C H A P T E R - 3

Long range correlations and co-operativity : Single-particle Schrödinger fluid

1. Long range order : Single particle Schrödinger fluid.

Apart from short ranged nucleon-nucleon correlations in the nuclear many body system there are models which involve the nucleons in collective and co-operative excitations for which ample experimental evidences exist. Notably the disposition of the vibrational and rotational energy levels of nuclei had led to Bohr's collective model of the nucleus. Furthermore, the EM transition probabilities between nuclear levels often show substantial enhancements over the single particle estimates which leads to the necessity of invoking correlated motion of many nucleons. With a view to gaining some experience and insight into the nature of such correlations, investigations have been carried out by the author into the relationship between the Schrödinger equation with hydrodynamics.

2. The 'Euler equation' in the coherent state basis :

Consider a particle moving in an oscillator potential ($\frac{1}{2}m\omega^2x^2$) described by the Hamiltonian $H = (p^2/2m + \frac{1}{2}m\omega^2x^2)$ where x and p are the coordinate and momentum operators respectively, and m is the mass of the particle and ω the classical oscillator frequency. Through the introduction of the operator, $a = (p - im\omega x)/(2m\omega\hbar)^{1/2}$ the Hamiltonian may, as usual, be written as $H = \hbar\omega(a^+a + \frac{1}{2})$. The state of the particle may be expressed in terms of the basic eigenstates $|n\rangle$ of the number operator a^+a possessing the energy eigenvalue $\hbar\omega(n+\frac{1}{2})$. We shall work in the coherent state basis [1]

$$|\alpha\rangle \equiv \exp\left(-\tfrac{1}{2}|\alpha|^2\right) \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \qquad \ldots \qquad (1)$$

where the complex number α is the eigenvalue of the annihilation operator a. The general (mixed) state of a single particle in an oscillator potential may be described by the density matrix, ρ, in this basis[2,3]. In terms of a weight function $P(\alpha)$ the density matrix is

$$\rho = \int d^2\alpha \, P(\alpha, t) \, |\alpha\rangle\langle\alpha| \qquad \ldots \qquad (2)$$

Given ρ as a function of a and a^+, the weight function P is readily obtained by writing ρ in the anti-normal ordered form $\rho^{(a)}(a, a^+)$, in which by inserting the complete set of coherent states the weight function is of the form $\rho^{(a)}(\alpha, \alpha^*)$. In the Heisenberg equation of motion for the density operator it becomes necessary to consider the commutators of the density operator with the annihilation and creation operators. For this purpose it is useful to note

$$a\rho = \int d^2\alpha \, |\alpha\rangle\langle\alpha| \, \alpha \, P(\alpha) \qquad \ldots \qquad (3a)$$

$$a^+\rho = \int d^2\alpha \, |\alpha\rangle\langle\alpha| \left(\alpha^* - \frac{\partial}{\partial\alpha}\right) P(\alpha) \qquad \ldots \qquad (3b)$$

$$\rho a = \int d^2\alpha \, |\alpha\rangle\langle\alpha| \left(\alpha - \frac{\partial}{\partial\alpha^*}\right) P(\alpha) \qquad \ldots \qquad (3c)$$

$$\rho a^+ = \int d^2\alpha \, |\alpha\rangle\langle\alpha| \, \alpha^* \, P(\alpha) \qquad \ldots \qquad (3d)$$

The Heisenberg equation of motion for the density operator may thus be readily translated into the P representation

$$\frac{\partial}{\partial t} P(\alpha, t) = - i\omega \left(\alpha^* \frac{\partial}{\partial \alpha^*} - \alpha \frac{\partial}{\partial \alpha}\right) P(\alpha, t) \qquad \ldots \qquad (4)$$

