3. MODIFIED TAMM-DANCOFF APPROXIMATION : BASIS STATES

Although the RPA is quite satisfactory for the description of one-phonon states but HRPA for the two-phonon states is open to severe criticism because of the nonorthonormality and redundancy of the four-quasiparticle basis states mentioned in the previous section. Therefore, one may legitimately doubt the reliability of the results obtained for the two-phonon states and is not sure whether one is gaining anything by doing HRPA. Therefore, one feels the necessity of developing a method which may be less accurate in principle but is not open to such criticism. Here we develop the method of constructing nonredundant and orthonormal set of four-quasiparticle basis states. In the first place, these states are classified according to the well-known shell-model method, which automatically insures an independent, orthonormal, and nonredundant basis. These forms are very well suited for evaluating the matrix elements in the pure quasiparticle subspaces. On the other hand, a one-to-one correspondence has been established between the shell-model basis states and the second-quantised states, which enables us to evaluate the matrix elements in the mixed quasiparticle subspaces. The next section is fully devoted to the evaluation of such matrix elements. In order to predict the detailed spectra of a nucleus one has to mix zero-, two-, and four-quasiparticle states. The effect of the spurious $0^+$ state is also removed from the two- and four-quasiparticle basis states by the method described in this section. This method will be subsequently called the modified Tamm-Dancoff approximation (MTDA). In the following section we develop the MTDA, along the lines mentioned earlier.
3.1 Zero- and Two-Quasiparticle Basis States and the Elimination of Spurious $0^+$ State

Zero-quasiparticle state is the quasiparticle vacuum $|\theta\rangle$ and the antisymmetric normalised two-quasiparticle states are just those produced by the doubles of the previous section operating on $|\theta\rangle$, viz.,

$$\hat{A}^\dagger(abJ^M)\langle\theta| = |abJ^M\rangle_{\theta} = \frac{1}{\sqrt{2}} \left\{ |a(u)\ b(a)\ J^M\rangle - (-1)^{Q^+ + b - J} \ |b(u)\ a(a)\ J^M\rangle \right\}. \quad (3.1)$$

The TDA method gives one extra state (spurious state) of $J = 0$ among all states containing a pair of quasiparticles. The RPA method has the property of insuring that the $0^+$ spurious state occurs at zero energy and thus eliminates the spurious state problem. The spurious state arises due to the nonconservation of the number of particles. Therefore, the $J = 0$ states require special treatment because the BCS ground-state (quasiparticle vacuum), $|\theta\rangle$, is not an eigenstate of the number operator $\hat{N}$. The state $\hat{N}|\theta\rangle$ is different from $|\theta\rangle$ and its component on two-quasiparticle state $J=0$ is spurious; therefore, only those two-quasiparticle states of zero angular momentum which are orthogonal to the spurious state should be considered in a calculation of the excited $0^+$ states. But the two-quasiparticle states resulting from the approximate diagonalisation of the Hamiltonian are usually not orthogonal to the spurious state, with the result that the spurious state is mixed with various percentages among all the states that one calculates. To find the spurious state, one has to find the part of $\hat{N}|\theta\rangle$ orthogonal to $|\theta\rangle$. We have
The normalisation is roughly $N$, the number of nucleons, and thus the first part of this expression is simply $\frac{1}{N} \sum_a [a] \langle \alpha \mid 0 \rangle = |0\rangle$. The second part, consisting of two-quasiparticle states is orthogonal to $|0\rangle$ and represents the spurious $0^+$ state. Denoting it by $|\phi_0\rangle$ (normalised), we get

$$|\phi_0\rangle = -\frac{1}{N_0} \sum_{\alpha} S_{\alpha} a_{\alpha} a_{\alpha}^+ |0\rangle,$$

where $N_0$ is the normalisation constant.

Writing

$$\sum_{\alpha} S_{\alpha} a_{\alpha}^+ a_{\alpha}^+ = \left[ \sum_{\alpha} \left[ a_{\alpha}^+ a_{-\alpha}^+ \right] S_{\alpha} a_{\alpha}^+(a_0 a_0) \right],$$

$|\phi_0\rangle$ can now be written as

$$|\phi_0\rangle = \frac{-1}{N_0} \sum_{\alpha} C_{\alpha} |a_{\alpha}^3 \rangle,$$

where $C_{\alpha} = \sqrt{\sum_{\alpha} S_{\alpha} a_{\alpha} a_{\alpha}^+}$ and $|a_{\alpha}^3 \rangle = a_{\alpha}^+(a_{\alpha} a_{\alpha}) |0\rangle$ is an antisymmetric normalised two-quasiparticle state of zero angular momentum.

To eliminate the spurious state one has to construct two-quasiparticle orthonormal states orthogonal to $|\phi_0\rangle$ by the Schmidt orthogonalisation procedure.

Suppose we have $n$ two-quasiparticle states of the type $|a_i^3 \rangle$ $(i = 1, 2, \ldots, n)$ as the basis states. The special linear combination $|\phi_0\rangle$ of these states

$$|\phi_0\rangle = \frac{1}{N_0 n} \sum_{i=1}^{n} C_{i} |a_i^3 \rangle.$$
is entirely spurious, where the normalisation $N_0(1)$ is given by

\[ N_0(1) = \left( \sum_{i=1}^{n} C_i^2 \right)^{1/2} \]

The new set of orthonormal states have to be constructed step by step.

Let us start with $|a_1^2\rangle$ and produce a state $|\phi_1\rangle$ (normalised), orthogonal to $|\phi_0\rangle$

\[ |\phi_1\rangle = \frac{1}{N_\phi(1)} \left[ -\frac{\langle \phi_0 | a_1^2 \rangle}{\langle \phi_0 | \phi_0 \rangle} |\phi_0\rangle + |a_1^2\rangle \right] \]

\[ = \frac{1}{N_\phi(1)} \left[ \frac{1}{N_0(1)} |a_1^2\rangle \sum_{i=2}^{n} C_i^2 - \frac{1}{N_0(1)} \sum_{i=2}^{n} C_i C_{i'} |a_i^2\rangle \right] \]

where the normalisation $N_\phi(1)$ is given by

\[ N_\phi(1) = \frac{1}{N_0(1)} \left\{ \sum_{i=2}^{n} C_i^2 \right\}^{1/2} \]

Thus $|\phi_1\rangle$ becomes

\[ |\phi_1\rangle = \frac{1}{N_0(1)} \left[ N_0(2) |a_1^2\rangle - \sum_{i=2}^{n} C_i C_{i'} |a_i^2\rangle \right] \tag{3.4} \]

with

\[ N_0(2) = \left\{ \sum_{i=2}^{n} C_i^2 \right\}^{1/2} \]

