}_2 + e_3\dot{e}_3 + e_4\dot{e}_4,
\]
\[
= -e_2^2 - be_3^2 - \sigma e_4^2 < 0, \quad \text{for} \quad e_2, e_3, e_4 \text{ not equal to zero as} \quad b, \sigma > 0. \quad (6.4)
\]

Therefore $e_2$, $e_3$ and $e_4$ must tend to zero as $t \to \infty$ i.e., $y_2 = x_2$, $y_3 = x_3$ and $y_4 = x_4$. Since, $e_2$, $e_3$ and $e_4 \to 0$ implies $e_1=\text{ constant}=c$. Therefore, $e_1 = x_1 - y_1 \to c$ where $c$ is some constant (dependent on the initial conditions of the full system). Each new set of initial conditions gives rise to different value of $c$. Therefore, the dynamics of the system of equations (6.2) is equivalent to the following four dimensional system
\[
\begin{align*}
\dot{x}_1 &= \sigma(x_2 - x_1 + c) + sx_4, \\
\dot{x}_2 &= r x_1(t-\tau) - x_2 - x_1(t-\tau)x_3, \\
\dot{x}_3 &= x_1(t-\tau)x_2 - bx_3, \\
\dot{x}_4 &= -x_1(t-\tau) - \sigma x_4. 
\end{align*}
\]  

(6.5)

The system (6.2) is a multistable system if the dynamical behavior of the system (6.5) varies qualitatively with the variation of the values of $c$. 

**Scheme-II:** Now we coupled two Lorenz-stenflo (LS) systems in slightly different way and consider the following type of delay coupling
\[
\begin{align*}
\dot{x}_1 &= \sigma(x_2 - y_1) + sx_4, \\
\dot{x}_2 &= r x_1(t-\tau) - y_2 - x_1(t-\tau)x_3, \\
\dot{x}_3 &= x_1(t-\tau)x_2 - bx_3, \\
\dot{x}_4 &= -x_1(t-\tau) - \sigma x_4, \\
\dot{y}_1 &= \sigma(x_2 - y_1) + sy_4, \\
\dot{y}_2 &= r x_1(t-\tau) - y_2 - x_1(t-\tau)y_3, \\
\dot{y}_3 &= x_1(t-\tau)x_2 - by_3, \\
\dot{y}_4 &= -x_1(t-\tau) - \sigma y_4.
\end{align*}
\]  

(6.6)

We now show that these eight dimensional dynamical system is a multistable system. We construct the dynamical equations for the synchronization errors $e_1 = x_1 - y_1$, $e_2 = x_2 - y_2$, $e_3 = x_3 - y_3$ and $e_4 = x_4 - y_4$ as
\[
\begin{align*}
\dot{e}_1 &= se_4, \\
\dot{e}_2 &= -x_1(t-\tau)e_3, \\
\dot{e}_3 &= -be_3, \\
\dot{e}_4 &= -\sigma e_4. 
\end{align*}
\]  

(6.7)
Which shows that $e_3$ and $e_4$ must tend to zero as $t \to \infty$ i.e., $y_3 = x_3$ and $y_4 = x_4$. Since, $e_3$ and $e_4 \to 0$ implies $e_1 = \text{constant} = c_1$ and $e_2 = \text{constant} = c_2$. Therefore, $e_1 = x_1 - y_1 \to c_1$ and $e_2 = x_2 - y_2 \to c_2$, where $c_1$ and $c_2$ are some constant (dependent on the initial conditions of the full system). Each new set of initial conditions gives rise to different value of $c_1$ and $c_2$. Therefore, the dynamics of the system of equations (6.6) is equivalent to the following four dimensional system

\begin{align*}
\dot{x}_1 &= \sigma(x_2 - x_1 + c_1) + sx_4, \\
\dot{x}_2 &= rx_1(t - \tau) - x_2 - x_1(t - \tau)x_3 + c_2, \\
\dot{x}_3 &= x_1(t - \tau)x_2 - bx_3, \\
\dot{x}_4 &= -x_1(t - \tau) - \sigma x_4.
\end{align*}

(6.8)

The system (6.6) is a multistable system if the dynamical behaviour of the system (6.8) varies qualitatively with the variation of the values of $c_1$ and $c_2$.