To make the passage to the hydrodynamical interpretation it may be recalled that

$$\langle \alpha | x | \alpha \rangle = - 2 \left(\frac{\hbar}{2m\omega} \right)^{1/2} \mathrm{Im}\, \alpha \qquad \ldots \qquad (5a)$$

$$\langle \alpha | p/m | \alpha \rangle = 2 \left(\frac{\hbar\omega}{2m} \right)^{1/2} \mathrm{Re}\, \alpha \qquad \ldots \qquad (5b)$$

The classical limit is effected by taking the limit $\hbar \to 0$, $|\alpha| \to \infty : \sqrt{\hbar}|\alpha| \to$ finite, whereby quantal correlations vanish and the imaginary and real parts of the complex variable α become proportional to the classical position coordinate(x) and the velocity (V) respectively. In the classical limit $P(\alpha, t) \to f(x, v, t)$, giving the distribution in phase space. The Heisenberg equation of motion, equation (4), becomes, in this limit,

$$\frac{\partial}{\partial t} f(x,v,t) = \left(\omega^2 x \frac{\partial}{\partial v} - v \frac{\partial}{\partial x}\right) f(x,v,t). \qquad \ldots \qquad (6)$$

Defining the mass density

$$\rho(x,t) \equiv m \int f(x,v,t)\, dv \qquad \ldots \qquad (7a)$$

and the 'local hydrodynamic velocity' $v^{(h)}$ through

$$v^{(h)}(x,t)\ \rho(x,t) \equiv m \int f(x,v,t)\ v\ dv, \qquad \ldots \quad (7b)$$

equation (6) may be integrated over velocity (v) space (with the distribution f vanishing at the boundaries) to give -

$$\frac{\partial}{\partial t}\rho(x,t) = -\frac{\partial}{\partial x}(\rho(x,t)v^{(h)}(x,t)). \qquad \ldots \quad (8)$$

This equation is easily generalized to three-dimensions to read

$$\frac{\partial}{\partial t}\rho + \nabla.\ (\rho v^{(h)}) = 0, \qquad \ldots \quad (9)$$

representing the equation of continuity. Proceeding in a similar manner, the equation of momentum conservation is readily derived :

$$\frac{\partial}{\partial t}(\rho v_i^{(h)}) = -\omega^2 \rho x_i - \frac{\partial}{\partial x_j}(\rho v_i^{(h)}\ v_j^{(h)}) \quad .. \ (10)$$

Combining the equations of continuity and momentum conservation we arrive at the Euler equation

$$\frac{\partial}{\partial t}v^{(h)} + (v^{(h)}\ .\nabla)\ v^{(h)} = -\omega^2 x, \qquad \ldots \quad (11)$$

where the term on the right-hand side represents the force (of the oscillator) on the fluid. Thus we have shown that the weight function of the density operator expressed in the coherent state basis admits in the classical limit an interpretation in terms of a 'single-particle Schrödinger fluid', and we have obtained the corresponding continuity and Euler equations.

3. The moment of inertia and the 'fluid'

One of the important collective parameters of the nucleus is its moment of inertia. Prescriptions have been given[4)] to calculate this measure of the response of the system to rotations. In various models[5)] the value of the moment of inertia turns out to be that of a rigid body. This result may also be derived in the present approach.

Consider an anisotropic harmonic oscillator described by the Hamiltonian

$$H_0 = \sum_{i=1}^{3} \left(\frac{p_i^2}{2m} + \tfrac{1}{2} m \omega_i^2 x_i^2 \right) = \sum_i \hbar \omega_i (a_i^+ a_i + \tfrac{1}{2}) \quad .. \quad (12)$$