Next we start with $|a_2^2\rangle$ and construct the state $|\phi_2\rangle$ (normalised) that is orthogonal to $|\phi_0\rangle$ and $|\phi_1\rangle$

\[ |\phi_2\rangle = \frac{1}{N_\phi(2)} \left[ -\frac{\langle \phi_0 | a_2^2 \rangle}{\langle \phi_0 | \phi_0 \rangle} |\phi_0\rangle - \frac{\langle \phi_1 | a_2^2 \rangle}{\langle \phi_1 | \phi_1 \rangle} |\phi_1\rangle + |a_2^2\rangle \right] \]

\[ = \frac{1}{N_\phi(2)} \left[ \frac{1}{N_0(2)} |a_2^2\rangle \sum_{i=3}^{n} C_i^2 - \frac{1}{N_0(2)} \sum_{i=3}^{n} C_i C_{i'} |a_i^2\rangle \right] \]
and the normalisation \( N_\Phi (2) \) is given by
\[
N_\Phi (2) = \frac{1}{N_0(2)} \left\{ \sum_{i=3}^{n} C_i^2 \right\}^{1/2}
\]

which gives \( |\phi_2\rangle \) as
\[
|\phi_2\rangle = \frac{1}{N_0(2) N_0(3)} \left[ N_0(3) |a_2^2 \rangle - \sum_{i=3}^{n} C_i C_i |a_i^2 \rangle \right]
\]

with
\[
N_0(3) = \left\{ \sum_{i=3}^{n} C_i^2 \right\}^{1/2}
\]

Proceeding in the same way one can construct other states which are normalised and orthogonal to \( |\phi_0\rangle, |\phi_1\rangle, |\phi_2\rangle, \ldots \). It is to be noticed that the number of these independent orthonormal states will now be one less number than the actual number \( n \) of states of the type \( |a_i^2 \rangle (i = 1, 2, \ldots, n) \). This is because one independent linear sum \( |\phi_0\rangle \) is now kept out of the basis. The \((n-1)\) independent orthogonal state \( |\phi_{n-1}\rangle \) (normalised) is given by
\[
|\phi_{n-1}\rangle = \frac{1}{N_0(n-1) N_0(n)} \left[ N_0(n) |a_{n-1}^2 \rangle - C_{n-1} C_n |a_n^2 \rangle \right]
\]

with
\[
N_0(k) = \left\{ \sum_{i=k}^{n} C_i^2 \right\}^{1/2}
\]

These new set of \((n-1)\) independent orthonormal states \( |\phi_0\rangle, |\phi_2\rangle, \ldots, |\phi_{n-1}\rangle \) should be used in the calculation of the excited \( 0^+ \) states in place of the original set \( |a_i^2 \rangle (i = 1, 2, \ldots, n) \). This procedure will remove the spurious \( 0^+ \) state from the two-quasi-
3.2 Four-Quasiparticle Basis States:

Four-quasiparticle basis states are treated by simple shell-model methods\(^{12}\) which, in turn, automatically insures an independent, orthogonal and nonredundant basis. The following different types of configurations in the four-quasiparticle subspace will occur:

\[
\begin{align*}
(1) | a^4 J \rangle, & \quad (2) | a^3 J_1, a^2 J \rangle, & \quad (3) | a^2 J_1, a^2 J_2, J \rangle, \\
(4) | a^3 J_1, a^2 J_2, J \rangle, & \quad (5) | a^3 a_2 J_1, a_2 J_2, J \rangle.
\end{align*}
\]

Here \( J \) is the total angular momentum, \( v \) is the seniority and the subscripted \( J \)'s are intermediate coupled angular momenta. The superscript on \( a, a_1, \) and \( a_2 \) denotes the number of quasiparticles present in that particular state.

This shell-model way of classifying the four-quasiparticle states is similar to the one, already discussed in Sec.3.1 of Part IB of this thesis in the case of four-particles. The same Sec. also gives the method of constructing antisymmetric and normalised states for the different cases of (3.7), and correspond to the expressions (3.2) and (3.5)-(3.8).

Second-Quantised Version of the Shell-Model States:

The second-quantised versions of four-quasiparticle states of the type (3), (4), and (5) of (3.7) is straightforward and are given as
These can be represented by a single general expression, the last Eq. of (3.8). The expressions for the states of the types (1) and (2) of (3.7) in second-quantised form is a little more complicated. The method will be outlined below. To be very explicit let us first consider the case of all the four quasiparticles in the same level (type(1)) where there is no repetition of the same $J$ with different seniorities.

Example 1. $|3/2 J=0 \gamma=0\rangle$.

In the second-quantised notation we can write

$$|3/2 J=0 \gamma=0\rangle_A = \mathcal{P}_{(00)} (3/2 3/2 3/2 3/2) = N_1 \left[ A^{\dagger}(3/2 3/2) A^{\dagger}(3/2 3/2) \right] |0\rangle,$$

where $N_1$ is the normalisation of the state of four quasiparticles in the same level with $J = 3/2$. In terms of single quasiparticle operators, it is written as

$$|3/2 J=0 \gamma=0\rangle = N_1 \sum_{\alpha \alpha'} \left[ \begin{array}{ccc} \alpha & 3/2 & 0 \\ \alpha' & 3/2 & 0 \end{array} \right] \left[ \begin{array}{ccc} \alpha & 3/2 & 0 \\ \alpha' & 3/2 & 0 \end{array} \right] a_\alpha^{\dagger} a^{\dagger}_\alpha a^{\dagger}_{\alpha'} a^{\dagger}_{\alpha'} |0\rangle.$$

In the above the right hand side sum is symmetric in $\alpha, -\alpha$ and in $\alpha', -\alpha'$; and hence
Here the sum has to be carried out for only positive values of $\alpha$ and $\alpha'$ with the restriction $\alpha \neq \alpha'$ (due to Pauli exclusion principle) and gives
\[
|\frac{3}{2} J^=0 \nu=0\rangle_A = N_1 i \sum_{\alpha, \alpha'} c_{\alpha\alpha'} a_\alpha^+ a_{\alpha'}^+ |0\rangle.
\]

The normalisation $N_1$ is given by
\[
4 N_1 \langle 0 | a_{3/2} a_{3/2} a_{-3/2} a_{-3/2} a_{3/2}^+ a_{-3/2}^+ a_{3/2}^+ a_{-3/2}^+ |0\rangle = 1
\]
or
\[
N_1 = \pm \frac{1}{4}.
\]

In fixing the sign of $N_1$, we will make the consistent assumption throughout, following Talmi and Deshalki's book that its sign is chosen to be the same as that of the corresponding two-particle-type fpc. Since, in this case the sign of the two-particle-type fpc
\[
\langle \frac{3}{2} (0), \frac{3}{2} (0) | 0 | \frac{3}{2} J^=0 \nu=0 \rangle
\]
is positive (e.g., see Table I of Appendix), therefore, the sign of $N_1$ will be taken positive, which is consistent with the above assertion. Hence
\[
|\frac{3}{2} J^=0 \nu=0\rangle_A = \frac{1}{2} \left[ A(3/2, 0) A^+(3/2, 0) \right]^{J^=0} |0\rangle.
\]

