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Spin-dependent Fermi liquid parameters and properties of polarized quark matter

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We calculate the spin-dependent Fermi liquid parameters (FLPs), single-particle energies, and energy densities of various spin states of polarized quark matter. The expressions for the incompressibility \( K \) and sound velocity \( c_1 \) in terms of the spin-dependent FLPs and polarization parameter \( \xi \) are derived. Estimated values of \( K \) and \( c_1 \) reveal that the equation of state of the polarized matter is stiffer than the unpolarized one. Finally, we investigate the possibility of the spin polarization (ferromagnetism) phase transition.

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I. INTRODUCTION

One of the important research areas of contemporary high-energy physics has been the study of matter under extreme conditions. Such matter in the laboratory can be produced by colliding heavy ions at ultra-relativistic energies. Because of the asymptotic freedom of quantum chromodynamics (QCD), it is predicted that hadronic matter at high temperature and/or density can undergo a series of phase transitions such as confinement-deconfinement and/or chiral phase transitions [1,2]. In the high-density regime QCD predicts the existence of a color superconducting state [3–5]. The possibility of a spin-polarized quark liquid (i.e., the existence of a ferromagnetic phase in a dense quark system) has also been suggested recently [6,7]. The properties of a dense quark system are particularly relevant for the study of various astrophysical phenomena.

Part of the motivation for studying the ferromagnetic phase transition in dense quark matter (DQM), as mentioned in Ref. [6], is provided by the discovery of “magnetars” [7], where an extraordinarily high magnetic field (~10^{15} G) exists. In Ref. [6], it is argued that the origin of such a high magnetic field can be attributed to the existence of spin-polarized quark matter [9]. To examine the possibility of ferromagnetism in DQM in Ref. [6] a variational calculation is performed in which it is observed that there exists a critical density below which spin-polarized quark matter is more energetically favorable than the unpolarized state. Subsequently, various other calculations were also performed to investigate this issue [2,4,5,7–10]. For example, in Ref. [5] it is shown that there is no contradiction between color superconductivity and ferromagnetism and both of these phases can coexist. In Ref. [10], the same problem was studied in the large-\( N_c \) and large-\( N_f \) limit while keeping \( N_f / N_c \) fixed, where it was shown that the spin-polarized state can exist; however, in the presence of magnetic screening, color superconductivity or dense chiral waves disappear. Such screening is now supported by the lattice calculation [10,11]. In Ref. [5] it is analytically shown that, if quarks are massless, ferromagnetism does not appear, which is consistent with the conclusion drawn in Ref. [10]. In Ref. [8] it is shown that ferromagnetism might appear

in quark matter with a Goldstone boson current, where the magnetization is shown to be related to triangle anomalies.

In the present work, we apply relativistic Fermi liquid theory (RFLT) to study the possibility of a para-ferro phase transition in DQM. The RFLT was developed by Baym and Chin [12], who have shown how the various physical quantities (e.g., chemical potential \( \mu \), incompressibility \( K \), and sound velocity \( c_1 \)) can be expressed in terms of the Landau parameters (LPs) calculated relativistically. However, the formalism developed in Ref. [12] is valid for unpolarized matter and LPs calculated there are spin averaged.

In this paper we extend the formalism of RFLT and the required LPs are calculated by retaining their explicit spin dependencies. As a result, here various combination of parameters such as \( f_{0LLLL}^{++} \), \( f_{0LLLL}^{-+} \), \( f_{0LLLL}^{+-} \), and \( f_{0LLLL}^{--} \) corresponding to scattering involving up-up, up-down, down-up, or down-down spins, appear [12]. Once determined, these parameters are used to calculate quantities such as chemical potentials for the spin-up and spin-down quarks or the total energy density of the system as a function of \( \xi = (n_u^+ - n_d^-)/n_u + n_d \) and \( n_u + n_d \) together with various other quantities, where \( n_u \) and \( n_d \) correspond to densities of spin-up and spin-down quarks, respectively, and \( n_u + n_d \) denotes total quark density [6]. We also compare some of our results with those presented in Ref. [6], where a more direct approach was adopted to calculate the total energy density from the loop. In addition, the present work is extended further to estimate incompressibility and sound velocity in a dense quark system for a given fraction of spin-up or spin-down quarks.

Furthermore, in dealing with the massless gluons, we find that a naive series expansion fails and one has to use a hard density loop (HDL) corrected gluon propagator to get the finite result for the LPs involving scattering of like spins [13]. This however does not cause any problem for the calculation of various physical quantities such as chemical potential, exchange energy, and incompressibility. We shall see, even though \( f_0 \) and \( f_2 \) (suppressing spin indices) individually remain divergent, what appears in our case is the particular combination of these parameters where such divergences cancel.

The plan of the paper is as follows. In Sec. II, as mentioned before, we extend the formalism of RFLT to include explicit spin dependence. In Sec. III, we derive spin-dependent LPs resulting from one-gluon exchange (OGE) for polarized quark

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matter. Subsequently, we calculate the chemical potential and energy density. We find the density dependence of incompressibility $K$ and first sound velocity $c_1$ with arbitrary spin polarization $\xi$. To compare with Ref. [6], we present ultra-relativistic and nonrelativistic results and study para-ferro phase transition of quark matter. Section IV is devoted to a summary and conclusion. In the Appendix, we calculate various LPs for unlike spin states of the scatterer.

II. FORMALISM

In Fermi liquid theory the total energy density $E$ of an interacting system is the functional of occupation number $n_p$ of the quasiparticle states of momentum $p$. The excitation of the system is equivalent to the change of occupation number by an amount $\delta n_p$. The corresponding energy density of the system is given by [12,14]

$$E = E^0 + \sum_p \int \frac{d^3p}{(2\pi)^3} \rho_0 (p) \delta n_p$$

$$+ \frac{1}{2} \sum_{p' p''} \int \frac{d^3p}{(2\pi)^3} \frac{d^3p''}{(2\pi)^3} \delta n_p \delta n_{p''},$$

where $E^0$ is the ground-state energy density and $s$ is the spin index, and the quasiparticle can be written as

$$\epsilon_{ps} = \epsilon_{ps}^0 + \sum_s \int \frac{d^3p}{(2\pi)^3} f_{ps, p''} \delta n_{p''}.$$

The compression modulus or incompressibility $K$ of the system is defined by the second derivative of total energy density $E$ with respect to the number density $n_q$, which is given by [16–20]

$$K = 9n_q \frac{\partial^2 E}{\partial n^2_q}.$$  

Now we introduce a polarization parameter $\xi$ by the equations $n_+ = n_q(1 + \xi)/2$ and $n_- = n_q(1 - \xi)/2$ under the condition $0 < \xi < 1$ [6]. The Fermi momenta in the spin-polarized quark matter then are $p_+ = p_f(1 + \xi)/3$ and $p_- = p_f(1 - \xi)/3$, where $p_f = (\pi^2 n_q)^{1/3}$ is the Fermi momentum of the unpolarized matter ($\xi = 0$). So, there are two Fermi surfaces corresponding to spin-up (+) and spin-down (−) states, such that $E = E(n_+, n_-)$. We have

$$\frac{\partial E}{\partial n_q} = \frac{\partial E}{\partial n_+} \frac{\partial n_+}{\partial n_q} + \frac{\partial E}{\partial n_-} \frac{\partial n_-}{\partial n_q}$$

$$= \frac{1}{2} \left(1 + \xi \right) \mu^+ + (1 - \xi) \mu^-.$$  

Using Eq. (11), the incompressibility becomes [20]

$$K = 9n_q \frac{1}{4} \left[ \frac{1 + \xi}{n_+} \frac{\partial n_+}{\partial n_q} + \frac{1 - \xi}{n_-} \frac{\partial n_-}{\partial n_q} \right]$$

$$= 9n_q \frac{1}{4} \left[ \frac{1 + \xi}{n_+} \frac{1 + F_+}{N^+(0)} + \frac{1 - \xi}{n_-} \frac{1 + F_-}{N^-(0)} \right].$$  

The Landau Fermi liquid interaction $f_{ps, p''}$ is related to the two-particle forward scattering amplitude via [12,14]

$$f_{ps, p''} = \frac{m_q}{\epsilon_p} m_q \epsilon_{p''} M_{ps, p''},$$

where $m_q$ is the mass of the quark and the Lorentz invariant matrix $M_{ps, p''}$ consists of the usual direct and exchange amplitude, which may, therefore, be evaluated by conventional Feynman rules. The dimensionless LPs are defined as $F_l = N^l(0) f_l^{\pm}$ [12], where $N^l(0)$ is the density of states at the Fermi surface, is given by

$$N^l(0) = \int \frac{d^3p}{(2\pi)^3} \delta(\epsilon_p - \mu^l)$$

$$= \frac{8\pi^2 \mu^l}{2\pi^2} \left( \frac{3p^l}{\delta(\epsilon_p - \mu^l)} \right) \int \frac{d^3p}{(2\pi)^3} \frac{d^3p''}{(2\pi)^3} f_{ps, p''} \delta(\epsilon_p - \mu^l).$$

Here $g_{deq}$ is the degeneracy factor. In our case $g_{deq} = N_c N_f$, where $N_c$ and $N_f$ are the color and flavor indices for quark matter. For spin-up (+) and spin-down (−) quarks, the density of states will change accordingly. In this expression $(\partial\mu^l/\delta n_p)_{\mu^l=\mu}$ is the inverse Fermi velocity $(1/v_f^l)$, which is related to the FLPs $F_l^\pm$ by

$$\frac{1}{v_f^l} = (\partial\mu^l/\delta n_p)_{\mu^l=\mu} = (\mu^l/|v_f^l|)(1 + F_l^\pm/3).$$

With Eq. (7) and Eq. (8) one reads the general relation as [16]

$$v_f^l = \mu^l (1 + 1/F_l).$$

The compression modulus or incompressibility $K$ of the system is defined by the second derivative of total energy density $E$ with respect to the number density $n_q$, which is given by [16–20]
where \[12\]

\[
\frac{\partial \mu^s}{\partial n_{qs}^s} = \frac{1 + F_0^s}{N_s^0}. \tag{13}
\]

Similarly, the relativistic first sound velocity is given by the first derivative of pressure \(P\) with respect to energy density \(E\). Since \(P = \sum \rho \mu^s_n n_{qs}^s - E\ [16,20]\), we have

\[
c_i^2 = \frac{\partial P}{\partial E} = \frac{\partial E}{\partial n_{qs}^s} \frac{\partial n_{qs}^s}{\partial E} = \left[ \frac{1 + (1 + \xi) \mu^s + (1 - \xi) \mu^-}{1 + (1 + \xi) \mu^s + (1 - \xi) \mu^-} \right] \left[ \frac{1 + F_0^s}{N_s^0} \right] + (1 - \xi)^2 \left[ \frac{1 + F_0^s}{N_s^0} \right]. \tag{14}
\]

In Eqs. (12) and (14), \(N_s^0(0)\) and \(F_0^s\) correspond to the density of states at the Fermi surface and the dimensionless LP for spin-up (+) and spin-down (−) quarks, respectively. For unpolarized matter, \(N_s^0 = 0\), implying \(\mu^s = \mu^-\), \(F_0^s = F_0^-\), and \(N_s^0(0) = N_s^0(0)\). From Eqs. (12) and (14) we have the well-known result that \(K = \sum \rho \mu^s \sum n_{qs}^s[16]\) and \(c_i^2 = \frac{1}{\mu} d_{n_{qs}^s}\) \([12]\).