In a rotating system (rotating about the first axis with angular velocity Ω) the Hamiltonian may be written as

$$H = H_0 + H_1 = H_0 + \frac{i\Omega\hbar}{2} \left[\left(\frac{\omega_2}{\omega_3} \right)^{1/2} (a_2 + a_2^+)(a_3 - a_3^+) - \left(\frac{\omega_3}{\omega_2} \right)^{1/2} (a_3 + a_3^+) (a_2 - a_2^+) \right] \quad ... \quad (13)$$

where the centrifugal term (quadratic in Ω) has been omitted since we are interested in the linear response. Introducing the three-mode coherent state $|\alpha_1, \alpha_2, \alpha_3\rangle$, eigenstates of

a_1, a_2, a_3 with eigenvalues $\alpha_1, \alpha_2, \alpha_3$ respectively, the density matrix $\rho = \rho_0 + \rho_1$ (with ρ_0 the density matrix of the non-rotating system), we may write down the Heisenberg equation of motion for the density operator in the P representation. To study the linear response it suffices to retain terms up to first order, and using the techniques expressed through equations (3), we have

$$\frac{\partial}{\partial t} P_1 = i\omega_k \left(\alpha_k \frac{\partial}{\partial \alpha_k} - \alpha_k^* \frac{\partial}{\partial \alpha_k^*}\right) P_1$$

$$+ \Omega\, \epsilon_{ijk} \left(\frac{\omega_j}{\omega_k}\right)^{1/2} \left(\alpha_k \frac{\partial}{\partial \alpha_j} + \alpha_k^* \frac{\partial}{\partial \alpha_j^*}\right) P_0 \qquad \ldots (14)$$

where repeated indices are to be deemed summed over. Expressing the complex numbers α_k in the phase representation $-i|\alpha_k| \exp(i\phi_k)$ we may cast equation (14) into the form

$$\left(\frac{\partial}{\partial t} - \omega_i \frac{\partial}{\partial \phi_i}\right) P_1 = \epsilon_{ijk}\, \Omega \left(\frac{\omega_k}{\omega_j}\right)^{1/2} \Big(\cos\phi_j \cos\phi_k |\alpha_j| \frac{\partial}{\partial |\alpha_k|}$$

$$- \sin\phi_j \sin\phi_k |\alpha_k| \frac{\partial}{\partial |\alpha_j|} \Big) P_0 \qquad \ldots (15)$$

This may be solved by the method of characteristics to yield the solution :

$$P_1 = \frac{\Omega}{2}\left[\frac{(\omega_2-\omega_3)}{(\omega_2+\omega_3)} \frac{\sin(\phi_2+\phi_3)}{(\omega_2\omega_3)^{1/2}} \left(|\alpha_2| \frac{\partial}{\partial|\alpha_3|} + |\alpha_3| \frac{\partial}{\partial|\alpha_2|}\right) \right.$$

$$\left. - \frac{(\omega_2+\omega_3)}{(\omega_2-\omega_3)} \frac{\sin(\phi_2-\phi_3)}{(\omega_2\omega_3)^{1/2}} \left(|\alpha_2| \frac{\partial}{\partial|\alpha_3|} - |\alpha_3| \frac{\partial}{\partial|\alpha_2|}\right)\right] P_0 \qquad \ldots. \; (16)$$

The first component of the angular momentum (L_1), which in a pure coherent state has the expectation value

$$\langle \alpha_1,\alpha_2,\alpha_3 | L_1 | \alpha_1,\alpha_2,\alpha_3\rangle = \frac{i\hbar}{2} \epsilon_{ijk} \left(\frac{\omega_k}{\omega_j}\right)^{1/2} (\alpha_j - \alpha_j^*)(\alpha_k + \alpha_k^*) \ldots \qquad (17)$$

will possess for the mixed state, described by the density matrix ρ, the average value (retaining terms up to order Ω)

$$\langle \bar{L}_1 \rangle = \tfrac{1}{2}\hbar\Omega \int \left(\frac{(\omega_2-\omega_3)^2 \left(|\alpha_2|^2 + |\alpha_3|^2\right)}{\omega_2\omega_3\ (\omega_2+\omega_3)} + \frac{(\omega_2+\omega_3)^2 \left(|\alpha_3|^2 - |\alpha_2|^2\right)}{\omega_2\omega_3\ (\omega_2-\omega_3)} \right) P_0 \prod_i d^2\alpha_i \qquad \ldots (18)$$

Imposing the self-consistency requirement that the potential shape follows the average density distribution, namely

$$\int |\alpha_1|^2 P_0 : \int |\alpha_2|^2 P_0 : \int |\alpha_3|^2 P_0 = 1/\omega_1 : 1/\omega_2 : 1/\omega_3 \qquad \ldots (19)$$

we arrive at the well known[6)] expression for the moment of inertia of a rigid structure.