We can also write the state $|\frac{3}{2} J^=0 \nu=0\rangle_A$ as
\[
|\frac{3}{2} J^=0 \nu=0\rangle_A = (b_{3/2})^0 \left[ A(3/2, 3/2, 3/2, 3/2) \right] = N_1' \left[ A(3/2, 3/2) A^+(3/2, 3/2) \right]^{J^=0} |0\rangle,
\]
where the normalisation $N_1'$ is different from $N_1$ above. In terms of single quasiparticle operators, we can write
Here the sum on the right hand side is symmetric in $M$ and $-M$ and therefore, it can be written in terms of the sums over the positive values of $M$. In the part corresponding to $M = 0$, the sum is also symmetric in $\alpha, -\alpha$ and $\alpha', -\alpha'$, therefore, this part is written as the sum over only positive values of $\alpha$ and $\alpha'$. Thus

$$|{3/2}^4 J = 0 \nu = 0\rangle_A = \sum_{M, \alpha, \alpha'} N_1 \sum_{\alpha, \alpha', \nu} \left[ \begin{array}{ccc} \alpha & 3/2 & 3/2 \\ M & -M & \nu \end{array} \right] \left[ \begin{array}{ccc} \alpha' & 3/2 & 3/2 \\ -M & \alpha & \nu \end{array} \right] \left( \begin{array}{ccc} 1/2 & 3/2 & 3/2 \\ \alpha & \alpha' & \nu \end{array} \right) a_{\alpha}^+ a_{M-\alpha}^+ a_{\alpha'}^+ a_{-M-\alpha'}^+ |0\rangle .$$

Once again, the sum over $M, \alpha$, and $\alpha'$ has to be carried, keeping in mind the Pauli exclusion principle. In the first part of the sum, corresponding to $M = 2, 1$, the values of $\alpha$ and $\alpha'$ are restricted to only the following values:

- For $M = 2$, $\alpha = \pm 3/2$ and $\alpha' = \pm 1/2, \pm 3/2$
- For $M = 1$, $\alpha = \pm 1/2$ and $\alpha' = \pm 1/2, \pm 3/2$

In the second part, the sum over $\alpha$ and $\alpha'$ is straightforward. After the sum is carried out with the substitution of the corresponding Clebsch-Gordan coefficients, we get

$$|{3/2}^4 J = 0 \nu = 0\rangle_A = \frac{10}{15} N_1 \sum_{\alpha, \alpha'} \left[ \begin{array}{ccc} \alpha & 3/2 & 3/2 \\ \alpha' & 3/2 & 3/2 \end{array} \right] \left( \begin{array}{ccc} 1/2 & 3/2 & 3/2 \\ \alpha & \alpha' & \nu \end{array} \right) a_{\alpha}^+ a_{\alpha'}^+ a_{3/2}^+ a_{-3/2}^+ |\nu\rangle .$$

where the normalisation $N_1$ is given by
The negative sign is chosen because the two-particle-type fpc
\[ \langle 3/2^2(1), 3/2^2(2) | \sigma \parallel 3/2^2 J = \sigma \nu \parallel 3/2^2 \rangle \] is negative (e.g., see Table I of Appendix) and thus
\[ |3/2^4 J = \sigma \nu = 0\rangle_A = - \frac{\sqrt{6}}{10} \left[ A^\dagger (3/2^2 1) A^\dagger (3/2^2 2) \right]^{J=0} |0\rangle \quad (3.12) \]

It is important to note that the states $\frac{1}{2} \left[ A^\dagger (3/2^2 0) A^\dagger (3/2^2 0) \right]^{J=0}$ and $- \frac{\sqrt{6}}{10} \left[ A^\dagger (3/2^2 1) A^\dagger (3/2^2 2) \right]^{J=0} |0\rangle$ are the same (as can be seen from their expressions in terms of single quasiparticle operators, see Eqs. (3.9) and (3.11)).

Next, let us consider the case of three quasiparticles in the same level (type (2)) where there is no repetition of the same \( J \) with different seniorities.

Example 2. \[ |3/2^3 J = 3/2 \nu = 1\rangle \]

In the second-quantised notation
\[ |3/2^3 J = 3/2 \nu = 1\rangle_A = N' \left[ A^\dagger (3/2^3 0) A^\dagger (3/2^3 0) \right]^{J=3/2} |0\rangle = N' \sum_{\alpha > 0} \alpha^{3/2 - \alpha} \left( \alpha^+ \alpha^+ \alpha^+ |0\rangle \right) \]

where \( N' \) is the normalisation of the state of three quasiparticles in the same level with \( j = 3/2 \). The sum over \( \alpha \) with the restriction \( \alpha \leq 3/2 \) (due to Pauli exclusion principle) gives
\[ |3/2^3 J = 3/2 \nu = 1\rangle_A = - N' \left( a_{1/2}^+ a_{1/2}^+ a_{3/2}^+ |0\rangle \right) \quad (3.13) \]
The normalisation $N'$ is
\[ N' = \pm 1 \]
Here, again the sign of $N'$ will be fixed by the same rule as was followed in the case of all the four quasiparticles in the same level, the only difference will be that in this case, the fpc of one-particle type will occur. Since, the one-particle-type $\langle \frac{3}{2}\alpha | 0 | \frac{3}{2}\alpha \rangle$ is positive (e.g., see the Appendix of Ref.12), therefore, the sign of $N'$ is taken positive. Thus
\[ | \frac{3}{2} J = \frac{3}{2} \alpha = \rangle \rightarrow [ A^\dagger (\frac{3}{2}\alpha) A_{\frac{3}{2}\alpha} ]^{J=\frac{3}{2}} \alpha \rangle \]
\[ (3.14) \]
One can also write the state $| \frac{3}{2} J = \frac{3}{2} \alpha = \rangle$ as
\[ | \frac{3}{2} J = \frac{3}{2} \alpha = \rangle \rightarrow N'' [ A^\dagger (\frac{3}{2}\alpha) A_{\frac{3}{2}\alpha} ]^{J=\frac{3}{2}} M=\frac{3}{2} \alpha \rangle \]
\[ = N'' \sum_{M=\frac{3}{2}} \left[ A_{\frac{3}{2}\alpha} A_{M=\frac{3}{2}\alpha} \right] \alpha \alpha \]
where the normalisation $N''$ is different from $N'$ above. The sum over $M$ ($M$ takes only the values $2, 1, 0$) and $\alpha$ can be carried out, keeping in mind the Pauli exclusion principle, and the substitution of the appropriate Clebsch-Gordan coefficients, finally, reduces it to
\[ | \frac{3}{2} J = \frac{3}{2} \alpha = \rangle \rightarrow \sqrt{N''} A^\dagger_{\frac{3}{2}\alpha} A^\dagger_{\frac{1}{2}\alpha} A^\dagger_{\frac{3}{2}\alpha} \alpha \rangle \]
\[ (3.15) \]
Here the normalisation $N''$ is
\[ N'' = -1/\sqrt{5} \]
The minus sign is chosen because the value of one-particle-type fpc $\langle \frac{3}{2}\alpha | 0 | \frac{3}{2}\alpha \rangle$ is negative (e.g., see the Appendix of Ref.12). Therefore,
\[ |3/2 \ J=5/2 \ v=1\rangle_A = -\frac{1}{15} \left[ A^\dagger (3/2^2) \ A^\dagger (3/2^2) \right]^{J=3/2} \ |0\rangle. \quad (3.16) \]

Once again, it is important to note that the states 
\[ [A^\dagger (3/2^2) \ A^\dagger (3/2^2)]^{J=3/2} \ |0\rangle \quad \text{and} \quad -\frac{1}{15} \left[ A^\dagger (3/2^2) \ A^\dagger (3/2^2) \right]^{J=3/2} \ |0\rangle \]
are same, as can be seen from their expressions in terms of single quasiparticle operators (Eqs. (3.13) and (3.15)).