III. LANDAU PARAMETERS FOR POLARIZED QUARK MATTER

In this section we calculate LPs for quark matter with explicit spin dependencies. We choose spin \(s\) along the \(z\) axis (i.e., \(s = (0,0,\pm1)\)) and represent spin-up and spin-down states by their signs. For a four-dimensional description of the polarization state, it is convenient to define a four-vector \(a^\mu\), which, in the rest frame of each quark, is the same as the three-dimensional vector \(a\); since \(s\) is an axial vector, \(a^\mu\) is a four-pseudovector. This four-vector is orthogonal to the polarization state, it is convenient to define a four-vector explicitly spin dependencies. We choose spin \(s\) along the axis \([i.e., s = (0,0,\pm1)]\) and represent spin-up and spin-down (−) quarks, respectively. For a four-dimensional description of the kinetic and potential energies is not influenced by the number of flavors. The number of flavors neither encourages nor discourages ferromagnetism [10].

Without loss of generality, for the calculation of energy density and other related quantities, we consider one-flavor quark matter. With the help of polarization density matrices given in Eq. (16), we have from Eq. (18) the interaction amplitude as [6]

\[
M_{a^\mu, a'^\nu}^{\text{ex}} = \sum_{\Pi} \hat{U}_\Pi (P') \hat{g}(s') \hat{y}^{\nu} U_\Pi (P) \times \frac{-g^{\mu\nu}}{(P - P')^2} \hat{U}_\Pi (P') \hat{g}(s' \nu) U_\Pi (P'). \tag{18}
\]

where \(a, b\) is the flavor index, \(i, j\) is the quark color index, \(t' = (\lambda, 2)\) is the color matrix, and \(s\) is the coupling constant. Since the gluon is flavor blind, the \(u\)-channel diagrams contribute only when \(a = b\) (i.e., the scattering of quarks with the same flavor [23]). This means that the Fermi sphere of each flavor makes an independent contribution. Thus the potential energy receives a factor \(N_f\). However, the quarks with different colors can take part in the exchange process, giving rise to a factor \(N_f^2\). Eventually, the potential energy density is proportional to \(N_f^2 N_c^2 s^2\). For the kinetic energy density, there arises an overall factor \(N_c N_f\). Thus, the factor \(N_c N_f\) factors out of the total energy density and the competition between the kinetic and potential energies is not influenced by the number of flavors. The number of flavors neither encourages nor discourages ferromagnetism [10].

We consider the color-symmetric forward scattering amplitude of the two quarks around the Fermi surface by the OGE interaction. The direct term does not contribute as it involves the trace of single color matrices such as \(T_i \lambda_j\), which vanish. Thus the leading contribution comes from the exchange (Fock) term [6]:

\[
M_{a^\mu, a'^\nu}^{\text{ex}} = \frac{g_{a^\mu, a'^\nu}}{2} \frac{1}{5} \sum \hat{U}_\Pi (P') \hat{g}(s') \hat{y}^{\nu} U_\Pi (P) \times \frac{-g^{\mu\nu}}{(P - P')^2} \hat{U}_\Pi (P') \hat{g}(s' \nu) U_\Pi (P'). \tag{18}
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\]
For (+, +) scattering the interaction parameter is given by

\[
\begin{align*}
&f_{pp''}^{++(p=p')} = \frac{g^2}{27e_f^2 p_f^2 (1 - \cos \theta)} \left[ 2m_q^2 - \frac{p_f^2}{3(\epsilon_f^+ + m_q)} \right] \\
&\quad \times \left[ \frac{p_f^2}{3(\epsilon_f^+ + m_q)} \right] (1 - \cos \theta) + \frac{1}{(\epsilon_f^+ + m_q)^2} \\
&\quad \times \left[ \frac{m_q (e_f^+ + m_q) p_f^2 (\cos^2 \theta_1 + \cos^2 \theta_2) + p_f^4 \cos \theta_1 \cos \theta_2}{(\epsilon_f^+ + m_q)^2} \right]. \quad (21)
\end{align*}
\]

where \( \epsilon_f^+ = \cos \theta_1, \epsilon_f^- = \cos \theta_2 \), and the Fermi energy is \( e_f^- = (p_f^2 + m_q)^{1/2} \). Since spin and momentum have no preferred direction, we have done an angular average of the spin-dependent parameter \([24]\):

\[
\begin{align*}
&f^{++} \equiv f_{pp''}^{++(p=p')} = \int \frac{d\Omega_1}{4\pi} \int \frac{d\Omega_2}{4\pi} f_{pp''}^{++(p=p')} \\
&= -\frac{g^2}{9e_f^2 p_f^2 (1 - \cos \theta)} \\
&\quad \times \left[ 2m_q^2 - \frac{p_f^2}{3(\epsilon_f^+ + m_q)} \right] (1 - \cos \theta) + \frac{1}{(\epsilon_f^+ + m_q)^2} \\
&\quad \times \left[ \frac{m_q (e_f^+ + m_q) p_f^2 (\cos^2 \theta_1 + \cos^2 \theta_2) + p_f^4 \cos \theta_1 \cos \theta_2}{(\epsilon_f^+ + m_q)^2} \right]. \quad (22)
\end{align*}
\]

With the help of Eq. (5) along with Eq. (22) one can find the LPs, but it is to be noted that \( f_{pp''}^{++} \) and \( f_{pp''}^{+-} \) are individually divergent because of the term \((1 - \cos \theta)\) in the denominator of the interaction parameter.\(^1\) This divergence disappears if one uses a Debye screening mass for gluons or equivalently an HDL corrected gluon propagator while evaluating the scattering amplitudes \([13,24]\). Note that the combination \( f_{pp''}^{++} \) is, however, finite as in this case the divergences cancel and we do not calculate the LPs separately. It would, however, be interesting to see how the results are modified if HDL calculations are performed to evaluate \( f_{pp''}^{++} \), \( f_{pp''}^{+-} \) and the corresponding physical quantities. The numerical estimates suggest that, for the results that we present here, the effect of HDL corrections are expected to be small.

\[\text{From Eq. (5), } \]

\[f_{pp''}^{++} - \frac{1}{4} f_{pp''}^{++} = -\frac{8^2}{18e_f^2 p_f^2} \int_{-1}^{1} \left[ 2m_q^2 - p_f^2 (1 - \cos \theta) \right] \left[ \frac{2m_q p_f^2}{3(\epsilon_f^+ + m_q)} \right] (\cos \theta) \]

\[\quad \times \left[ \frac{m_q (e_f^+ + m_q) p_f^2 (\cos^2 \theta_1 + \cos^2 \theta_2) + p_f^4 \cos \theta_1 \cos \theta_2}{(\epsilon_f^+ + m_q)^2} \right]. \quad (23)\]

This combination will appear in the calculation of the chemical potential and other relevant quantities. For \((+, -)\) scattering, the angular averaged interaction parameter yields

\[
\begin{align*}
&f^{+-} \equiv f_{pp''}^{++(p=p')} = -\frac{g^2}{9e_f^2 p_f^2} \left[ 1 - \frac{m_q p_f^2}{3(\epsilon_f^+ + m_q)} \right] \\
&\quad \times \left[ \frac{1}{(\epsilon_f^+ + m_q)} \right] (1 - \cos \theta) + \frac{1}{(\epsilon_f^+ + m_q)^2} \\
&\quad \times \left[ \frac{m_q (e_f^+ + m_q) p_f^2 (\cos^2 \theta_1 + \cos^2 \theta_2) + p_f^4 \cos \theta_1 \cos \theta_2}{(\epsilon_f^+ + m_q)^2} \right]. \quad (24)
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It is to be noted that individual LPs for scattering of unlike spin states are finite (i.e., free of divergences), in contrast to the case involving scattering of like spin states. (For details see the Appendix.)

### A. Chemical potential

Now we proceed to calculate the chemical potential, which, in principle, will be different for spin-up and spin-down quarks, denoted by \( \mu^+ \) with \( s \) (or \( s' \)) = +, − for matter containing unequal densities of up and down quarks. To determine the chemical potential with arbitrary polarization \( \xi \), we take the distribution function with explicit spin index (s or \( s' \)), so that variation of the distribution function gives \([14,20,25]\):

\[
\delta n_q^\xi = -N_q^\xi \left[ \sum_{\nu} f_{\nu s}^\xi \delta n_q^\nu - \delta \mu^\xi \right]. \quad (25)
\]

where \( N_q^\xi \) is given by Eq. (7). Equation (25) yields

\[
\frac{\partial \mu^\xi}{\partial n_q^\xi} = \frac{1}{N_q^\xi} \sum_{\nu} f_{\nu s}^\xi \frac{\partial n_q^\nu}{\partial n_q^\xi}. \quad (26)
\]

Separately for spin-up and spin-down states we have

\[
\begin{align*}
&\frac{\partial \mu^+}{\partial n_q} = \frac{1}{N_q^+} \left[ f_{++} - f_{--} \right] \left( \frac{\partial n_q^+}{\partial n_q} \right), \\
&\frac{\partial \mu^-}{\partial n_q} = \frac{1}{N_q^-} \left[ f_{++} - f_{--} \right] \left( \frac{\partial n_q^-}{\partial n_q} \right), \quad (27)
\end{align*}
\]

where the superscripts ++ and −− denote scattering of quasiparticle with up-up and up-down spin states. For unpolarized matter the upper and lower components become equal, which gives rise to the well-known result \([12]\). We get

\[
\mu = -\frac{g^2}{27e_f^2 p_f^2} \int_{-1}^{1} \left[ 2m_q^2 - p_f^2 (1 - \cos \theta) \right] \left[ \frac{2m_q p_f^2}{3(\epsilon_f^+ + m_q)} \right] (\cos \theta) \]

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In this equation the term in the first set of square brackets arises from the scattering of like spin states \((++\)) and the term in curly braces comes from the scattering of unlike spin states \((-\))

Similarly, for the spin-down quark, one may determine \(\mu^-\) by replacing \(p^f_+\) with \(p^f_-\) and with \(e^f_+\) in Eq. (29).

For the numerical estimation of these quantities, following Refs. [6,26], we take \(a_8 = g_1/4\pi = 2.2\) as the fine structure constant of QCD and \(m_q = 300\) MeV. In Fig. 1 we plot the chemical potential for spin-up and spin-down quarks as a function of density with order parameter \(\xi = 0.5\). In real astrophysical calculations, the chemical potentials are determined by the \(\mu\text{-equilibrium conditions, where the condition of charge neutrality is also imposed. In Fig. 1, however, we use density } n_q \text{ and polarization parameter } \xi \text{ as input parameters and Eq. (29) is used to determine } \mu \text{ for a system with one flavor.}

**B. Energy density**

Once the value of \(\mu\) is determined, one can readily calculate the exchange energy density by evaluating \([12,17,25]\)

\[
E_{\text{ex}} = \int d\mathbf{q} (\mu - e_f).
\]

After summing over the color degrees of freedom and evaluating over the Fermi surfaces, we have the exchange energy density. The latter, consisting of all types of scattering amplitudes, can be written as

\[
E_{\text{ex}} = E_{\text{ex}}^{++} + E_{\text{ex}}^{--} + E_{\text{ex}}^{+-},
\]
interaction gives no contribution because of charge neutrality, the Fock exchange interaction gives the leading contribution as the interaction energy. For spontaneous ferromagnetism, the interaction energy dominates over the kinetic energy \[6,21,28\]. Therefore, if the exchange energy from the OGE interaction is negative and becomes greater than the kinetic energy at some density, the quark matter becomes polarized, giving rise to ferromagnetism \[6\].