### 6.3 Result and discussion

To show the effectiveness of our technique we solve the system (6.5) numerically for different values of $c$. We keep the parameters $\sigma = 1$, $b = 0.7$ and $s = 1.5$ are fixed for our simulations. Time evolution of $x_3$ of the system (6.5) for $\tau = 0.01$ and $r = 25$ are depicted in figure 6.1(a) to 6.1(c) respectively for $c = 0$, $c = 2$ and $c = 2.3$. Chaos to steady state transition is observed with the increase of $c$ in figure 6.1 keeping all other parameters fixed. For $\tau = 0.02$ and $r = 25$ time evolution of $x_3$ of the system (6.5) are plotted in figure 6.2(a) to 6.2(c) respectively for $c = 0$, $c = 2.35$ and $c = 2.4$. Therefore, with increase of $c$ chaos to steady state transition is observed from figure 6.2 keeping all other parameters fixed. Thus, we observed that the system (6.6) has multistable behaviour.

Secondly we solve the system (6.8) numerically for suitable set of parameter values $c_1$ and $c_2$. We analyse the simulation results of the system (6.8) through time evolution diagrams for different set of $c_1$ and $c_2$. Time plot of $x_3$ of the system (6.8) for $\tau = 0.01$ and $r = 25$ are depicted in figure 6.3(a) to 6.3(c) respectively for $c_1 = 0$, $c_2 = 0$; $c_1 = 0$, $c_2 = 8.5$ and $c_1 = 0$, $c_2 = 8.7$ with increase of $c_2$ for fixed $c_1 = 0$ chaos to steady state transition is noticed from figure 6.3, all other parameters are kept fixed. Figure 6.4(a) to 6.4(c) respectively represent the time evolution diagram of $x_3$ of the system (6.8) for $c_1 = 0$, $c_2 = 0$; $c_1 = 2.2$, $c_2 = 0$ and $c_1 = 2.4$, $c_2 = 0$ fixing $\tau = 0.01$ and $r = 25$. With increase of $c_1$ for fixed $c_2 = 0$ chaos to steady state transition is found from figure 6.4. In figure 6.5(a) to 6.5(c) respectively represent the time evolution diagram of $x_3$ of the system (6.8) for $c_1 = 1$, $c_2 = 1$; $c_1 = 1$, $c_2 = 5.2$ and $c_1 = 1$, $c_2 = 5.4$ fixing $\tau = 0.01$ and $r = 25$. With increase of $c_2$ for fixed $c_1 = 1$ chaos to steady state transition is observed.
Figure 6.1: Time evolution of $x_3$ for fixed $\tau = .01$, $\sigma = 1$, $b = 0.7$, $s = 1.5$ in (a) $c = 0$, in (b) $c = 2$ and in (c) $c = 2.3$ of the system (6.5).

Therefore we observed that the system (6.6) has multistable behaviour.

6.4 Conclusion

We have proposed a technique for designing multistable systems with delay coupling. We have discussed our technique considering two delay coupled Lorenz-Stenflo systems. Efficiency of our technique is shown with numerical simulation results. Most important observation is that amplitude death and chaotic oscillations are observed in a delay coupled system with the variation of initial conditions (keeping all other parameters fixed) only. Multistable time delayed system may be useful to understand mechanism for memory storage and temporal pattern recognition in neuronal systems.
Figure 6.2: Time evolution of $x_3$ for fixed $\tau = .02, \sigma = 1, b = 0.7, s = 1.5$ in (a) $c = 0$, in (b) $c = 2.35$ and in (c) $c=2.4$ of the system (6.5).

Figure 6.3: Time evolution of $x_3$ for fixed $\tau = .01, \sigma = 1, b = 0.7, s = 1.5$ in (a) $c_1 = 0, c_2 = 0$, in (b) $c_1 = 0, c_2 = 8.5$ and in (c) $c_1 = 0, c_2 = 8.7$ of the system (6.8).
Figure 6.4: Time evolution of $x_3$ for fixed $\tau = 0.01, \sigma = 1, b = 0.7, s = 1.5$ in (a) $c_1 = 0, c_2 = 0$, in (b) $c_1 = 2.2, c_2 = 0$ and in (c) $c_1 = 2.4, c_2 = 0$ of the system (6.8).

Figure 6.5: Time evolution of $x_3$ for fixed $\tau = 0.01, \sigma = 1, b = 0.7, s = 1.5$ in (a) $c_1 = 1, c_2 = 1$, in (b) $c_1 = 1, c_2 = 5.2$ and in (c) $c_1 = 1, c_2 = 5.4$ of the system (6.8).