Following the method outlined above, one will find that the different states
\[ \Phi^\dagger (J'J'') \ J_M (aa,aa), \]
and also
\[ \Phi^\dagger (J'J'') \ J_M (aa,ab) \]
for a given \( J \) (\( M \) may have any projection value), obtained by all permissible \( J' \) and \( J'' \) are equivalent. For \( j < 7/2 \), one can find the following, if there is no repetition of \( J \) with different seniorities.

\[ |5/2 \ j=0 \ v=0\rangle_A = -\frac{1}{4} \sqrt{2} \left[ A^\dagger (5/2^0) \ A^\dagger (5/2^0) \right]^{J=0} \ |0\rangle \]
\[ = \frac{1}{2} \sqrt{2} \left[ A^\dagger (5/2^0) \ A^\dagger (5/2^0) \right]^{J=0} \ |0\rangle \]
\[ = \frac{1}{2} \sqrt{2} \left[ A^\dagger (5/2^1) \ A^\dagger (5/2^1) \right]^{J=0} \ |0\rangle \]

\[ |5/2 \ j=2 \ v=2\rangle_A = \frac{1}{4} \sqrt{2} \left[ A^\dagger (5/2^1) \ A^\dagger (5/2^1) \right]^{J=2} \ |0\rangle \]
\[ = \frac{7}{40} \left[ A^\dagger (5/2^1) \ A^\dagger (5/2^1) \right]^{J=2} \ |0\rangle \]
\[ = -\frac{7}{18} \left[ A^\dagger (5/2^2) \ A^\dagger (5/2^2) \right]^{J=2} \ |0\rangle \]
\[ = -\frac{7}{4} \sqrt{2} \left[ A^\dagger (5/2^2) \ A^\dagger (5/2^2) \right]^{J=2} \ |0\rangle. \]
\[
15/4 \quad J=4 \quad \nu=2 \langle A = \frac{1}{2} \sqrt{3} \left[ A^+ (5/2) \right] A^+(5/2 \ 4) \right] \langle j=4 \rangle l0\rangle \\
= -\frac{1}{4} \sqrt{3} \left[ A^+ (5/2 \ 2) A^+(5/2 \ 2) \right] \langle j=4 \rangle l0\rangle \\
= -\frac{1}{4} \sqrt{3} \left[ A^+ (5/2 \ 2) A^+(5/2 \ 2) \right] \langle j=4 \rangle l0\rangle \\
= -\frac{1}{4} \sqrt{3} \left[ A^+ (5/2 \ 2) A^+(5/2 \ 2) \right] \langle j=4 \rangle l0\rangle \\
= -\frac{1}{2} \sqrt{143} \left[ A^+ (5/2 \ 4) A^+(5/2 \ 4) \right] \langle j=4 \rangle l0\rangle \\
\]

\[
17/2 \quad J=0 \quad \nu=0 \langle A = \frac{1}{4} \left[ A^+ (7/2) A^+ (7/2) \right] \langle j=0 \rangle l0\rangle \\
= -\frac{1}{4} \left[ A^+ (7/2) A^+ (7/2) \right] \langle j=0 \rangle l0\rangle \\
= -\frac{1}{4} \left[ A^+ (7/2) A^+ (7/2) \right] \langle j=0 \rangle l0\rangle \\
= -\frac{1}{4} \left[ A^+ (7/2) A^+ (7/2) \right] \langle j=0 \rangle l0\rangle \\
= -\frac{1}{4} \left[ A^+ (7/2) A^+ (7/2) \right] \langle j=0 \rangle l0\rangle \\
And \\
\[
5/2 \quad J=5/2 \quad \nu=1 \langle A = -\frac{1}{2} \left[ A^+ (5/2) \right] \langle j=5/2 \rangle l0\rangle \\
= -\frac{1}{2} \left[ A^+ (5/2) \right] \langle j=5/2 \rangle l0\rangle \\
= -\frac{1}{2} \left[ A^+ (5/2) \right] \langle j=5/2 \rangle l0\rangle \\
= -\frac{1}{2} \left[ A^+ (5/2) \right] \langle j=5/2 \rangle l0\rangle \\
5/2 \quad J=3/2 \quad \nu=3 \langle A = -\frac{1}{2} \left[ A^+ (5/2) \right] \langle j=3/2 \rangle l0\rangle \\
= -\frac{1}{2} \left[ A^+ (5/2) \right] \langle j=3/2 \rangle l0\rangle \\
= -\frac{1}{2} \left[ A^+ (5/2) \right] \langle j=3/2 \rangle l0\rangle \\
5/2 \quad J=9/2 \quad \nu=3 \langle A = -\frac{1}{2} \left[ A^+ (5/2) \right] \langle j=9/2 \rangle l0\rangle \\
= -\frac{1}{2} \left[ A^+ (5/2) \right] \langle j=9/2 \rangle l0\rangle \\
= -\frac{1}{2} \left[ A^+ (5/2) \right] \langle j=9/2 \rangle l0\rangle
From the basic definition of seniority and fractional parentage coefficients, it can be verified that the seniority-classified antisymmetric and normalised states of three and four quasiparticles in the same j-state in second-quantised notation, given above, can conve-
niently be expressed in a very compact forms through the very simple relations, in terms of coefficients of fractional parentage.

\[ |3J^M\rangle \equiv \left[ \sum_{j_1} \langle 3j_1, j_3; J^3J^M_\nu \rangle \right]^{-1} \left[ A^+(3j_1) a_{j_1}^+ \right]_M^{J^3J^M_\nu} |0\rangle \]

and

\[ |4J^M\rangle = \left[ \sum_{j_1} \langle 4j_1, j_3; J^4J^M_\nu \rangle \right]^{-1} \left[ A^+(4j_1) A^+(4j_3) \right]_M^{J^4J^M_\nu} |0\rangle . \]  

In the case of repeated \( J \) corresponding to different seniorities, the lowest seniority state can uniquely be written down, and the higher seniority state (same \( J \)) can be found by taking the linear combination with different intermediate angular momentum coupling values in such a way that this new state becomes normalised and orthogonal to the lower seniority state. The following example will make this procedure explicit:

Example 1. Case of three quasi-particles in the same level with \( j=9/2 \)