4. Unobserved modes, viscosity and the Navier-Stokes equation :

The concept of viscosity arises when certain unobserved modes which provide channels for irreversible leakage of energy exist. Thus we append to the system under consideration (the particle in a harmonic oscillator potential) a set of oscillators (reservoir) with frequencies $\{\omega_k\}$ described by annihilation and creation operators $\{b_k\}$ and $\{b_k^+\}$ and governed by the Hamiltonian

$$H = \hbar\omega a^+a + \sum_k \hbar\omega_k b_k^+ b_k + \sum_k (\lambda_k a^+ b_k + \lambda_k^* a b_k^+) \quad \ldots (20)$$

The coupling between the oscillator and the reservoir oscillators has been taken to be of a particularly simple form[7] to emphasize the main features, and is archetypal of a large class of damping mechanisms. The reservoir modes $\{\omega_i\}$ will be assumed to be closely spaced in frequency with a density $g(\omega_i)$. The equation of motion for the density operator ρ in the interaction picture is readily obtained using the Wigner-Weisskopf approximation[7]. The reduced density operator, S, for the oscillator, which is the trace over the reservoir modes of ρ, is then found to satisfy the equation

$$\frac{\partial S}{\partial t} = \frac{1}{2}\gamma\left[2aSa^+ - a^+aS - Sa^+a\right] + \gamma\bar{n}\left[a^+Sa + aSa^+ - a^+aS - Saa^+\right] \quad \ldots (21)$$

where γ, the inverse of the relaxation time, is given by

$$\gamma = 2\,g(\omega)\,\pi\,|\lambda_\omega|^2 \quad \ldots (22)$$

and $\bar{n}$ is the occupancy of the reservoir mode with frequency

$$\bar{n} = 1/\left[\exp\left(\hbar\omega/kT\right) - 1\right] \quad \ldots \quad (23)$$

The reservoir has been taken at a temperature T and provides the irreversible channel for the transfer of energy from the oscillator, inflicting on it a complex shift in frequency, the imaginary part (γ) of which parametrizes the rate of energy loss to the unobserved modes. This rate is proportional to the level density of the unobserved modes at the frequency of the mode suffering viscous losses and to the coupling between the two. There is also a real shift in the frequency, which we omit from the present discussion since it does not play a crucial role. The equation of motion for the reduced density operator, equation (21), is next expressed in the coherent state representation and we revert to the Schrödinger picture. Making use of the identities appearing in equations (3), we arrive at the equation of motion for the reduced density operator in the Schrödinger picture :

$$\frac{\partial}{\partial t} S_S\left(\alpha, \alpha^*, t\right) = -i\omega\left(\alpha^*\frac{\partial}{\partial\alpha^*} - \alpha\frac{\partial}{\partial\alpha}\right) S_s + \gamma\left[1 + \frac{1}{2}\left(\alpha\frac{\partial}{\partial\alpha} + \alpha^*\frac{\partial}{\partial\alpha^*}\right) + \bar{n}\frac{\partial^2}{\partial\alpha\,\partial\alpha^*}\right] S_s \ldots \quad (24)$$

Generalizing to three dimensions and defining the 'fluid density' and 'hydrodynamic velocity' in a manner analogous to the procedure adopted in our derivation of the Euler equation, the equations of continuity and momentum conservation follow :

