C H A P T E R - 4

Long range correlations and co-operativity : The Lipkin model and coherent state

1. The Lipkin Model :

The nature of co-operativity in nuclei, arising from long range correlations has been investigated in a simple model viz. the Lipkin Model[1]. It is an exactly soluble model of a system of N fermions occupying two levels (each possessing an N-fold degeneracy) separated by an energy spacing ϵ. Let $a_{p,\sigma}$ be the annihilation operator for a particle in the state labelled by the quantum number p (enumerating the substates 1, ... N in each level or 'shell') and σ(adopting values $\pm$ 1) is the dichotomic level index. The Hamiltonian of the system is written as -

$$H = (\epsilon/2) \sum_{p,\sigma} \sigma\, a^{+}_{p,\sigma} a_{p,\sigma} + (V/2) \sum_{\substack{p,p' \\ \sigma}} a^{+}_{p,\sigma} a^{+}_{p',\sigma} a_{p',-\sigma} a_{p,-\sigma} \qquad \ldots (1)$$

where a two-body 'monopole-monopole' interaction (of strength V), scattering a pair of particles from one 'shell' to the other, without changing the 'subshell' quantum number p, is introduced. In terms of 'quasi-spin' operators

$$J_{\pm} = \sum_{p} a^{+}_{p,\pm 1} a_{p,\mp 1} \qquad \ldots (2a)$$

$$J_z = 1/2 \sum_{p,\sigma} \sigma\, a^{+}_{p,\sigma} a_{p,\sigma} , \qquad \ldots (2b)$$

satisfying angular momentum commutation relations, the Hamiltonian may be cast into a particularly elegant form, to wit,

$$H = \epsilon J_z + \frac{1}{2} V (J_+^2 + J_-^2) \qquad \ldots (3)$$

It may be remarked that, m, the eigenvalue of J_z is simply half the difference between the number of particles in the upper and the lower states, and consequently its maximum value, namely j, equals N/2. It is clear that the unperturbed (V=0) ground state is $|j = N/2, m = -N/2\rangle$ possessing the unperturbed energy $E_0 = -\epsilon N/2$. The interaction mixes states within the same j - multiplet corresponding to different numbers of 'hole-particle' pairs.

The Hamiltonian exhibits the following symmetries :-

(a) invariance under a rotation of π about the Z-axis in quasi-spin space.

(b) $H \rightarrow -H$ under a rotation of π about an axis lying in the XY-plane making an angle of $\pi/4$ with respect to the X and Y axes.

2. The Bloch States :

The familiar angular momentum states $|jm\rangle$, with eigenvalues j(j+1) and m for the operators J^2 and J_z, are superposed to construct a convenient basis, variously called coherent atomic states, Bloch or Radcliffe states[2)], thus

$$|\alpha, j\rangle = N_0 \exp(\alpha J_+) \, | j, -j\rangle$$

$$= (1+|\alpha|^2)^{-j} \alpha^{j} \sum_{m=-j}^{+j} \alpha^m \binom{2j}{j+m}^{1/2} |j, m\rangle, \quad \ldots(4a)$$

where α is, in general, a complex parameter specifying the state, N_0 the appropriate normalisation factor and $\binom{2j}{j+m}$

the binomial combinatorial. This basis has been extensively used[3] in various areas of Quantum Optics such as laser theory, super-radiance and resonance propagation.

It is to be noted that the basis is not orthogonal i.e.,

$$\langle \beta, j \mid \alpha, j \rangle = \frac{(1+\beta^*\alpha)^{2j}}{(1+|\alpha|^2)^j\,(1+|\beta|^2)^j} \qquad \ldots (4b)$$