To check whether our results for the total energy density are consistent with Ref. \[6\], we consider two limiting cases, corresponding to the ultra-relativistic (UR) and nonrelativistic (NR) regimes. In the UR limit, \( p_f \gg m_q \), then using Eq. (29) we have

\[
\mu_{\pm,UR} = p_f^+ + \frac{\alpha_e}{3\pi} \left[ p_f^+ + \frac{p_f^+ p_f^+}{p_f^+} \right].
\]

Similarly one can find \( \mu_{-,-} \) by replacing \( p_f^+ \) with \( p_f^- \).

One can arrive at the same expression \( \mu_{\pm,UR} \) by taking the UR limit of the scattering amplitude. For \((+,+)\) scattering one gets the interaction parameter as

\[
f_{p\rightarrow p}^+ = \frac{g^2}{9p_f^2} (1 + \cos \theta_1 \cos \theta_2).
\]

After taking the angular average of the interaction parameter and with the help of Eq. (5), we find that \( f_{p\rightarrow p}^+ \) vanishes. Thus we have

\[
f_{p\rightarrow p}^+ = f_{p\rightarrow p}^- = \frac{g^2}{9p_f^2}.
\]

Similarly, for \((+,-)\) scattering, the interaction parameter yields

\[
f_{p\rightarrow p}^+ = \frac{g^2}{9p_f^2} (1 - \cos \theta_1 \cos \theta_2).
\]

The only existing LP is \( \mu_{\pm,UR} \) and other higher order LPs do not contribute. Hence we get

\[
f_{p\rightarrow p}^+ = f_{p\rightarrow p}^- = \frac{g^2}{9p_f^2}.
\]

It is observed that, in the UR limit, all the LPs are finite. Now the chemical potential for the spin-up quark is found to be

\[
\mu_{\pm,UR} = p_f^+ + \frac{\alpha_e}{3\pi} \left[ p_f^+ + \frac{p_f^+ p_f^+}{p_f^+} \right].
\]

The chemical potential \( \mu_{\pm,UR} \) can be obtained by replacing \( p_f^+ \) with \( p_f^- \) in Eq. (39).

Using Eqs. (30) and (31), the exchange energy densities are given by

\[
\begin{align*}
E_{ex}^{+,UR} &= \frac{\alpha_e}{8\pi^2} p_f^+ \left[ (1 + \xi)\xi^{3/2} + (1 - \xi)\xi^{3/2} + 2(1 - \xi^2)\xi^{3/2} \right], \\
E_{ex}^{-,-} &= \frac{\alpha_e}{8\pi^2} p_f^- \left[ (1 + \xi)\xi^{3/2} + (1 - \xi)\xi^{3/2} + 2(1 - \xi^2)\xi^{3/2} \right].
\end{align*}
\]

This result is the same as in Ref. \[6\]. Similarly, from Eq. (32), the kinetic energy density in the UR limit takes the following form \[6\]:

\[
E_{kin} = \frac{3}{2} \frac{p_f^+}{8\pi^2} \left[ (1 + \xi)\xi^{3/2} + (1 - \xi)\xi^{3/2} \right].
\]

In the NR limit, \( p (or p') \ll m_q \), the interaction parameter reduces to a simple form

\[
f_{p\rightarrow p'} = \frac{-g^2}{9p_f^2} \left[ \frac{1 + s \cdot s'}{(1 - \cos \theta)} \right].
\]

For a spin antiparallel interaction \( s = -s' \), then \( f_{p\rightarrow p'} \) vanishes. Thus the contribution from the scattering of quarks with unlike spin states vanishes and the dominant contribution to the energy density comes from the parallel spin states \((s = s')\). For \((s, s)\) scattering, the interaction parameter yields

\[
f_{p\rightarrow p'} = \frac{-g^2}{9p_f^2} \left[ \frac{1 + s \cdot s'}{(1 - \cos \theta)} \right]^{-1}.
\]

The NR chemical potential \( \mu_{\pm,NR} \) is given by

\[
\mu_{\pm,s} = m_q - \frac{g^2}{3\pi^2} p_f^s.
\]

Using Eq. (30), the exchange energy density for \((+,+)\) scattering is given by

\[
E_{ex}^{+,NR} = -\frac{g^2}{8\pi^2} p_f^s \left[ (1 + \xi)\xi^{3/2} \right].
\]

Similarly, for \((-,-)\) scattering, we have

\[
E_{ex}^{-,-} = -\frac{g^2}{8\pi^2} p_f^s \left[ (1 - \xi)\xi^{3/2} \right].
\]

As in the NR limit, \( E_{ex}^{+,NR} = E_{ex}^{-,-} = 0 \) as mentioned before, so from Eq. (31) the exchange energy density yields

\[
E_{ex}^{+,NR} = \frac{\alpha_e}{8\pi^2} p_f^s \left[ (1 + \xi)\xi^{3/2} + (1 - \xi)\xi^{3/2} \right].
\]

Thus the density, in this limit, becomes negative. The kinetic energy density turns out to be \[6\]

\[
E_{kin} = \frac{3}{2} \frac{p_f^s}{20\pi^2 m_q} \left[ (1 + \xi)\xi^{3/2} + (1 - \xi)\xi^{3/2} \right].
\]

In the NR limit, ferromagnetism can appear as a consequence of competition between the kinetic energy and the Coulomb potential energy \[10\]. The latter favors spin alignment owing to quantum effects. When the energy gain from the spin alignment dominates over the increase in the kinetic energy at some density, the unpolarized state suddenly turns into the completely polarized state \[29\]. In contrast, in the UR limit, the contribution to the energy density comes not only from the like spin states but also from the unlike spin states of the scatterer (see Ref. \[6\] for a detailed discussion).
To check the consistency we compare our result derived in the RFLT approach with that of Ref. [6] derived from two loop ring diagrams. In Fig. 4 we plot $E_{\text{tot}}/n_q$ as a function of polarization parameter $\xi$. The results clearly show that, for lower density ($<0.14 \text{ fm}^{-3}$), total energy favors $\xi = 1$, which indicates a completely polarized state, whereas, at higher density, the system becomes unpolarized ($\xi = 0$). Thus the polarization parameter suddenly changes from $\xi = 1$ to $\xi = 0$ as one increases the number density of the system. So the phase transition is first order and the critical density $n^c_q$ is around 0.14 fm$^{-3}$.

In Fig. 5 we show the total energy as a function of the polarization parameter for different densities. In every plot, there is a minima, which corresponds to a possible metastable state. We notice that when density increases metastable state arises for lower values of polarization parameter $\xi$. For example, at a density of $\sim 0.2 \text{ fm}^{-3}$ a minima arises at $\xi = 0.1$ whereas at a density of $\sim 0.35 \text{ fm}^{-3}$ a minima arises at $\xi = 0.03$. Thus the metastable state shows a tendency to disappear as the density increases.

IV. SUMMARY AND CONCLUSION

In this work we have applied RFLT to study the properties of dense quark matter. Accordingly, we calculate the FLPs by retaining their explicit spin dependencies. We also show how the physical quantities such as the chemical potential of spin-up and spin-down states, their energy densities, and incompressibility and sound velocity for polarized quark matter can be expressed in terms of these spin-dependent FLPs. For scattering involving like spin states, the LPs $f_{0j}^{+}$ and $f_{0j}^{-}$ are found to diverge. However, we show that for the combination in which they appear in the calculation of the physical quantities such divergences cancel. For scattering involving unlike spin states no such divergence appears. The appearance of such divergences is related to the unscreened gluonic interaction between the quarks, and invoking a hard dense loop corrected gluon propagator may eliminate these divergences. We do not perform such a calculation here and postpone this for a future investigation. As far as the equation of state is concerned, in the present model we find that the equation of state for polarized quark matter is stiffer than that for unpolarized matter. In addition, we also show that there exists a metastable state that disappears at higher density, although it seems that the effect is tiny.

We reconfirm that DQM can exhibit ferromagnetism at low density, as was originally suggested in Ref. [6]. However, the density at which the spin-polarized ferromagnetic state in the present model might appear depends strongly on the quark mass. The critical density increases with increasing mass. In Fig. 4, we observe that states with $\xi$ appear only below
or around normal nuclear density where deconfined quark matter is not likely to exist. We cannot, however, ascertain the critical density from the present analysis in which we restrict ourselves only to OGE diagrams and a one-flavor system. In this regime, multigluon exchange processes [10T might play an important role. Furthermore, the correlations as given by the ring diagrams can also change the conclusion. Further work therefore is necessary to understand the existence of ferromagnetic quark matter in real multiflavor systems that might appear in astrophysics.

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APPENDIX

In the text the interaction parameter \( f_{f_f} \) for unlike spin states was calculated. Here we give a detailed expression of the Landau parameters. With the help of Eq. (5), the LPs are given by

\[
f_{f_f} = \frac{g^2}{18e_f^2e_f} \left[ 6 - \frac{2m_q(e_f^2p_f^2 + m_q(p_f^2 + p_f^2) + e_f^2p_f^2)}{p_f^2p_f^2(m_q + e_f)(m_q + e_f)} \right] \]

and

\[
\frac{1}{3} f_{f_f} = \frac{g^2}{18e_f^2e_f} \left[ 2 - \frac{m_q(p_f^2)^2}{3(e_f^2 + m_q)} + \frac{m_q(p_f^2)^2}{3(e_f^2 + m_q)} \right] \]

Using Eqs. (A1) and (A2) we have

\[
f_{f_f} - \frac{1}{3} f_{f_f} = \frac{g^2}{18e_f^2e_f} \left[ - \frac{2}{p_f^2p_f^2} + \frac{(p_f^2p_f^2 + m_e^2 - e_f^2e_f)}{(p_f^2p_f^2)} \right] \ln \left( \frac{m_q + p_f^2p_f^2 - e_f^2e_f}{m_q - p_f^2p_f^2 - e_f^2e_f} \right) \]


Ground state energy of spin polarized quark matter with correlation

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We calculate the ground-state energy of cold and dense spin-polarized quark matter with corrections due to correlation energy ($E_{\text{corr}}$). Expressions for $E_{\text{corr}}$ both in the nonrelativistic and ultrarelativistic regimes have been derived and compared with the exchange and kinetic term present in the perturbation series. It is observed that the inclusion of correlation energy does not rule out the possibility of the ferromagnetic phase transition at low density within the model proposed by Tatsumi [Phys. Lett. B489, 280 (2000)]. We also derive the spin stiffness constant in the high-density limit of such a spin-polarized matter.

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I. INTRODUCTION

The possibility of ferromagnetic phase transition in dense quark matter was first discussed by Tatsumi [1] where it was shown that quark liquid interacting through one gluon exchange shows spontaneous magnetic instability at low densities. Such an investigation was motivated by the observation of a strong magnetic field in neutron star. Moreover, the theoretical conjectures about the possible existence of quark stars provide additional impetus to examine this issue further [2–6].