The \( \nu = 1 \) state can uniquely be shown to be

\[ |\Psi_a\rangle \equiv |q/2^3 J=9/2 \nu=1\rangle_A = -\frac{\sqrt{15}}{4} \left[ A^+(q/2^3) a_{q/2}^+ \right]^{J=9/2} |0\rangle \]

\[ = \frac{\sqrt{2}}{2} \left[ a_{q/2}^+ a_{q/2} a_{q/2} - a_{5/2}^+ a_{5/2} a_{q/2} + a_{3/2}^+ a_{3/2} a_{q/2} - a_{1/2}^+ a_{1/2} a_{q/2} \right] |0\rangle . \]

We will write the state \( |\Psi_b\rangle \) as the linear combination of states with intermediate angular momentum coupled values 2 and 4 (although one can take any intermediate angular momentum coupled values allowed by the angular momentum coupling rule), i.e.,

\[ |\Psi_b\rangle \equiv C_1 \left[ A^+(q/2^2) a_{q/2}^+ \right]^{J=9/2} |0\rangle + C_2 \left[ A^+(q/2^4) a_{q/2}^+ \right]^{J=9/2} |0\rangle . \]
In terms of single quasiparticle operators $|\psi_b\rangle$ can be found as:

$$
\begin{align*}
|\psi_b\rangle &\equiv (\frac{3}{11} \eta c_1 + \frac{10\sqrt{6}}{13x55} \eta c_2) a^+_{\eta/2} a_{\eta/2}^+ a_{\eta/2}^+ |\phi\rangle - (\frac{4}{11} \eta c_1 + \frac{18\sqrt{6}}{13x55} \eta c_2) a^+_{\eta/2} a_{\eta/2}^+ a_{\eta/2}^+ |\phi\rangle \\
+ (\frac{3}{11} \eta c_1 + \frac{8\sqrt{6}}{13x55} \eta c_2) a^+_{\eta/2} a_{\eta/2}^+ a_{\eta/2}^+ |\phi\rangle + (\frac{4}{11} \eta c_1 + \frac{4\sqrt{6}}{13x55} \eta c_2) a^+_{\eta/2} a_{\eta/2}^+ a_{\eta/2}^+ |\phi\rangle \\
+ (\frac{3}{11} \eta c_1 - \frac{7}{13x13} \eta c_2) a^+_{\eta/2} a_{\eta/2}^+ a_{\eta/2}^+ |\phi\rangle + (\frac{4}{11} \eta c_1 + \frac{3\sqrt{6}}{13x13} \eta c_2) a^+_{\eta/2} a_{\eta/2}^+ a_{\eta/2}^+ |\phi\rangle \\
+ (\frac{5}{11} \eta c_1 - \frac{2\sqrt{6}}{13x13} \eta c_2) a^+_{\eta/2} a_{\eta/2}^+ a_{\eta/2}^+ |\phi\rangle.
\end{align*}
$$

One can see that $|\psi_b\rangle$ is not a $\nu = 3$ state, as it contains a part of $|\psi_a\rangle$ ($\nu = 1$ state), therefore, to obtain a $\nu = 3$ state one has to eliminate this part of $\nu = 1$ state from $|\psi_b\rangle$, by suitably choosing $c_1$ and $c_2$: This is achieved by making use of the fact that $|\psi_b\rangle$ should be orthogonal to $|\psi_a\rangle$, and normalised to unity. The orthogonal condition $\langle \psi_b | \psi_a \rangle = 0$ is equivalent to

$$
C_2 = -\frac{15}{3} C_1.
$$

Putting the value of $c_2$ (in terms of $c_1$) in $|\psi_b\rangle$, and normalising it to unity i.e. $\langle \psi_b | \psi_b \rangle = 1$ gives the value of $c_1$ as

$$
C_1 = \frac{\sqrt{11x13}}{8 \sqrt{6}}.
$$

and thus the $\nu = 3$ state (normalised and orthogonal to $\nu = 1$ state) is given by

$$
|q/2\rangle_J^{J=3} = \frac{1}{\sqrt{11x13}} \left\{ \left[ A^+ (q/2 \lambda) A_{\lambda/2} \right]^J_{J=3} \right\} |\phi\rangle.
$$

(3.19)
Example 2. Case of four quasiparticles in the same level with \( j = 7/2 \).

The \( \nu = 2 \) state of angular momentum 2 can uniquely be written down as:

\[
|\Psi_3\rangle = \frac{1}{\sqrt{2}} \left[ A^+(\eta/2) A^*(\eta/2) \right]^{J=2} |\ell\rangle
\]

Again we express the state \( |\Psi_b\rangle \) as a linear combination of the states given below:

\[
|\Psi_b\rangle = c_1' \left[ A^+(\eta/2) A^*(\eta/2) \right]^{J=2} |\ell\rangle + c_2' \left[ A^+(\eta/2) A^*(\eta/2) \right]^{J=2} |\ell\rangle
\]

one can take any intermediate angular momentum coupled values, allowed by the angular momentum coupling rule. In terms of single quasiparticle operators \( |\Psi_b\rangle \) can be shown to be

\[
|\Psi_b\rangle = (-\frac{2}{3} \eta^2 c_1' + \frac{32}{41 \times 42} c_1') a^+_{\frac{7}{2}} a^+_{\frac{5}{2}} a^+_{\frac{7}{2}} a^+_{\frac{7}{2}} |\ell\rangle + (\frac{1}{3} \eta^2 c_1' - \frac{16}{41 \times 42} c_2') x a^+_{\frac{5}{2}} a^+_{\frac{5}{2}} a^+_{\frac{5}{2}} a^+_{\frac{5}{2}} |\ell\rangle
\]

It can be noticed like previous example that \( |\Psi_b\rangle \) is not a \( \nu = 4 \) state, as it contains a part of \( |\Psi_a\rangle \) \( (\nu = 2 \) state). Therefore, the effect of \( \nu = 2 \) state has to be eliminated from \( |\Psi_b\rangle \), in order to obtain a \( \nu = 4 \) state, by taking \( |\Psi_b\rangle \) to be normalised and
orthogonal to $|\psi_2\rangle$. This determines the coefficients $c_1'$ and $c_2'$ suitably.