The following relations can easily by verified :

$$\langle \beta, j \mid J_z \mid \alpha, j \rangle = \frac{(1+\beta^*\alpha)^{2j}\; j\,(\beta^*\alpha - 1)}{(1+|\alpha|^2)^j\,(1+|\beta|^2)^j\,(1+\beta^*\alpha)} \qquad \ldots (4c)$$

$$\langle \beta, j \mid J_+ + J_- \mid \alpha, j \rangle = \frac{(1+\beta^*\alpha)^{2j}}{(1+|\alpha|^2)^j\,(1+|\beta|^2)^j} \frac{\times\; 2j\,(\alpha+\beta^*)}{(1+\beta^*\alpha)} \qquad \ldots (4d)$$

$$\langle \beta, j \mid J_+^2 + J_-^2 \mid \alpha, j \rangle = \frac{(1+\beta^*\alpha)^{2j}}{(1+|\alpha|^2)^j\,(1+|\beta|^2)^j} \frac{\times\; 2j\,(2j-1)\,(\alpha^2+\beta^{*2})}{(1+\beta^*\alpha)^2} \qquad \ldots (4e)$$

$$\langle \beta, j \mid J_+J_- + J_-J_+ \mid \alpha, j \rangle = \frac{(1+\beta^*\alpha)^{2j}}{(1+|\alpha|^2)^j\,(1+|\beta|^2)^j} \frac{\times\; 2j\,[1+4j\,\beta^*\alpha + (\beta^*\alpha)^2]}{(1+\beta^*\alpha)^2} \qquad \ldots (4f)$$

3. The Lipkin Model and Bloch States :

With respect to Lipkin Model, Bloch states serve as a correlated basis function :-

$$\langle \alpha, j | = \sum_{m=-j}^{+j} C_m \, | j, m \rangle \qquad \ldots \quad (5)$$

Here the co-efficients C_m are to be found out from a minimum of energy expectation value. Employing the Bloch state as basis and using equations (4c) and (4e) the energy expectation value of the Lipkin Hamiltonian becomes

$$\langle \alpha, j | H | \alpha, j \rangle = -\epsilon j \left(1-|\alpha|^2\right)/\left(1+|\alpha|^2\right) + V j (2j-1) \left(\alpha^2+\alpha^{*2}\right)/\left(1+|\alpha|^2\right) \qquad \ldots \quad (6a)$$

and introducing $\rho \equiv V(2j-1)/\epsilon$, minimisation with respect to α yields the solutions

$$\alpha^2 = (\rho+1)/(\rho-1) \quad \text{for } \rho < -1, \qquad \ldots \quad (6b)$$

$$\alpha^2 \equiv -\alpha_m^2 = -(\rho-1)/(\rho+1) \text{ for } \rho > +1, \qquad \ldots \quad (6c)$$

and $$\alpha^2 = 0 \qquad \text{for } -1 < \rho < +1 \qquad \ldots \quad (6d)$$

Regarding ρ, which is proportional to the interaction strength and the number of particles, as a control parameter, it is thus seen that for $\rho > +1$ (as also, mutatis mutandis for $\rho < -1$) we have arrived at a minimum, which is immediately recognised to be the 'deformed' Hartree-Fock ground state as obtained by Agassi[4], possessing the energy

$$E_0 = -(N\epsilon/2)\cos\chi_m - [N(N-1)/4]\, V \sin^2\chi_m \qquad \ldots (7a)$$

through the identification

$$\cos\chi_m = (1-\alpha_m^2)/(1+\alpha_m^2) = \epsilon/[V(N-1)] \qquad \ldots (7b)$$

The corresponding ground state is best expressed as -

$$|i\alpha_m, j\rangle = \prod_{p=1}^{N} b_p^{+} \,|\,\text{Vacuum}\rangle \qquad \ldots (8)$$

where b_p is a 'rotated' single particle operator defined by -

$$b_p = [\cos(\chi_m/2)]\, a_{p,-1} - i[\sin(\chi_m/2)]\, a_{p,+1} \qquad \ldots (9)$$

The 'deformed' Hartree-Fock description obtains for potential strengths greater than a certain critical value given by $V_c = \epsilon/(N-1)$, while (within the realm of Bloch states for $|V| < V_c$ it is the unperturbed ground state ($\alpha = 0$) which is stabilised, and then the situation is in fact better described by the Random Phase Approximation (RPA). The different branches of the roots of the minimisation condition realised in various ranges of the control parameter ρ is depicted through the bifurcation diagram shown in Fig. 1.