The underlying mechanism of such a phase transition for slow-moving massive quark is similar to what one observes in case of interacting electron gas [7,8] in a neutralizing positive charge background where the electrons interact only by the exchange interaction and the contribution of the direct term cancels with the background contribution. In case of interacting electron gas, the kinetic energy is minimum in unpolarized state; the exchange energy, however, favors spin alignment. These are two competing phenomena that also depend on density. It is seen that the kinetic energy dominates at some density and as the density is lowered the exchange energy becomes larger. At some point turning the electron gas suddenly into a completely polarized state. This is the mechanism of ferromagnetism in electron gas interacting via Coulomb potential [9].

The exchange energy for quark matter interacting via one gluon exchange (OGE) is also attractive and becomes dominant at some density giving rise to ferromagnetism [1–4]. However, there are similarities and differences between quark matter and electron gas as discussed in Ref. [1].

For slow-moving massive quarks the dynamics are very similar to what happens in electron gas, while in the relativistic case a completely different mechanism works when the spin-dependent lower component of the Dirac spinor becomes important. It should also be noted that the exchange energy is negative for massive strange quark at low densities while it is always positive for massless $u$ and $d$ quarks, as observed in Ref. [10] and subsequently in Refs. [1,11].

The magnetic property of the quark matter was also studied in Ref. [2] by evaluating the effective potential by employing magnetic moment of a quark and treating this as an order parameter. Unlike in Ref. [1], in this model, $u$, $d$, and $s$ quarks, i.e., all of these flavors, show ferromagnetic phase transition at various densities. In Ref. [4], we revisited this problem and have evaluated Fermi liquid parameters for spin-polarized quark matter that were subsequently used to derive single-particle spectrum and total energy density as a function of density within the model proposed by Tatsumi [Phys. Lett. B489, 280 (2000)]. We also derive the spin stiffness constant in the high-density limit of such a spin-polarized matter.

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E_{\text{corr}} = E - E_{\text{ex}} - E_{\text{kin}}. \tag{1}

Here, $E_{\text{corr}}$, $E_{\text{ex}}$, and $E_{\text{kin}}$ correspond to correlation, exchange, and kinetic energy density, respectively. In general, for electron gas interacting via Coulomb force it takes the following form [7,8]

E_{\text{corr}} = A \ln r_s + C + O(r_s). \tag{2}

At large Fermi momentum ($p_F$), i.e., in the limit $r_s = 0$, the result becomes exact [12,13]. For the case of electron gas, the inverse density is set equal to $\frac{1}{\pi r_s^3}$ and the dimensionless parameter $r_s$ is defined as $r_s$ divided by Bohr radius $\alpha$. Here...
we derive a similar expression for the dense quark matter with arbitrary spin polarization with appropriate modifications. The model adopted in the present work is same as that of Ref. [1] except here we go beyond $O(g^2)$ and include ring diagrams to evaluate the correlation energy of spin-polarized quark matter. This, together with the contribution of $E_{\text{ex}}$ and $E_{\text{el}}$, as we shall see, has the small $g$ expansion

$$\Pi_{\mu\nu} = \frac{N_f g^2}{2} \int \frac{d^3p}{(2\pi)^3} \sum_{\mu=\pm} \Theta^p \left[ \frac{K^\mu}{K^2 - 4(P \cdot K)^2} \right. \left. \sum_{s=\pm} \left[ \mathcal{M}^\mu_{\mu\nu}(P + K, P) + \mathcal{M}^\mu_{\mu\nu}(P, P - K) \right] \right. \left. - \frac{2(P \cdot K)}{K^2 - 4(P \cdot K)^2} \sum_{s=\pm} \left[ \mathcal{M}^\mu_{\mu\nu}(P + K, P) \right. \right. \left. \left. - \mathcal{M}^\mu_{\mu\nu}(P, P - K) \right] \right).$$

Here, $E_{\text{ex}}$ is related to the Compton-scattering amplitude as shown in Fig. 2. To derive Eq. (4), following Refs. [1,4] we use projection operator $\mathcal{P}(\alpha) = \{1 + \gamma_4\} \alpha$ at each vertex. The momentum integration is performed at the Fermi surface restricted by $\Theta^p = \Theta(p_F - |p|)$. Now we choose $K = (k_0, 0, 0, |k|)$, $P = (p_F, |p| \sin \theta \cos \phi, |p| \sin \theta \sin \phi, |p| \cos \phi)$, and $s = (1, -1, -1, -1)$. Note that the upper- and lowercase fonts are used to distinguish between four and three vectors.

From Fig. 2(a) the scattering amplitude becomes

$$\mathcal{M}_{\mu\nu}^{\text{dir}, \text{ex}}(P + K, P) = -b_\mu P_\nu(a \cdot K) - g_{\mu\nu}(b \cdot P)(a \cdot K) + b_\nu K_\mu(a \cdot P)$$

$$+ a_\mu P_\nu(b \cdot K) - g_{\mu\nu}(a \cdot P)(b \cdot K) + P_\mu(-K_\nu(a \cdot b - 1)$$

$$- 2P_\nu(a \cdot b - 1) + b_\nu(a \cdot K) + 2b_\nu(a \cdot P) + a_\nu(b \cdot K))$$

$$+ a_\mu K_\nu(b \cdot P) + 2a_\mu P_\nu(b \cdot P) - 2g_{\mu\nu}(a \cdot P)(b \cdot P)$$

$$- K_\mu(P_\nu(a \cdot b - 1) - b_\nu(a \cdot P) + a_\nu(b \cdot P)) + (K \cdot P)$$

$$\times [g_{\mu\nu}(a \cdot b - 1) - b_\mu a_\nu - a_\mu b_\nu].$$

The components of the four-pseudovector $b_\mu(\alpha a_\mu)$ in a frame in which the particle is moving with momentum $p(\text{or } p + k)$ are found by the Lorentz transformation from the rest frame as given by [1,4],

$$a_0 = \frac{(p + k) \cdot s}{m_q}; \quad \vec{a} = s + \frac{(p + k)(p + k) \cdot s}{m_q(\vec{p} + k + m_q)};$$

$$b_0 = \frac{p \cdot s'}{m_q}; \quad \vec{b} = s' + \frac{p(p \cdot s')}{m_q(\vec{p} + m_q)}.$$

II. GLUON SELF-ENERGY IN POLARIZED MATTER

To calculate the correlation energy of spin-polarized quark matter one needs to calculate the gluon self-energy in matter with arbitrary spins. This spin-dependent gluon polarization arises from the quark loop shown in Fig. 1 [18]. Mathematically [19,20],

$$\Pi_{\mu\nu} = \frac{N_f g^2}{2} \int \frac{d^3p}{(2\pi)^3} \sum_{\mu=\pm} \Theta^p \left[ \frac{K^\mu}{K^2 - 4(P \cdot K)^2} \right. \left. \sum_{s=\pm} \left[ \mathcal{M}^\mu_{\mu\nu}(P + K, P) + \mathcal{M}^\mu_{\mu\nu}(P, P - K) \right] \right. \left. - \frac{2(P \cdot K)}{K^2 - 4(P \cdot K)^2} \sum_{s=\pm} \left[ \mathcal{M}^\mu_{\mu\nu}(P + K, P) \right. \right. \left. \left. - \mathcal{M}^\mu_{\mu\nu}(P, P - K) \right] \right).$$

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$$+ a_\mu P_\nu(b \cdot K) - g_{\mu\nu}(a \cdot P)(b \cdot K) + P_\mu(-K_\nu(a \cdot b - 1)$$

$$- 2P_\nu(a \cdot b - 1) + b_\nu(a \cdot K) + 2b_\nu(a \cdot P) + a_\nu(b \cdot K))$$

$$+ a_\mu K_\nu(b \cdot P) + 2a_\mu P_\nu(b \cdot P) - 2g_{\mu\nu}(a \cdot P)(b \cdot P)$$

$$- K_\mu(P_\nu(a \cdot b - 1) - b_\nu(a \cdot P) + a_\nu(b \cdot P)) + (K \cdot P)$$

$$\times [g_{\mu\nu}(a \cdot b - 1) - b_\mu a_\nu - a_\mu b_\nu].$$

The components of the four-pseudovector $b_\mu(\alpha a_\mu)$ in a frame in which the particle is moving with momentum $p(\text{or } p + k)$ are found by the Lorentz transformation from the rest frame as given by [1,4],

$$a_0 = \frac{(p + k) \cdot s}{m_q}; \quad \vec{a} = s + \frac{(p + k)(p + k) \cdot s}{m_q(\vec{p} + k + m_q)};$$

$$b_0 = \frac{p \cdot s'}{m_q}; \quad \vec{b} = s' + \frac{p(p \cdot s')}{m_q(\vec{p} + m_q)}.$$
Similarly, from Fig. 2(b) we have,

\[ M_{\alpha}^{\text{flip}}(P, P - K) = -\hat{b}_\alpha P_\alpha(\hat{b} \cdot K) + \hat{g}_{\mu \nu}(\hat{d} \cdot P)(\hat{d} \cdot K) + b_\alpha K_{\alpha}(\hat{d} \cdot P) \]

\[ -\hat{a}_\mu P_\mu(\hat{b} \cdot K) + \hat{g}_{\mu \nu}(\hat{d} \cdot P)(\hat{d} \cdot K) - P_\mu(\hat{b} \cdot K) \]

\[ + 2P_\mu(\hat{b} \cdot K - 1) + \hat{b}_{\mu}(\hat{d} \cdot K) - 2\hat{b}_{\mu}(\hat{a} \cdot P - \hat{a}_{\mu}(\hat{b} \cdot K)) \]

\[ - \hat{a}_{\mu} K_{\mu}(\hat{d} \cdot P) + 2\hat{b}_{\mu} P_{\mu}(\hat{b} \cdot P) - 2\hat{g}_{\mu \nu}(\hat{a} \cdot P)(\hat{d} \cdot P) \]

\[ - K_{\mu}(-P_\mu(\hat{b} \cdot K - 1) + \hat{b}_{\mu}(\hat{a} \cdot P + \hat{a}_{\mu}(\hat{b} \cdot P))] \]

\[ - (K \cdot P)(\hat{g}_{\mu \nu}(\hat{d} \cdot K - 1) - \hat{b}_{\mu} \alpha_{\mu} - \hat{a}_{\mu} \alpha_{\mu}), \]  

(7)

where

\[ \hat{a}_0 = \frac{P \cdot s}{m_q}; \quad \hat{a} = \hat{s} + \frac{P(\hat{P} \cdot s)}{m_q(\hat{P} \cdot s + m_q)} \]

\[ \hat{b}_0 = \frac{(p - k) \cdot s}{m_q}; \quad \hat{b} = \hat{s} + \frac{(P - k)(p - k) \cdot s'}{m_q(p - k) \cdot s'} \]  

(8)

Now we define matrix elements \( M_{\mu \nu}^{\text{flip}} \) in terms of flip (f) and nonflip (nf) interactions where \( M_{\mu \nu}^{\text{flip}} = M_{\mu \nu}^{\text{flip}} \) and \( M_{\mu \nu}^{\text{nonflip}} = M_{\mu \nu}^{\text{flip}} \). Using Eq. (5) and Eq. (7) we have

\[ M_{\mu \nu}^{\text{flip}}(P + K, P) + M_{\mu \nu}^{\text{nonflip}}(P, P - K) = 8\pi^2 \]

\[ M_{\mu \nu}^{\text{flip}}(P + K, P) - M_{\mu \nu}^{\text{nonflip}}(P, P - K) = -4(P \cdot K) \]

\[ M_{\mu \nu}^{\text{flip}}(P + K, P) + M_{\mu \nu}^{\text{flip}}(P, P - K) = 8\pi^2 \sin^2 \theta \]

\[ M_{\mu \nu}^{\text{nonflip}}(P + K, P) - M_{\mu \nu}^{\text{nonflip}}(P, P - K) = 4P^2 \cos^2 \theta \]

\[ + (P \cdot K) \],

(9)

and

\[ M_{\mu \nu}^{\text{flip}}(P + K, P) + M_{\mu \nu}^{\text{flip}}(P, P - K) = M_{\mu \nu}^{\text{flip}}(P + K, P) + M_{\mu \nu}^{\text{flip}}(P, P - K) = 4(P \cdot K) \].