The orthogonality of $|\psi_b\rangle$ with $|\psi_2\rangle$ gives the relation

$$c_2 = \frac{4}{\sqrt{33}} c_1'. $$

Substituting $c_2'$ (in terms of $c_1'$) in $|\psi_b\rangle$ and normalising it to unity gives $c_1'$ as

$$c_1' = \frac{\sqrt{33}}{14}.$$

Finally, the $\psi=4$ state (normalised and orthogonal to $\psi=2$ state) becomes

$$|\psi^4_{1/2 \ J=2 \ \nu=4}\rangle_A = \frac{\sqrt{33}}{14} \left\{ \left[ A^\dagger(\gamma_{1/2}^2) A^\dagger(\gamma_{1/2}^2) \right]^{J=2} + \frac{4}{\sqrt{33}} \left[ A^\dagger(\gamma_{1/2}^2) A^\dagger(\gamma_{1/2}^2) \right]^{J=2} \right\} |\psi_2\rangle.$$

Similarly, one can show that the repeated $J=4$ state with different seniorities, is

$$|\psi^4_{1/2 \ J=4 \ \nu=1}\rangle_A = \frac{1}{12} \left[ A^\dagger(\gamma_{1/2}^2) A^\dagger(\gamma_{1/2}^2) \right]^{J=4} |\psi_2\rangle$$

and

$$|\psi^4_{1/2 \ J=4 \ \nu=4}\rangle_A = \frac{3}{16} \left\{ \left[ A^\dagger(\gamma_{1/2}^2) A^\dagger(\gamma_{1/2}^2) \right]^{J=4} + \frac{12}{16} \left[ A^\dagger(\gamma_{1/2}^2) A^\dagger(\gamma_{1/2}^2) \right]^{J=4} \right\} |\psi_2\rangle.$$

Conversely, all the second-quantised states having three and four quasiparticles in the same level can be expressed as the linear combination of the corresponding shell-model states, with same $J$ having different seniorities. It is found that the following relations hold in the case of repeated $J$ with different seniorities.
\[ [A^+(J_1 J_2) A^+_f] \, J \, \psi = \frac{\mathcal{A}}{3!} \left\{ \langle J^2(J_1), J; J J J J \rangle | J^3 J \psi \rangle 
- \langle J^2(J_1), J; J J J \rangle | J^3 J \psi' \rangle \right\} \]

and
\[ [A^+(J_1 J_2) A^+(J_2 J_2)] \, J \, \psi = \frac{\mathcal{A}}{3!} \left\{ \langle J^2(J_1), J^2(J_2); J J J J \rangle | J^4 J \psi \rangle 
- \langle J^2(J_1), J^2(J_2); J J J J \rangle | J^4 J \psi' \rangle \right\} \]

where
\[ \psi \langle \psi' \]

### 3.3 Elimination of the Effect of Spurious $0^+$ State From Four-Quasiparticle Basis States

Let the creation operators for the new set of $(n-1)$ nonspurious states $|\phi_1\rangle, |\phi_2\rangle, \ldots, |\phi_{n-1}\rangle$, described in the Sec. 3.1 are $A^+_1, A^+_2, \ldots, A^+_n$, which are linear combinations of the creation operators for the original set $A^+_i(d_{i0}) (i=1,2,\ldots,n)$.

Here we lay the emphasis on the fact that the spurious effects in four-quasiparticle basis arise from the same $0^+$ two-quasiparticle state due to the appearance of this original set $A^+_i(d_{i0}) (i=1,2,\ldots,n)$ as one of the two-quasiparticle state, which then coupled to the other two-quasiparticle state, gives the four-quasiparticle state. There is only one $0^+$ two-quasiparticle spurious state which has to be systematically removed from both the two-quasiparticle states and from all the four-quasiparticle states. Therefore, if a particular method of eliminating spurious effects works in the two-quasiparticle states, the same method should work for the four-quasiparticle states too.
Thus, we will be consistent in removing this spurious $0^+$ effect of the two-quasiparticle state from the four-quasiparticle basis. The recipe is, therefore, first to write the expression for spurious state and then construct the states orthogonal to this spurious state and orthogonal to each other. This procedure eliminates the effect of spurious $0^+$ state from the four-quasiparticle basis. A slightly special treatment has to be given in only one case of four-quasiparticles coupled to $0^+$. The proof of this assertion, together with the special case mentioned just now, is given below.

We have

$$N\,A^\dagger(mnJM)|0\rangle = \left\{ \sum_\alpha \left( U_{a\alpha} - V_{a\alpha} \right) a_{\alpha}^\dagger a_{\alpha} + \sum \left[ a \right] V_{a} \right\} A^\dagger(mnJM)|0\rangle + \sum_\alpha \epsilon_{\alpha} U_{\alpha} V_{\alpha} \left( a_{\alpha}^\dagger a_{\alpha}^\dagger + a_{\alpha} a_{\alpha} \right) A^\dagger(mnJM)|0\rangle.$$ 

Now

$$\sum_\alpha \left( U_{a\alpha} - V_{a\alpha} \right) a_{\alpha}^\dagger a_{\alpha} A^\dagger(mnJM)|0\rangle = \sum_\alpha \left[ a \right] \left( U_{a\alpha} - V_{a\alpha} \right) \left[ N\left(a\alpha\alpha\right), A^\dagger(mnJM)\right]|0\rangle N(mn) = \left( U_{m\alpha} - V_{m\alpha} + U_{n\alpha} - V_{n\alpha} \right) A^\dagger(mnJM)|0\rangle.$$

Thus,

$$N\,A^\dagger(mnJM)|0\rangle = \left\{ U_{m\alpha} - V_{m\alpha} + U_{n\alpha} - V_{n\alpha} + \sum \left[ a \right] V_{a} \right\} A^\dagger(mnJM)|0\rangle + \left( A^\dagger + A_0 \right) A^\dagger(mnJM)|0\rangle.$$ 

(3.24)

The first line on the right hand side of (3.24) represents a constant times the same two-quasiparticle state $A^\dagger(mnJM)|0\rangle$ that appears
on the left hand side. The operation of the number operator $N$ on this state should have given only this term, had there been a strict conservation of number in the theory. Because of the nonconservation of number, we get the second line on the right hand side of (3.24) in which $A_0^+$ corresponds to the creation of spurious state $|\Phi_0\rangle$, given in the Sec. 3.1; $A_0$ is the corresponding destruction operator. When $J \neq 0$

$$A_0^+ A^+(mnJM) |\psi\rangle = 0,$$

and hence from (3.24) the four-quasiparticle state $A_0^+ A^+(mnJM) |\psi\rangle$ with $J \neq 0$ is entirely spurious. The non-spurious states, which are orthogonal to this one and orthogonal to each other, can easily be constructed by Schmidt orthogonalisation procedure. This is in agreement with the assertion we made above. We next consider the case of $J = 0$. In this case, if we use $(n-1)$ non-spurious set $A_\beta^+ |\psi\rangle$, ($\beta = 1, 2, \ldots, (n-1)$) instead of $A^+(m^2i)$, ($m = a_i$, $i = 1, 2, \ldots, n$) then also $A_0 A_\beta^+ |\psi\rangle$ is still zero, and hence it follows from (3.24) that $A_0^+ A_\beta^+ |\psi\rangle$, ($\beta = 1, 2, \ldots, n-1$) are entirely spurious. However, the special treatment mentioned earlier, is needed in this case, the reason being the spurious states $A_0^+ A_\beta^+ |\psi\rangle$ are not orthogonal to each other. Because of this we need to construct, by suitable linear combinations, the set of states that are orthogonal to the spurious set $A_0^+ A_\beta^+ |\psi\rangle$, and orthogonal to each other. Although, this procedure needs a little algebraic manipulation, but it is quite straightforward to carry out in practice. This procedure is made clear by taking some specific example.
Let us start with the example of Ni-isotopes where we have three single particle states $2p_{3/2}$, $2p_{1/2}$, and $1f_{5/2}$. The creation operators for the nonspurious states $A_1^+$, $A_2^+$, along with the creation operator for the spurious state $A_0^+$ are given as (all operators are unnormalised):

$$\begin{align*}
A_0^+ &= c_1 A_t^+(5/2^0) + c_2 A_t^+(3/2^0) + c_3 A_t^+(1/2^0), \\
A_1^+ &= (c_1^2 + c_2^2) A_t^+(5/2^0) - c_1 \{ c_2 A_t^+(3/2^0) + c_3 A_t^+(1/2^0) \}, \\
A_2^+ &= c_3 A_t^+(3/2^0) - c_2 A_t^+(1/2^0).
\end{align*}$$