Confining our attention to the branch corresponding to $\rho > 1$ wherein $\alpha^2 = -\alpha_m^2$, it may be observed that the states $|i\alpha_m, j\rangle$ and $|-i\alpha_m, j\rangle$ are degenerate. The minimum at $\alpha = 0$ for the region $|\rho| < 1$ splits into these two minima much as what occurs in a typical broken symmetry situation, for example, in ϕ^4 field theory as the co-efficient of quadratic 'mass' term changes sign. The Lipkin Hamiltonian through the interaction term mixes only even number of particle-hole pairs, as a consequence of the symmetry of the Hamiltonian. Recently Kümmel[5] has shown in a different approach that the new, rotated wavefunction like equation (8) is better in the sense that it has maximum overlap with the exact wave function.

The states $|i\alpha_m, j\rangle$ and $|-i\alpha_m, j\rangle$ contain arbitrary numbers (even as well as odd) of such pairs, and does not possess the symmetry enjoyed by the Hamiltonian, and the corresponding symmetry is broken. However the gerade (or symmetric) combination of states

$$|\alpha, j\,; g\rangle \equiv N_+[\,|\alpha, j\rangle + (-)^{2j}\,|-\alpha, j\rangle\,] \quad \ldots \quad (10)$$

has a lower energy and represents a better candidate for the ground state. N_+ is the normalisation constant. The expectation value of the Hamiltonian is

$$\langle \alpha, j; g| H |\alpha, j, g\rangle = \langle \alpha, j | H | \alpha, j\rangle \left[(1 + \cos^{2j-2}\chi)/(1 + \cos^{2j}\chi) \right] \quad \ldots (11)$$

where $\cos\chi$ is defined as $(1 - \alpha^2)/(1 + \alpha^2)$. Inserting for χ the value χ_m given by eqn. (7b), the resulting expression for the ground state energy is identical to the one obtained by Agassi[4] and called the 'projected' Hartree-Fock result. The corresponding state differs from the 'unprojected' wave function through the absence of admixtures of odd numbers of particle-hole pairs. For large values of $|\alpha|$, however, $\sin\chi \to 0$ and the projected and unprojected versions become identical[6].

Just as the ground state was obtained as the symmetric combination of Bloch states, the present approach admits of a simple description of the first excited state, through the orthogonal antisymmetric (or ungerade) combination,

$$|\alpha, j; u\rangle = N_{-} \left[|\alpha, j\rangle - (-)^{2j} |-\alpha, j\rangle \right] \quad \ldots (12)$$

which has odd numbers of hole-particle pairs unlike the gerade (or ground) state which has even numbers of the same. The nature of correlations present in the projected ground and excited states, manifested through the basis chosen, is further revealed by the study of the transition probability between these states. Within the confines of this model the simplest 'transitions' are caused by what may be called the 'monopole' transition operators J_x and J_y, so named because they do not change the p-quantum number. The relevant transition matrix elements are readily calculated to yield

$$\langle \alpha, j; g | J_x | \alpha, j; u \rangle = j \sin\chi \left[1 - (\cos\chi)^{4j} \right]^{-1/2} \quad \text{... (13a)}$$

$$\langle \alpha, j; g | J_y | \alpha, j; u \rangle = 0 \quad \text{... (13b)}$$

for the case where α is real. This, and other results, for real α may readily be extended to complex α through the observation

$$\exp(i\theta J_z) \, |\alpha, j\rangle = \exp(i\theta j) \, |\alpha \exp(i\theta), j\rangle \quad \text{... (14)}$$

following from the definition, eqn. (4), of Bloch states, where the operation involved is a rotation by an angle θ about the Z-axis in quasi-spin space. In the limit of large N , as also in the strong-coupling limit (V large), it may be seen from eqn. (7b), that $\cos\chi \to 0$ and the transition matrix element becomes proportional to N. The transition probability is thus proportional to the square of the number of particles, in sharp contrast to what occurs in the Random Phase Approximation where the transition probability is proportional to the number of particles. Indeed, a strong parallelism exists between what has been obtained here and what occurs in the theory of super-radiance in atomic physics[7], wherein a collection of N atoms initially prepared in some excited state under suitable conditions return to their ground states by emitting electromagnetic radiation, of intensity I proportional to the square of the number of atoms, in a super-radiant pulse, in contrast to the normal situation where $I \sim N$. This phenomenon has been extensively discussed (in the atomic context) in terms of Bloch states.