(10)

The detailed expressions of the matrix element \( M_{\mu \nu}^{\text{flip}} \) are given in the Appendix. In the present work we consider one flavor quark matter. Generalization for the multiflavor system is straightforward. Using Eq. (4) and Eqs. (9)–(11) we get

\[ \Pi_{11} = \frac{8\pi^2}{3} \sum_{i=\pm} \int_{0}^{\rho_f} \frac{d^3p}{p^3} \left[ \frac{K^2p^2 \sin^2 \theta - 2(P \cdot K)^2}{K^4 - 4(P \cdot K)^2} \right], \]

(12)

\[ \Pi_{00} = \frac{8\pi^2}{3} \sum_{i=\pm} \int_{0}^{\rho_f} \frac{d^3p}{p^3} \left[ \frac{K^2p^2 \sin^2 \theta + (P \cdot K)^2}{K^4 - 4(P \cdot K)^2} \right], \]

(13)

\[ \Pi_{33} = \frac{8\pi^2}{3} \sum_{i=\pm} \int_{0}^{\rho_f} \frac{d^3p}{p^3} \times \left[ \frac{K^2p^2 \cos^2 \theta - 2pk \cos \theta}{K^4 - 4(P \cdot K)^2} \right], \]

(14)

We are interested in evaluating longitudinal (\( \Pi_L \)) and transverse (\( \Pi_T \)) components of the polarization tensor. We define \( \Pi_L = -\Pi_{00} + \Pi_{11} + \Pi_{11} = \Pi_{22} \). In the long-wavelength limit \( |p| \approx p_f \) and \( |k| \ll p_f \), i.e., for low-lying excitation near the Fermi surface, \( K^2 \) can be neglected compared to \( 4(P \cdot K)^2 \) in the denominators of Eqs. (12)–(14) [21]. The longitudinal and transverse polarization in this limit are determined to be

\[ \Pi_L = \frac{8\pi^2}{4\pi^2}(C_0^2 - 1) \sum_{i=\pm} \rho_f^2 \left[ 1 - \frac{C_0}{2\nu_f} \right] \ln \left( \frac{C_0 + \nu_f}{C_0 - \nu_f} \right), \]

(15)

\[ \Pi_T = \frac{8\pi^2}{16\pi^2}C_0 \sum_{i=\pm} \rho_f^2 \left[ \frac{2C_0}{\nu_f} + \left( 1 - \frac{C_0^2}{\nu_f^2} \right) \ln \left( \frac{C_0 + \nu_f}{C_0 - \nu_f} \right) \right]. \]

(16)

Here, we take \( C_0 = k_0/|k| \) and \( \nu_f = p_f(1 \pm |\xi|)/(|\xi| + 2m^2)^{1/2} \) to cast the results in a more familiar form as presented in Ref. [21] for \( \xi = 0 \). It might be noted here that, although the final expressions for the longitudinal and transverse polarization look rather similar to what one obtains in the case of unpolarized matter [21] with only a difference in \( \nu_f^2 \) and summation over the spins, the calculation of the matrix elements with explicit spin dependencies are rather involved (see the Appendix).

\( \Pi_L \) and \( \Pi_T \) have two limiting values, corresponding to the nonrelativistic (nr) and the ultrarelativistic (ur) regime. In the nonrelativistic limit \( \epsilon_f^2 \approx m_q \)

\[ \Pi_{\text{nr}} = \frac{8\pi^2}{4\pi^2}m_q \sum_{i=\pm} \rho_f^2 \left[ 1 - \frac{C_0}{2\nu_f} \right] \ln \left( \frac{C_0 + \nu_f}{C_0 - \nu_f} \right), \]

(17)

\[ \Pi_{\text{ur}} = \frac{8\pi^2}{16\pi^2}C_0 \sum_{i=\pm} \rho_f^2 \left[ \frac{2C_0}{\nu_f} + \left( 1 - \frac{C_0^2}{\nu_f^2} \right) \ln \left( \frac{C_0 + \nu_f}{C_0 - \nu_f} \right) \right]. \]

(18)

Here \( \nu_f^2 = p_f^2/m_q \). These expressions were derived in Refs. [12,22] for unpolarized electron gas. In this limit, the current-current interaction is inherently small, for which this term can be neglected compared to the Coulomb interaction to calculate correlation energy. Here, \( \Pi_{\text{nr}} \sim (k_0/|k|)^2 \) and \( \Pi_{\text{ur}} \sim (k_0/|k|)^2 \) when both \( k_0 \rightarrow 0 \) and \( |k| \rightarrow 0 \). It is apparent from this behavior of \( \Pi_T \) that the current-current interaction remain unscreened at zero frequency [22].

In the ultrarelativistic limit \( (\epsilon_f^2 \approx p_f^2) \) the polarization tensors take the following forms:

\[ \Pi_{\text{ur}} = \frac{8\pi^2}{4\pi^2} \sum_{i=\pm} \rho_f^2 \sin^2 \theta \epsilon_f (1 - \theta_f \cot \theta_f), \]

(19)

\[ \Pi_{\text{ur}} = \frac{8\pi^2}{8\pi^2} \sum_{i=\pm} \rho_f^2 \left[ 1 - \sin^2 \theta \epsilon_f (1 - \theta_f \cot \theta_f) \right], \]

(20)

with \( \theta_f = \tan^{-1}(|k|/k_0) \). For \( \xi = 0 \), these results are same as those of Ref. [21]. In the next section, Eqs. (15)–(20) are used to evaluate the contribution of the ring diagrams.
It might not be out of context here to mention that once we have the expressions of $n^r$, a one-loop-corrected gluon propagator in polarized quark matter can easily be constructed. This forms the basis for calculation of various physical quantities, including the FLPs, which, without such medium corrections, suffer from infrared divergences [4,11,16].

III. GROUND-STATE ENERGY WITH CORRELATION

The leading contributions to the ground-state energy are given by the three terms viz. kinetic, exchange, and correlation energy densities, i.e.,

$$E = E_{\text{kin}} + E_{\text{ex}} + E_{\text{corr}} + O(r_q).$$

In the high-density limit $O(r_q)$ vanish, the result becomes exact [13]. $E_{\text{kin}}$ is given by [1,4]

$$E_{\text{kin}} = \frac{3}{16n_v^2} \left[ p_f(1 + \xi)^{3/2} \sqrt{p_0^2(1 + \xi)^{3/2} + m^2_q} \right. \times \left[ 2p_f^2(1 + \xi)^{3/2} + m^2_q \right]$$

$$- m_q^2 \ln \left( \frac{p_f(1 + \xi)^{3/2} + \sqrt{p_0^2(1 + \xi)^{3/2} + m^2_q}}{m_q} \right)$$

$$+ \left\{ [\xi \to -\xi] \right\}. \tag{22}$$

where $\xi$ is the polarization parameter with the condition $0 < \xi < 1$. Here $n^+_q$ and $n^-_q$ represent densities of spin-up and spin-down quarks respectively and $n_q = n^+_q + n^-_q$ denote total quark density. Then the Fermi momenta in the spin-polarized quark matter are defined as $p_f^\pm = \sqrt{p_0^2(1 + \xi)^{3/2} + m^2_q}$, where $p_f = (p^2n_q)^{1/2}$, is the Fermi momentum of the unpolarized matter (\(\xi = 0\)).

In the nonrelativistic (nr) and the ultrarelativistic (ur) limit kinetic energy density becomes [1,4],

$$E_{\text{kin}}^{\text{nr}} \approx \frac{3p_0^4}{20\pi^2 m_q} [(1 + \xi)^{3/2} + (1 - \xi)^{3/2}], \tag{23}$$

$$E_{\text{kin}}^{\text{ur}} \approx \frac{3p_0^2}{8\pi^2} [(1 + \xi)^{3/2} + (1 - \xi)^{3/2}]. \tag{24}$$

The first correction due to interaction to the ground-state energy is given by the exchange energy density. This arises from two quarks interchanging positions in the Fermi sea by exchanging a virtual gluon [23]. The exchange energy density was calculated in Ref. [4] within Fermi liquid theory approach. One can directly evaluate the loop diagram to calculate $E_{\text{ex}}$ as shown in Fig. 3 [1].

For polarized quark matter, $E_{\text{ex}}$, consists of two terms $E_{\text{ex}} = E_{\text{ex}}^f + E_{\text{ex}}^\nu$. Here [1],

$$E_{\text{ex}}^f = \frac{9}{2} \sum_{s = \pm} \int \frac{d^3p}{(2\pi)^3} \frac{d^3p'}{(2\pi)^3} \delta(p_f^+ - |p|)\delta(p_f^- - |p'|) f_{\nu'}p_f$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{d^3p'}{(2\pi)^3} \delta(p_f^+ - |p|)\delta(p_f^- - |p'|) f_{\nu'}p_f^{\nu}, \tag{25}$$

where $p_f^{\nu} = \sqrt{p_0^2(1 + \xi)^{3/2} + m^2_q}$ is the two-particle forward-scattering amplitude which is given by [1,4],

$$p_f^{\nu} = \frac{2g^2}{9\pi^2 m_q (p - p')^2} \left( \frac{1}{2m^2_q - p \cdot p' - (p \cdot s)(p' \cdot s')} \right)$$

$$+ \frac{1}{(e_p + m_q)(e_p' + m_q)} \times [ (m_q(e_p + m_q)(e_p' + m_q)]$$

$$+ m_q(e_p + m_q)(p \cdot s)(p' \cdot s')$$

$$+ (p \cdot p')(p \cdot s)(p' \cdot s') \right). \tag{27}$$

Now we come to the central aim of the present work, i.e., the evaluation of the correlation energy of dense quark matter with arbitrary spin polarization; the leading contribution to $E_{\text{corr}}$ can be obtained by adding the contributions of ring diagrams as shown in Fig. 4. It is to be noted that each of these diagrams are infrared divergent while their sum is finite [7,8,21,23,24] and are given by:

$$E_{\text{corr}} = E_{\text{corr}}^f + E_{\text{corr}}^\nu = \frac{1}{2} \int \frac{d^4K}{(2\pi)^4} \left[ \ln(1 - D^0\Pi_L) + D^0\Pi_L \right]$$

$$+ 2\ln(1 - D^0\Pi_T) + D^0\Pi_T \right]. \tag{30}$$

FIG. 3. Two-loop contribution to exchange energy density. Solid line represents the quark propagator and the wavy line represents gluon.

$$E_{\text{ex}}^f = \int \frac{d^3p}{(2\pi)^3} \frac{d^3p'}{(2\pi)^3} \delta(p_f^+ - |p|)\delta(p_f^- - |p'|) f_{\nu'}p_f^{\nu}, \tag{26}$$