(1) Case I: $J \neq 0$

Let us consider for $A_t^+(mnJM)$, the state $A_t^+(5/2^2)$. In all, there will be three four-quasiparticle states

1. $[A_t^+(5/2^0) A_t^+(5/2^2)] J=2 \Rightarrow |5/2, J=2; \nu=2\rangle$

by multiplying with a suitable constant.

2. $[A_t^+(3/2^0) A_t^+(5/2^2)] J=2 \Rightarrow |3/2^0, 5/2^2; J=2\rangle$

3. $[A_t^+(1/2^0) A_t^+(5/2^2)] J=2 \Rightarrow |1/2^0, 5/2^2; J=2\rangle$.

The four-quasiparticle spurious state (unnormalised) is given by
We want to construct two nonspurious states (normalized) $|\Psi_1\rangle$ and $|\Psi_2\rangle$, which are orthogonal to $|\Psi_0\rangle$ and orthogonal to each other. These are given by

$$|\Psi_1\rangle = \frac{1}{\sqrt{\left(\frac{1}{3}c_1^2 + c_2^2 + c_3^2\right)\left(c_2^3 + c_3^3\right)}} \left\{ \left(\begin{array}{c} c_2 \left| 3/2, 5/2 ; J=2 \right> + c_3 \left| 1/2, 5/2 ; J=2 \right> \\
\end{array}\right) + c_1 \left| 5/2, 2 ; J=2 \right> \right\}$$

and

$$|\Psi_2\rangle = \frac{1}{\sqrt{\left(\frac{1}{3}c_1^2 + c_2^2 + c_3^2\right)\left(c_2^3 + c_3^3\right)}} \left[ c_2 \left| 3/2, 5/2 ; J=2 \right> - c_3 \left| 1/2, 5/2 ; J=2 \right> \right].$$

It is to be noticed that the counting of the number of states in the space of $|\Omega_{IV}\rangle$ is not altered; corresponding to the one state $|5/2, 2 \rangle$ we still get one state with ofcourse a little bit of subtraction of another states of the type $|a_0^2, a_1^2 ; J=2\rangle$ but the two states in the space of $|a_0^2, a_1^2 ; J=2\rangle$ have been reduced by one.

Next let us demonstrate the case of $\mathcal{A}^{(5\frac{1}{2}; J=2, J=2)}$. In this case there will be three four-quasiparticle states:

$$(\text{c}) \left[ \mathcal{A}^{(5\frac{1}{2})} \left(5\frac{1}{2}; J=2\right) \right] \left|0\right> \text{can be related to} \left|5\frac{1}{2}; 5\frac{1}{2}, 3/2 ; J=2\right>$$

by multiplying with a suitable constant.
(11) \[ \mathcal{A}^{(3/2a)} \mathcal{A}^{(5/23/2)} \] \( J=2 \) can be related to \( |3/2_{3/2}, 5/2_{3/2}; J=2 \rangle \) by multiplying with a suitable constant.

(111) \[ \mathcal{A}^{(1/2a)} \mathcal{A}^{(5/23/2)} \] \( J=2 \) \( \Rightarrow \) \( |1/2_{3/2}, 5/2_{3/2}; J=2 \rangle \)

The four-quasiparticle spurious state (unnormalised) is

\[ |\Psi_0\rangle = A_0 \mathcal{A}^{(5/23/2)} \langle 1/2; 3/2, 5/2; J=2 \rangle = \frac{1}{\sqrt{2}} c_1 \left\{ \begin{array}{l} \langle 3/2; 3/2, 5/2, 5/2; J=5/2, \nu=1 \rangle \\ \times |5/2_{3/2}, 3/2; J=2 \rangle - \frac{1}{\sqrt{2}} c_2 \left\{ \begin{array}{l} \langle 3/2; 3/2, 5/2, 5/2; J=3/2, \nu=1 \rangle \\ \times |3/2_{3/2}, 5/2; J=2 \rangle + c_3 |1/2_{3/2}, 5/2_{3/2}; J=2 \rangle + c_3 |1/2_{0}, 5/2_{3/2}; J=2 \rangle \end{array} \right. \right\} \]

The nonspurious states (normalised) \( |\Psi_1\rangle \) and \( |\Psi_2\rangle \), orthogonal to \( |\Psi_0\rangle \) and orthogonal to each other can be written down as

\[ |\Psi_1\rangle = \frac{1}{\sqrt{\left\{ \left( \frac{3}{2} c_1^2 + \frac{1}{2} c_2^2 + c_3^2 \right) \left( \frac{3}{2} c_1^2 + c_2^2 + c_3^2 \right) \right\} \sqrt{2}} \left[ \left( \frac{1}{\sqrt{2}} c_2 c_1 \right) |3/2_{3/2}, 3/2, 5/2; J=2 \rangle + c_3 |1/2_{0}, 5/2_{3/2}; J=2 \rangle \right] \]

and

\[ |\Psi_2\rangle = \frac{1}{\sqrt{\left( c_3^2 + \frac{1}{2} c_2^2 \right)}} \left[ c_3 |3/2_{3/2}, 5/2; J=2 \rangle + \frac{1}{\sqrt{2}} c_2 |1/2_{0}, 5/2_{3/2}; J=2 \rangle \right] \]

These new states have the structure of the type \( |\alpha_1 J, \nu_1, \alpha_2 J \rangle \) with a suitable linear combination of the others. Thus once again, the number of the states of the type \( |\alpha_1 J, \nu_1, \alpha_2 J \rangle \) remains the same, i.e.
two (the states, however, receive a little bit of modification of the other states), but the number of the states of the type \(|a_1^2 a_3 J_3; J\rangle\) is reduced by one (from one to zero).

(2) Case II \(J = 0\)

In this case there will be two spurious states

(1) \(|\tilde{\Psi}_1\rangle \equiv A_0^+ A_1^+ |0\rangle\) and (2) \(|\tilde{\Psi}_2\rangle \equiv A_0^+ A_2^+ |0\rangle\).