It is instructive to consider an extended version of the Lipkin model where the Hamiltonian is

$$H = \epsilon J_z + (V/2)(J_+^2 + J_-^2) + (W/2)(J_+J_- + J_-J_+) \quad \dots (15)$$

Minimisation of the expectation value of this Hamiltonian in the Bloch state yields, for the parameter α, the values

$$\alpha^2 = 0 \;,\; (\rho+\gamma+1)/(\rho+\gamma-1) \;,\; -(\rho-\gamma-1)/(\rho-\gamma+1) \quad \dots (16)$$

for control parameters satisfying $|\rho+\gamma| < 1$, $(\rho+\gamma) < -1$ and $(\rho-\gamma) > 1$ respectively, where $\gamma \equiv (2j-1)W/\epsilon$ is defined analogous to ρ. A perusal of the minimisation condition reveals that in the absence of the V terms ($\rho = 0$) there exists a degeneracy with respect to the phase of α which is a consequence of the invariance of the Hamiltonian (when

V = 0) with respect to arbitrary rotations in the quasi-spin space about the Z - axis. This symmetry is broken by the very presence of V and the phase of α can then only adopt the values 0 and $\pm \pi/2$.

4. Conclusion :

Thus it is shown that the Bloch or angular momentum 'coherent' states furnish a useful basis for the discussion of the Lipkin model of the nucleus. The Hartree-Fock and the projected Hartree-Fock results have been readily obtained in this framework. Contact with the phenomena of super-radiance in atomic physics has been made. While the results so far are confined to a model system it is expected that this work should provide guidelines for application to more realistic models of the nucleus[8]. However, the large transition matrix element (proportional to N) stemming from co-operativity may be[9] over-emphasized in this model since there are only two single particle states for each particle, and each particle interacts equally with every other particle. Nevertheless, the change from the spherical to the deformed Hartree state wave function as the control parameter passes the bifurcation point is reminiscent of the sudden onset of deformation as one passes through the critical value 82 of the neutron number[5].

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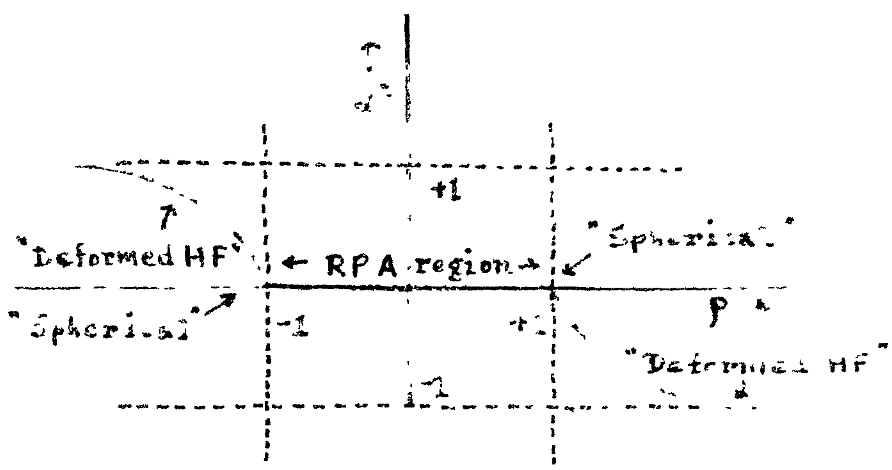

FIG.1.

BIFURCATION DIAGRAM SHOWING STABLE SCAN HF.