FIG. 4. The series of ring diagrams.
Here \( D^0 \) is the free gluon propagator. The spatial integral of Eq. (30) can be reduced to one for the radial variable only, because all the polarization propagators are independent of the direction of three-momentum transfer \( k \). A Wick rotation is performed on the fourth component of the integration momentum \( k_0 \rightarrow i\kappa_0 \) so that space metric becomes Euclidean \([21,25]\). With \( K^2 \equiv k_0^2 + k^2 = -k^2 \) and \( \tan \theta_E = |k|/k_0 \), Eq. (30) becomes,

\[
E_{\text{cor}} = \frac{1}{(2\pi)^3} \int_0^{\infty} \int_0^{\infty} K^2 dK \int_0^{\pi/2} \sin^2 \theta_E d\theta_E \times \left\{ \ln \left( 1 + \frac{\Pi_L(K^2, \theta_E)}{K^2} \right) - \frac{\Pi_L(K^2, \theta_E)}{K^2} \right\} + 2 \frac{\Pi_T(K^2, \theta_E)}{K^2} \left\{ \ln \left( 1 + \frac{\Pi_T(K^2, \theta_E)}{K^2} \right) - \frac{\Pi_T(K^2, \theta_E)}{K^2} \right\} \right\} \right\}.
\]

Infrared divergences would arise in Eq. (31), if we were to expand the logarithms in powers of \( n \), because of the nonzero value of \( n_0(K_0, \theta_E) \) at \( K_0 = 0 \). This can be isolated by writing \( K_0 = 0 \) whenever possible in the integrand. Following Refs. \([23,26]\), we have

\[
E_{\text{cor}} \simeq \frac{1}{(2\pi)^3} \int_0^{\infty} K^2 dK \int_0^{\pi/2} \sin^2 \theta_E d\theta_E \times \left\{ \ln \left( 1 + \frac{\Pi_L(K^2, \theta_E)}{K^2} \right) - \frac{\Pi_L(K^2, \theta_E)}{K^2} \right\} + 2 \frac{\Pi_T(K^2, \theta_E)}{K^2} \left\{ \ln \left( 1 + \frac{\Pi_T(K^2, \theta_E)}{K^2} \right) - \frac{\Pi_T(K^2, \theta_E)}{K^2} \right\} \right\}.
\]

Performing \( K^2 \) integration the ring energy becomes \([21,23,26]\),

\[
E_{\text{cor}} \simeq \frac{1}{(2\pi)^3} \int_0^{\pi/2} \frac{1}{2} \sin^2 \theta_E d\theta_E \left\{ \Pi_L \left[ \ln \left( \frac{\Pi_L}{\theta_E^2} \right) - \frac{1}{2} \right] + 2 \Pi_T \left[ \ln \left( \frac{\Pi_T}{\theta_E^2} \right) - \frac{1}{2} \right] \right\}.
\]

To proceed further, we first express \( \Pi_L \) and \( \Pi_T \) in terms of polar variables. From Eq. (15) and Eq. (16) we obtain

\[
\Pi_L = \frac{g^2}{4\pi^2} \sum_{p^2 \leq E_F} \frac{p_L^2}{p^2} \coth \left[ \frac{\cos \theta_E}{v_F} \tan^{-1} (v_F \tan \theta_E) \right],
\]

\[
\Pi_T = \frac{g^2}{8\pi^2} \sum_{p^2 \leq E_F} \frac{p_T^2}{p^2} \coth \left[ \frac{\cos \theta_E}{v_F} \tan^{-1} (v_F \tan \theta_E) \right] + \frac{1}{2} \cos^2 \frac{\theta_E}{v_F} \tan^{-1} (v_F \tan \theta_E).\]

These are then inserted in Eq. (33) and \( \theta_E \) integration is performed numerically to estimate \( E_{\text{cor}} \) for various \( \xi \) as shown in Fig. 5.

![Fig. 5. Correlation energy \( E_{\text{corr}} \) as a function of density for different polarization parameters.](image)
FIG. 6. Comparison of exchange and correlation energy in polarized quark matter as a function of density for different polarization parameters.

contribute with opposite sign while in the ultrarelativistic limit, both of them contribute with same sign.

Using Eq. (34) and Eq. (35) correlation energy is estimated numerically that is valid for all the kinematic regimes. For this, following Refs. [1,4], we take $a_s = g^2/4\pi = 2.2$ and $m_q = 300$ MeV. In Fig. 5 we plot density dependence of correlation energy for various $\xi$. This shows that at a given density, with higher value of $\xi$, $E_{cont}$ increases. In Fig. 6, we compare exchange and correlation energy density. It shows system becomes more bound when quark matter changes its phase from unpolarized to polarized matter. With increasing $\xi$, $E_{cont}$ remains attractive; however, its value decreases as observed in both Figs. 5 and 6. In Fig. 7 we plot ground-state energy as a function of polarization parameter $\xi$. Hence we conclude that the quark matter interacting via OGE becomes polarized at lower density, whereas at higher density it becomes unpolarized. This clearly shows phase transition is first order and critical density is still around normal nuclear matter density $n_{q} = 0.16$ fm$^{-3}$ [1,4]. In this regime, it is seen that $E_{cont}$ makes the system more bound.

To derive the spin stiffness constant in the high-density limit using Eqs. (24), (29), and (39) we have

$$\beta_s = \frac{\partial^2 E}{\partial \xi^2} = \beta_s^{s+s} + \beta_s^{\pi+} + \beta_s^{\pi-}$$

$$= \frac{\rho_s^{\pi+}}{3\pi^2} \left[ 1 - \frac{g^2}{6\pi^2} - \frac{g^4}{384\pi^4} (\ln \rho_s - 0.286) \right]. \quad (40)$$

Here, the logarithmic term arises from the correlation correction.

FIG. 7. Total energy of quark liquid as a function of polarization parameter at $n_q = 0.1$ fm$^{-3}$ and $n_q = 0.25$ fm$^{-3}$. The critical density is around $n_q = 0.16$ fm$^{-3}$ in this case.

IV. SUMMARY AND CONCLUSION

In this work we derive the expressions for the gluon self-energy in spin-polarized quark matter and calculate the ground-state energy of such a system up to term $O(g^4)$ that includes corrections due to correlation effects. The analytical expressions for the correlation energy in two limiting cases (non-ultrarelativistic) are presented and compared with $E_{ex}$ and $E_{kin}$. It is shown that the correlation energy for polarized quark matter is comparatively larger than the unpolarized one, although it is always attractive. We find that numerically the contribution of $E_{cont}$ to the total energy is not found to be large and therefore, although qualitatively important, it is not the main factor in determining whether quark matter is ferromagnetic. With out this, however, the results remain incomplete because of the associated divergences of the terms beyond exchange diagrams [12,13]. Furthermore, this is an important first step to include the corrections due to correlations to the spin susceptibility [28-30]. In this work we present spin stiffness constant $\beta_s$ of dense quark system only in the high-density limit. A detailed study of this is now underway and shall be reported elsewhere [31].

The inclusion of correlation energy, as shown here, does not rule out the possibility of ferromagnetic phase transition in quark matter at low density; rather, it makes it more probable within the model and parameter set used by Tatsumi [1] that was borrowed from the bag model and was also used in Ref. [10]. Clearly the critical density at which the spin-polarized ferromagnetic state might appear depends strongly on the quark mass and the critical density increases with increasing mass; this might change our numerical estimates.

Further uncertainty to the estimation of the critical density from the present analysis comes from the fact that we here restrict ourselves only to OGE diagrams and one flavor system. In this regime, multigluon exchange processes [5] might play an important role. More work in this direction is therefore necessary to examine this issue, especially for multilavor systems that might appear in astrophysics. Leaving aside these questions, the evaluation of the gluon self-energy and the estimation of correlation energy in polarized matter, as mentioned in the text, nevertheless constitutes an important component for the study of the properties of dense quark system.

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APPENDIX

In the text Compton-scattering amplitudes are given as a sum of flip and nonflip terms. Here we give detail expression of $\mathcal{M}_{11}^s(P + K, P) + \mathcal{M}_{11}^f(P, P - K)$ with explicit spin indices. With the help of Eq. (5) and Eq. (7) we have

$$\mathcal{M}_{11}^s(P + K, P) + \mathcal{M}_{11}^f(P, P - K) = T_1 + T_3 + T_4 + T_5 + T_6,$$  \quad (A1)
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where

\[ T_1 = (b \cdot P)(a \cdot K) - (\bar{b} \cdot P)(\bar{a} \cdot K) \]
\[ = \frac{k^2 e_p}{m_q^2 (e_p + m_q)}[(p \cdot s)(p \cdot s') - (p \cdot s)(k \cdot s')] \]
\[ + \frac{k^2}{m_q^2 (e_p + m_q)}[(p \cdot s)(p \cdot s') - (p \cdot s)(k \cdot s')] \]
\[ + \frac{k^2}{m_q^2 (e_p + m_q)}[(p \cdot s)(p \cdot s') - (p \cdot s)(k \cdot s')] \]

\[ T_2 = (a \cdot P)(b \cdot K) - (\bar{a} \cdot P)(\bar{b} \cdot K) \]
\[ = \frac{\epsilon_p k^2}{m_q^2}(p \cdot s')(k \cdot s) + (p \cdot s)(k \cdot s') \]
\[ + \frac{\epsilon_p k^2}{m_q^2}(p \cdot s')(k \cdot s) + (p \cdot s)(k \cdot s') \]
\[ + \frac{\epsilon_p k^2}{m_q^2}(p \cdot s')(k \cdot s) + (p \cdot s)(k \cdot s') \]

\[ T_3 = 2P^2 sin^2 \theta \left[ \frac{1}{m_q^2}[(p \cdot s)(p \cdot s') + (p \cdot s')(k \cdot s) \right]
\[ - \frac{(p \cdot s)(k \cdot s') - 2(p \cdot s')(k \cdot s)}{m_q^2 (e_p + m_q)} \]
\[ \times [4(p \cdot s)(p \cdot s') + 2(p \cdot s)(k \cdot s')] \]
\[ - \frac{p^2}{m_q^2 (e_p + m_q)^2}[(p \cdot s)(p \cdot s') - (p \cdot s)(k \cdot s')] \]
\[ + (p \cdot s')(k \cdot s) - \frac{m^2 (e_p + m_q)^2}{m^2 (e_p + m_q)^2}[(p \cdot s)(p \cdot s') + (p \cdot s')(k \cdot s)] \]
\[ T_6 = \langle \mathbf{K} (a - \mathbf{b} - \mathbf{a} \cdot \mathbf{b}) \rangle = \frac{\langle \mathbf{K} \rangle}{m_0^2} \left[ \langle (p \cdot s)(k \cdot s') + (p \cdot s')(k \cdot s) \rangle \right. \\
+ \frac{2m_q}{(\epsilon_q + m_q)} \left[ \langle p \cdot s \rangle(k \cdot s') + \langle p \cdot s' \rangle(k \cdot s) \right] \\
- \frac{p^2}{(\epsilon_q + m_q)^2} \left[ \langle p \cdot s \rangle(k \cdot s')(p \cdot s')(k \cdot s) \right] \]

Similarly, one can derive terms like \[ \{ \mathcal{M}_{11}(P + K, P) - \mathcal{M}_{11}(P, P - K) \}, \{ \mathcal{M}_{22}(P + K, P) \pm \mathcal{M}_{22}(P, P - K) \}, \text{ etc.} \]

The help of Eq. (6) and Eq. (8). After explicit calculation of those terms, \( \Pi_{L,T} \) can be evaluated.
Spin susceptibility of degenerate quark matter

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The expression for spin susceptibility $\chi$ of degenerate quark matter is derived with corrections up to $O(g^4 \ln g^2)$. It is shown that, at low density, $\chi^{-1}$ changes sign and turns negative, indicating a ferromagnetic phase transition. To this order, we also calculate sound velocity $c_1$ and incompressibility $K$ with arbitrary spin polarization. The estimated values of $c_1$ and $K$ show that the equation of state of the polarized matter is stiffer than that of unpolarized matter. Finally, we determine the finite temperature corrections to the exchange energy and derive corresponding results for the spin susceptibility.