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v^{(h)}) - \tfrac{1}{2}\gamma \nabla \cdot (\rho x) - \frac{\gamma \bar{n} \hbar}{2m\omega} \nabla^2 \rho = 0 \qquad \ldots \qquad (25)$$

$$\frac{\partial}{\partial t}(\rho v_k^{(h)}) = - \omega^2 x_k \rho - \frac{\partial}{\partial x_i}(\rho v_i^{(h)} v_k^{(h)})$$

$$+ \frac{\gamma}{2} \frac{\partial}{\partial x_i}(x_i v_k^{(h)} \rho) - \tfrac{1}{2}\gamma v_k^{(h)} \rho + \frac{\gamma \bar{n} \hbar}{2m\omega} \frac{\partial^2}{\partial x_i^2}(v_k^{(h)} \rho) \quad \ldots \quad (26)$$

Through the use of the equation of continuity the momentum conservation equation may be recast in the form

$$\frac{\partial}{\partial t} v^{(h)} = -\omega^2 x - (v^{(h)} \cdot \nabla) v^{(h)} + \tfrac{1}{2}\gamma (x \cdot \nabla) v^{(h)} - \tfrac{1}{2}\gamma v^{(h)}$$

$$+ \frac{\gamma \bar{n} \hbar}{m\omega\rho} (\nabla \rho \cdot \nabla) v^{(h)} + \frac{\gamma \bar{n} \hbar}{2m\omega} \nabla^2 v^{(h)} \qquad \ldots \qquad (27)$$

The essence of the physical content of these equations lies in the phenomena of viscosity and diffusion. Considering the system as a mixture of two fluids corresponding to the single particle of interest (density ρ) and the 'fluid' representing the reservoir (density ρ_R) a diffusion flux exists[8] with a current

$$i_D = - (\rho + \rho_R) \, D \, \nabla \frac{\rho}{\rho + \rho_R} \qquad \ldots \qquad (28)$$

With $\rho_R \gg \rho$; one obtains an expression for the diffusion current

$$i_D \simeq -D\nabla\rho - \frac{m\omega}{\bar{n}\hbar} D x \rho \qquad \ldots \qquad (29)$$

if one makes use of the fact that the reservoir distribution, expressing as it does a state of thermal (or gaussian) noise, satisfies the equation

$$\nabla \rho_R = -\frac{x}{\bar{n}} \frac{m\omega}{\hbar} \rho_R \qquad \ldots \qquad (30)$$

Examining the equation of continuity, equation (25), it is clear that the unfamiliar terms (containing γ and arising from the interaction) arise from the diffusion flux governed by a diffusion constant $D = \gamma\bar{n}\hbar/2m\omega$. The momentum conservation equation, equation (27), is the Navier-Stokes equation with additional diffusional drift terms. The co-efficient of viscosity η can be read off from the last term and is given by

$$\eta = (\gamma\bar{n}\hbar\rho)/(2m\omega) = D\rho$$

Thus the coherent state basis has facilitated the introduction of the concept of a single-particle Schrödinger fluid. By introducing unobserved modes, the Navier-Stokes equation with viscous and diffusional effects has been extracted.

R E F E R E N C E S

1. E. Schrödinger, Naturewiss 14 (1926) 664.

2. R. J. Glauber, Phys. Rev. 131 (1963) 2766

3. E. C. G. Sudarshan, Phys. Rev. Lett. 10 (1963) 277.

4. D. R. Inglis, Phys. Rev. 96 (1954) 1059.

5. A. Bohr and B. R.Mottelson, K. Danske Vidensk. Selsk, Mat-Fys. Meddr. 30 (1955) 1.

6. B. R. Mottelson, Many body Problem, Les Houche Summer School ed C de Witt (New York : Wiley) (1958) P 309.

7. R. J. Glauber, Quantum Optics,Proc. Int. School of Physics 'Enrico Fermi' ed R.J.Glauber (1969) 15 - 56.

   W. H. Louisell (same as above) pp 691 - 5.

8. L. D. Landau and E. M. Lifschitz, Fluid Mechanics (1959) 219 - 29.

****