First of all these spurious states should be expressed as a linear combinations of four-quasiparticles states coupled to the total \(J=0\), i.e., \(|5/2 J=0,+0\rangle; |3/2 J=0,+0\rangle; |5/2 J=0,+0\rangle; |5/2 J=0,+0\rangle\), etc. Let us write

\(|\tilde{\Psi}_1\rangle \equiv a_5 |5/2 J=0,+0\rangle + a_4 |3/2 J=0,+0\rangle + a_3 |5/2 J=0,+0\rangle + a_4 |5/2 J=0,+0\rangle + a_3 |3/2 J=0,+0\rangle + a_5 |5/2 J=0,+0\rangle + a_4 |3/2 J=0,+0\rangle + a_3 |5/2 J=0,+0\rangle + a_4 |5/2 J=0,+0\rangle + a_3 |3/2 J=0,+0\rangle

and

\(|\tilde{\Psi}_2\rangle \equiv b_4 |3/2 J=0,+0\rangle + b_3 |5/2 J=0,+0\rangle + b_2 |5/2 J=0,+0\rangle + b_1 |3/2 J=0,+0\rangle + b_4 |5/2 J=0,+0\rangle + b_3 |3/2 J=0,+0\rangle + b_2 |5/2 J=0,+0\rangle + b_1 |3/2 J=0,+0\rangle + b_4 |5/2 J=0,+0\rangle + b_3 |3/2 J=0,+0\rangle

where the coefficients \(a's\) and \(b's\) can be found by writing the expression for \(A_0^+ A_1^+\) and \(A_0^+ A_2^+\) and are given below:

\[
\begin{align*}
a_5 &= c_1 (c_2^3 + c_3^3) \times \frac{1}{3} \{ \langle 5/2^4 (0); 5/2^4 (0) ; 0 \rangle | 5/2^4 J=0,+0\rangle \} = -\frac{2}{13} c_1 (c_2^3 + c_3^3), \\
a_4 &= -c_1 c_2^2 \times \frac{1}{4} \{ \langle 3/2^4 (0); 3/2^4 (0) ; 0 \rangle | 3/2^4 J=0,+0\rangle \} = -c_1 c_2^3, \\
a_3 &= c_2 \left( c_2^2 + c_3^2 - c_1^2 \right), \\
a_2 &= c_3 (c_2^2 + c_3^2 - c_1^2), \\
a_1 &= -c_1 c_2 c_3,
\end{align*}
\]
and
\[ b_4 = c_3 c_3 \times \frac{1}{2} \left\{ 6! \right\} \langle \frac{3}{2}, \frac{3}{2} | 0 \rangle \langle \frac{3}{2}, \frac{3}{2} | 0 \rangle \langle c_1 c_3 \rangle = c_2 c_3 \]
\[ b_3 = c_1 c_3 , \quad b_2 = -c_1 c_1 , \quad b_1 = -c_2 + c_3 . \]

We now want to construct three nonspurious states \(| \Phi_1 \rangle, | \Phi_2 \rangle \) and \(| \Phi_3 \rangle \) which should be orthogonal to \(| \tilde{\Psi}_1 \rangle \) and \(| \tilde{\Psi}_2 \rangle \) and orthogonal to each other. Let

\( | \Phi_1 \rangle \equiv \frac{1}{5/2, 1/2, 1/2} | 0 \rangle + c_1 \langle 5/2, 1/2, 1/2 | 0 \rangle + c_2 \langle 5/2, 3/2, 3/2 | 0 \rangle + c_3 \langle 5/2, 3/2, 3/2 | 0 \rangle . \)

The coefficients \( d_2 \) and \( d_3 \) are determined from:
\[ \langle \tilde{\Psi}_1 | \Phi_1 \rangle = a_1 + a_3 d_1 + a_3 d_3 \]
\[ \langle \tilde{\Psi}_2 | \Phi_1 \rangle = b_1 + b_2 d_2 + b_3 d_3 . \]

These equations give
\[ d_3 = \frac{b_1 a_2 - a_1 b_2}{b_2 a_3 - a_1 b_3} \quad \text{and} \quad d_2 = -\frac{a_1 + a_3 d_3}{a_2} . \]

\( | \Phi_2 \rangle \equiv \frac{1}{3/2, 1/2, 1/2} | 0 \rangle + e_2 \langle 3/2, 1/2, 1/2 | 0 \rangle + e_3 \langle 3/2, 3/2, 3/2 | 0 \rangle + e_4 \langle 3/2, 3/2, 3/2 | 0 \rangle . \)

The coefficients \( e_2 \), \( e_3 \), and \( e_4 \) are determined from
\[ \langle \tilde{\Psi}_1 | \Phi_2 \rangle = a_1 + a_2 e_2 + a_3 e_3 + a_4 e_4 \]
\[ \langle \tilde{\Psi}_2 | \Phi_2 \rangle = b_1 + b_2 e_2 + b_3 e_3 + b_4 e_4 \]
\[ \langle \Phi_1 | \Phi_2 \rangle = c_1 + c_2 e_2 + c_3 e_3 . \]

These equations give the values of \( e_2 \), \( e_3 \), and \( e_4 \) as
\[ e_3 = \frac{d_2(b_1a_4-a_1b_4)+(a_2b_4-b_2a_4)}{d_2(a_3b_4-b_3a_4)-d_3(a_2b_4-b_2a_4)}, \quad e_2 = -\frac{1+d_3e_3}{d_2}, \quad e_4 = -(a_1+a_2e_2+a_3e_3)/a_4 \]

and

\[
\Phi_3 \equiv |3/2^o, 1/2^o; J=0> + f_2 |5/2^o, 1/2^o; J=0> + f_3 |5/2^o, 3/2^o; J=0> + f_4 |3/2^o, J=0 \nu=0> + f_5 |5/2^o, J=0 \nu=0>
\]

once again, these coefficients \( f_2, f_3, f_4, \) and \( f_5 \)

will be determined from

\[
\langle \Phi'_1 | \Phi_3 \rangle = 0 = a_1 + f_2a_2 + f_3a_3 + f_4a_4 + f_5a_5
\]

\[
\langle \Phi'_2 | \Phi_3 \rangle = 0 = b_1 + f_2b_2 + f_3b_3 + f_4b_4
\]

\[
\langle \Phi'_1 | \Phi_3 \rangle = 0 = 1 + f_2d_2 + f_3d_3
\]

\[
\langle \Phi'_2 | \Phi_3 \rangle = 0 = 1 + f_2e_2 + f_3e_3 + f_4e_4
\]

The coefficients \( f's \) are found from above equations as

\[
f_2 = \frac{(b_3 e_4-b_4 e_3)-d_3(b_1 e_4-b_2 e_4)}{d_3(b_2 e_4-b_4 e_2)-d_2(b_3 e_4-b_4 e_3)}, \quad f_3 = -\frac{1+d_3f_2}{d_3}
\]

\[
f_4 = -\frac{1+e_2f_2+e_3f_3}{e_4}, \quad f_5 = -(a_1+a_2f_2+a_3f_3+a_4f_4)/a_5
\]

This completes the determination of \( |\Phi_1>, |\Phi_2>, \) and \( |\Phi_3> \) (all are unnormalised).

In all our calculations, described in this report, the effect of the spurious \( 0^+ \) state has thus been completely removed from the two- (see Sec. 3.1) and four-quasiparticle basis states.