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I. INTRODUCTION

One of the active areas of high-energy physics research is the exploration of the so-called quantum chromodynamics (QCD) phase diagram. In particular, with the advent of ultrarelativistic heavy ion beams at the Relativistic Heavy Ion Collider (RHIC) and CERN, and with the upcoming facilities at GSI where compressed baryonic matter is expected to be produced, such studies assume special importance. In addition to the laboratory experiments, various astrophysical objects, such as neutron stars and quark stars, provide natural sites where many of the theoretical conjectures about the various phases of quark matter can be tested. The latter, in the present context, is more relevant here, as we study the possibility of a para-ferro phase transition in a dense quark system interacting via one-gluon exchange.

The original idea of a para-ferro phase transition in quark matter was proposed recently in Ref. [1] where the possibility of a Bloch-like phase transition [2] was studied; it was shown that spin-polarized quark matter might exist at low density [3]. The underlying mechanism of such a phase transition is analogous to what was originally proposed for the degenerate electron gas [2]. There, for the Coulomb interaction, it was shown that the exchange correction to energy is attractive, which at low density wins over the kinetic energy, thus giving rise to a ferromagnetic state [2]. In Ref. [1], a variational calculation was performed to show that it is indeed possible to have spin-polarized quark matter at low density of a strange quark system, whereas for the light quark matter this never happens [1]. Similar differences between both light and strange quark matter, albeit in a different context, were observed earlier [4]. However, in Ref. [3] it was shown that both the light and heavy flavor systems can exhibit such phase transitions, although the critical density for the strange matter is higher than that for the light quark systems. Such investigations were also performed in Refs. [5-8] and also in Refs. [9,10], where the calculation was extended to include thermal effects. A Bloch-like phase transition for the strange quark was also reconfirmed in Ref. [11].

One shortcoming of all these works, including Ref. [11], is that the calculations are restricted to the Hartree-Fock level and the terms beyond the exchange diagrams, commonly known as the correlation energy [12-16], are ignored. Without such corrections, however, the calculations are known to remain incomplete since the higher order terms are plagued with infrared divergences that arise out of the exchange of massless gluons. This indicates the failure of the naive perturbation series. We know that this problem can be cured by reorganizing the perturbation theory where a particular class of diagrams, that is, the bubbles, are resummed in order to obtain a finite result. Originally, as is well known, this was done by Gell-Mann and Brueckner [17] while calculating the ground-state energy of a degenerate electron gas. The contribution of the bubbles involve terms of $O(g^4 \ln g^2)$, indicating the nonperturbative nature of the correction [18-21].

In the present work, we calculate the spin susceptibility $\chi$ of a dense quark system with corrections due to correlations, that is, containing terms up to $O(g^4 \ln g^2)$. This requires knowledge of the ground-state energy (GSE) of spin-polarized matter with the inclusion of bubble diagrams. The GSE of the polarized quark matter was calculated only recently in Ref. [17], which serves as the starting point of the present article. This work is very similar to that of Brueckner and Swada [22] and that of Refs. [23,24] and is applied to the case of QCD matter. Unlike degenerate electron gas, however, we have both the electric and magnetic interactions; the calculation is performed relativistically, while the nonrelativistic results appear as a limit.

The spin susceptibility $\chi$, for quark matter up to $O(g^2)$, was already calculated in Ref. [1], which we only briefly discuss. Subsequently, the non-Fermi-liquid corrections to $\chi$ were also studied in Refs. [9,10]. These studies provided further motivation for undertaking the present endeavor to include correlation corrections, without which, as mentioned already, the perturbative evaluation of $\chi$ remains incomplete. In addition, we also calculate the incompressibility and sound velocity for spin-polarized quark matter with correlation corrections that involve the evaluation of a single-particle energy at the Fermi surface. These quantities are of special interest for applications to astrophysics. Moreover, we also evaluate the exchange energy density at nonzero temperature.
and determine the corresponding corrections to the spin susceptibility.

The plan of the article is as follows. In Sec. II we calculate spin susceptibility with correlation corrections for degenerate quark matter. Analytic expressions are presented both in ultrarelativistic (UR) and nonrelativistic (NR) limits. In Sec. III, we evaluate exchange energy density and spin susceptibility at nonzero temperature. In Sec. IV we summarize and conclude. Detailed expressions of the intermediate expressions from which \( \chi \) is derived are relegated to the Appendix.

II. SPIN SUSCEPTIBILITY

The spin susceptibility of quark matter is determined by the change in energy of the system as its spins are polarized [22]. We introduce a polarization parameter \( \xi = (n^+ - n^-)/n_q \) with the condition \( 0 \leq \xi \leq 1 \), where \( n^+ \) and \( n^- \) correspond to the densities of spin-up and spin-down quarks, respectively, and \( n_q = n^+ + n^- \) denotes total quark density. The Fermi momenta in the spin-polarized quark matter are then \( p_F(1 + \xi) \) and \( p_F(1 - \xi) \), where \( p_F = (\pi n_q)^{1/3} \), is the Fermi momentum of the unpolarized matter (\( \xi = 0 \)). In the small-\( \xi \) limit, the GSE behaves as [1]

\[
E(\xi) = E(\xi = 0) + \frac{1}{2} \beta_\xi \xi^2 + O(\xi^4). \tag{1}
\]

Here, \( \beta_\xi = \frac{\partial^2 E(\xi)}{\partial \xi^2} \big|_{\xi=0} \) is defined as the spin stiffness constant analogous to Refs. [16,21]. The spin susceptibility \( \chi \) is proportional to the inverse of the spin stiffness; mathematically, \( \chi = 2\beta_\xi^{-1} \) [26]. Note that in Eq. (1), the first term corresponds to unpolarized matter energy.

Now, the leading contributions to the GSE are given by three terms, that is, kinetic, exchange, and correlation energy densities [16]:

\[
E = E_{\text{kin}} + E_{\text{ex}} + E_{\text{corr}}. \tag{2}
\]

The total kinetic energy density for spin-up and spin-down quarks becomes [1,11]

\[
E_{\text{kin}} = \frac{3}{16\pi^2} \left\{ p_F(1 + \xi)^{3/2} \times \sqrt{p_F^2(1 + \xi)^{3/2} + m_q^2} \left[ 2p_F^2(1 + \xi)^{3/2} + m_q^2 \right] \right. \\
- m_q^4 \ln \left( \frac{p_F(1 + \xi)^{3/2} + \sqrt{p_F^2(1 + \xi)^{3/2} + m_q^2}}{m_q} \right) \left. + [\xi \to -\xi] \right\}, \tag{3}
\]

where \( m_q \) is the quark mass.

The exchange energy density \( E_{\text{ex}} \) was calculated in Ref. [11] within a Fermi liquid theory approach. One can also directly evaluate the two loop diagrams [1] to obtain

\[
E_{\text{ex}}^\partial = \frac{2}{3} \sum_{s_{\text{flip}}} \int \int \frac{d^3 p}{(2\pi)^3} \frac{d^3 p'}{(2\pi)^3} \frac{\Delta(p_f' - |p|)\Delta(p_f' - |p'|) f_{p_f'}^{\text{dir}}}{f_{p_f}^{\text{dir}}}, \tag{4}
\]

\[
E_{\text{ex}}^\epsilon = \frac{9}{2} \int \int \frac{d^3 p}{(2\pi)^3} \frac{d^3 p'}{(2\pi)^3} \frac{\Delta(p_f' - |p|)\Delta(p_f' - |p'|) f_{p_f'}^{\text{flip}}}{f_{p_f}^{\text{flip}}}, \tag{5}
\]

where \( f_{p_f'}^{\text{dir}} \) and \( f_{p_f'}^{\text{flip}} \) stand for nonflip \( (s = s') \) and flip \( (s = -s') \) forward scattering amplitudes given in Refs. [1,11,16]. Here, \( E_{\text{ex}} = E_{\text{ex}}^\partial + E_{\text{ex}}^\epsilon \) can be estimated numerically. However, an analytical evaluation of these integrals is possible in the UR and NR limits, as reported in Refs. [1,11,16].

The next higher order correction to the GSE beyond the exchange term is the correlation energy \( E_{\text{corr}} \) [12–15]. A detailed calculation of the correlation energy for spin-polarized matter was derived in Ref. [16], which we quote here:

\[
E_{\text{corr}} \approx \frac{1}{(2\pi)^2} \int_0^{\pi/2} \sin^2 \theta_d d\theta_d \left\{ \frac{\Pi_L}{\xi_f} \left[ \ln \left( \frac{\Pi_L}{\xi_f} \right) - \frac{1}{2} \right] \right. \\
+ \left. 2 \Pi_T \left[ \ln \left( \frac{\Pi_T}{\xi_f} \right) - \frac{1}{2} \right] \right\}, \tag{6}
\]

with \( \theta_d = \tan^{-1}(k_f/k_0) \). The relevant \( \Pi_L \) and \( \Pi_T \) are determined to be [16]

\[
\Pi_L = \frac{g^2}{8\pi^2} \sum_{s_{\text{flip}}} P_{s_{\text{flip}}}^2 \sin^2 \theta_d \left[ 1 - \frac{\cot \theta_d}{\xi_f^2} \tan^{-1} \left( \frac{v_f}{\xi_f} \tan \theta_d \right) \right], \tag{7}
\]

\[
\Pi_T = \frac{g^2}{8\pi^2} \sum_{s_{\text{flip}}} P_{s_{\text{flip}}}^2 \cot \theta_d \left[ \frac{\cot \theta_d}{\xi_f^2} + \left( 1 + \frac{\cot^2 \theta_d}{\xi_f^4} \right) \tan^{-1} \left( \frac{v_f}{\xi_f} \tan \theta_d \right) \right]. \tag{8}
\]

The spin susceptibility is given by Ref. [1] as

\[
\chi^{-1} = \frac{1}{2} \frac{\partial^2 E(\xi)}{\partial \xi^2} \bigg|_{\xi=0}. \tag{9}
\]

We have \( \chi^{-1} = \chi_{\text{kin}}^{-1} + \chi_{\text{ex}}^{-1} + \chi_{\text{corr}}^{-1} \). The kinetic and exchange contributions, evaluated in Ref. [1], are given by

\[
\chi_{\text{kin}} = \frac{\Pi_L}{\xi_f^2}, \tag{10}
\]

\[
\chi_{\text{ex}} = -\frac{g^2 p_F^3}{18\pi^2} \left[ 2 - \frac{6P_f^3}{e_f^2} - \frac{3P_f e_f - m_q^2}{e_f^2} \ln \left( \frac{p_f + e_f}{m_q} \right) \right], \tag{11}
\]

To determine the correlation corrections to spin susceptibility we expand the terms in curly braces from Eq. (6) in powers of the polarization parameter \( \xi \), which gives

\[
\Pi_L \left[ \ln \left( \frac{\Pi_L}{\xi_f^2} \right) - \frac{1}{2} \right] + 2 \Pi_T \left[ \ln \left( \frac{\Pi_T}{\xi_f^2} \right) - \frac{1}{2} \right] = (\Delta_{\text{UL}} + B_{\text{UL}}) + \xi^2(\Delta_{\text{UL}} + B_{\text{UL}}) + O(\xi^4). \tag{12}
\]
Here, \( A_{\text{ql}} \) and \( B_{\text{or}} \) correspond to unpolarized matter terms and the detailed expressions of \( A_{\text{ql}} \) and \( B_{\text{or}} \) are given in the Appendix. \( \chi_{\text{cor}}^{-1} \) is

\[
\chi_{\text{cor}}^{-1} = \frac{1}{2} \left. \frac{\partial^2 E_{\text{cor}}(\xi)}{\partial \xi^2} \right|_{\xi=0} \\
\approx \frac{1}{2} \int_{0}^{\pi/2} \sin^2 \theta \rho d\theta (A_{\text{ql}} + B_{\text{or}}).
\]  
(13)

From this expression, and with the help of the expressions presented in the Appendix, can be estimated numerically. The results for the two limiting cases, however, can be obtained analytically, as we present in the following two sections.

### A. UR limit

In the UR limit, the kinetic, exchange, and correlation energies are [16]

\[
E_{\text{kin}}^{\text{UR}} = \frac{3}{8\pi^2} (1 + \xi)^{4/3} + (1 - \xi)^{4/3},
\]

\[
E_{\text{ex}}^{\text{UR}} = \frac{g^2}{32\pi^4} [3(1 + \xi)^{4/3} + (1 - \xi)^{4/3} + 2(1 - \xi^2)^{2/3}],
\]

\[
E_{\text{cor}}^{\text{UR}} = \frac{g^4}{2048\pi^6} [3(1 + \xi)^{4/3} + (1 - \xi)^{4/3} + 2(1 - \xi^2)^{2/3}].
\]

With the help of Eq. (1) each energy contribution to the susceptibility is

\[
\chi_{\text{kin}}^{-1} = \frac{\rho_f^2}{6\pi^2},
\]

\[
\chi_{\text{ex}}^{-1} = \frac{g^2}{32\pi^4},
\]

\[
\chi_{\text{cor}}^{-1} = -\frac{g^4}{2048\pi^6} (\ln r_s - 0.286),
\]

where \( r_s = g^2(2\pi)^3 \). From Eq. (15), the sum of all the contributions to the susceptibility can be written as [16]

\[
\chi = \chi_F \left[ 1 - \frac{g^2}{32\pi^4} + \frac{g^4}{2048\pi^6} (\ln r_s - 0.286) \right]^{-1},
\]

where \( \chi_F \) is the noninteracting susceptibility [23,24].

### B. NR limit

Now, we use the NR limit to calculate spin susceptibility in order to compare our results with those of a dense electron gas [22-24,26] interacting via the static Coulomb potential. In this limit, kinetic and exchange energy densities are [1,11,16]

\[
E_{\text{kin}}^{\text{NR}} = \frac{3}{8\pi^2} (1 + \xi)^{4/3} + (1 - \xi)^{4/3},
\]

\[
E_{\text{ex}}^{\text{NR}} = \frac{g^2}{32\pi^4} [3(1 + \xi)^{4/3} + (1 - \xi)^{4/3} + 2(1 - \xi^2)^{2/3}],
\]

\[
E_{\text{cor}}^{\text{NR}} = \frac{g^4}{2048\pi^6} [3(1 + \xi)^{4/3} + (1 - \xi)^{4/3} + 2(1 - \xi^2)^{2/3}].
\]

The contribution to the susceptibility from kinetic and exchange energy densities yields

\[
\chi_{\text{kin}}^{-1} = \frac{\rho_f^2}{6\pi^2 m_q},
\]

\[
\chi_{\text{ex}}^{-1} = -\frac{g^2}{2048\pi^6} m_q P_f^3.
\]

We calculate the contribution to the spin susceptibility beyond the exchange correction. For this we must first evaluate the correlation energy in this limit.

The dominant contribution to the correlation energy is found to be

\[
E_{\text{cor}}^{\text{NR}} = \frac{g^4}{2048\pi^6} m_q P_f^3 [1 - \ln 2 - 0.286],
\]

\[
\chi_{\text{cor}}^{-1} = \frac{g^4}{2048\pi^6} m_q P_f^3.
\]

Note that, here, the correlation energy is independent of spin polarization \( \xi \). For the spin parallel interactions, \( \xi \)-dependent terms contribute to an opposite sign and therefore cancel each other out. For \( \xi = -\xi' \), the integral on \( x \) takes the form

\[
I = \int_{0}^{\pi/2} x \ln \left[ \frac{1 - \xi' \ln \left( x + 1 + 1 \right) - 1}{x - 1 - \xi} \right] \theta \left[ 1 - x \left( \frac{1 + \xi}{1 - \xi} \right) \right] dx.
\]

Expanding the natural logarithm in terms of \( \xi \) and retaining up to \( O(\xi^2) \), we have

\[
I \simeq \frac{2}{3} \left[ (1 - \ln 2) - \frac{1}{6} \xi^2 \right].
\]

Using Eqs. (19), (21), and (22) we have

\[
E_{\text{cor}}^{\text{NR-\xi}} \simeq \frac{g^4}{128\pi^6} m_q P_f^3 \left[ (1 - \ln 2) - \frac{1}{6} \xi^2 \right].
\]

It should be mentioned that similar expressions for a degenerate electron gas interacting via a static Coulomb potential can be found in Ref. [26]. From Eqs. (20) and (23) it is clear that spin antiparallel states are attractive, in contrast to the parallel states obtained by the Pauli exclusion principle. In this limit the correlation contribution to the susceptibility is found to be

\[
\chi_{\text{cor}}^{-1} = \frac{g^4}{2048\pi^6} m_q P_f^3.
\]
The single-particle energy at the Fermi surface or the chemical potential of spin-up quark turns out to be
\[ \mu_+ = \mu_0 + \mu_+^{\text{kin}} + \mu_+^{\text{corr}}. \]
Similarly, \( \mu_- \) can be obtained by replacing \( \mu_+ \) with \( \mu_- \) in Eq. (28). In Ref. [11], the chemical potential was determined within the Fermi liquid theory approach up to \( O(g^2) \). However, here we calculate \( \mu_+ \) with a different approach up to \( O(g^4 \ln g^2) \).

Using Eqs. (27) and (28), the incompressibility becomes
\[
K^{\text{cor}} = \frac{3}{2} \rho \left\{ (1 + \xi) \frac{y}{3\rho} + (1 - \xi) \right\} \\
+ \frac{g^2}{12\pi^2} \left\{ (1 + \xi) \frac{y}{3\rho} + (1 - \xi) \frac{y}{3\rho} + 2(1 - \xi^2) \frac{y}{3\rho} \right\} \\
+ \frac{g^4 \ln g^2}{768\pi^4} \left\{ (1 + \xi) \frac{y}{3\rho} + (1 - \xi) \frac{y}{3\rho} + 2(1 - \xi^2) \frac{y}{3\rho} \right\}. 
\]

Another interesting quantity to calculate is the first sound velocity, which is given by the first derivative of pressure with respect to energy density. Mathematically [11],
\[
\frac{\partial \mu_+}{\partial n_q^+} = \frac{2\pi^2}{3} \rho \left\{ (1 + \xi) \frac{y}{3\rho} + (1 - \xi) \frac{y}{3\rho} \right\} \left( (1 + \xi) \frac{y}{3\rho} + (1 - \xi) \frac{y}{3\rho} \right) \\
+ \frac{g^4 \ln g^2}{768\pi^4} \left\{ (1 + \xi) \frac{y}{3\rho} + (1 - \xi) \frac{y}{3\rho} + 2(1 - \xi^2) \frac{y}{3\rho} \right\}. 
\]

The second and last terms in the curly braces correspond to the exchange and correlation contributions, respectively. Similarly, \( \frac{\partial \mu_+}{\partial n_q^-} \) can be obtained by replacing \( \xi \) with \( -\xi \).

In Fig. 2, we plot the density dependencies of the incompressibility with a correlation correction. This shows that, for a higher value of the order of parameter \( \xi \), the incompressibility becomes higher for the same value of the density. Thus, numerical values for incompressibility and sound velocity show that the equation of state for polarized quark matter is stiffer than for unpolarized matter [11].
III. SUSCEPTIBILITY AT NONZERO TEMPERATURE

In this section we calculate the exchange energy density $E_{\text{ex}}$ at low temperature ($T < \frac{m}{\sqrt{2}}$), for which we replace $\theta (p' \cdot p - p \cdot p')$ of Eqs. (4) and (5) with a proper Fermi distribution function. In the UR limit, the angular averaged interaction parameter is given by \[ j_\text{ur} = \frac{g^2}{2 m^2} \int \frac{d^3 p}{(2\pi)^3} \frac{d^3 p'}{(2\pi)^3} \rho_p n_p(T) n_{p'}(T) \]

\[ + \frac{g^2}{8\pi^2} \cdot \frac{T^2}{3} \cdot \frac{p^2}{m} \cdot \frac{1 + \xi}{1 - \xi} \cdot \frac{1}{3} \cdot \frac{1}{3}. \] (32)

The spin nonflip contribution to the exchange energy density is

\[ E_{\text{ex}} = \frac{g^2}{2 m^2} \int \frac{d^3 p}{(2\pi)^3} \frac{d^3 p'}{(2\pi)^3} \rho_p n_p(T) n_{p'}(T) \]

\[ \times \frac{1 + \xi}{1 - \xi} \cdot \frac{1}{3} \cdot \frac{1}{3}. \] (33)

Here $n_p(T)$ is the Fermi distribution function. Similarly, $E_{\text{ex}}$ can be evaluated. The total $E_{\text{ex}}$ at low temperature is found to be

\[ E_{\text{ex}} = \frac{g^2}{32\pi^4} \int \frac{d^3 p}{(2\pi)^3} \int \frac{d^3 p'}{(2\pi)^3} \rho_p n_p(T) n_{p'}(T) \]

\[ + \frac{g^2}{24\pi^4} \cdot \frac{T^2}{3} \cdot \frac{p^2}{m} \cdot \frac{1 + \xi}{1 - \xi} \cdot \frac{1}{3} \cdot \frac{1}{3}. \] (34)

The kinetic energy density can be written as

\[ E_{\text{kin}} = \frac{g^2}{8\pi^2} \int \frac{d^3 p}{(2\pi)^3} \left[ 1 + \xi \right] \cdot \frac{1 + \xi}{1 - \xi} \cdot \frac{1}{3} \cdot \frac{1}{3}. \] (35)

From Eq. (1) each energy contribution to the susceptibility is

\[ \chi_{\text{ex}} = \frac{g^2}{6\pi^2} \left( 1 - \frac{\pi^2 T^2}{3 p_f^2} \right), \]

\[ \chi_{\text{kin}} = -\frac{g^2}{3\pi^2} \left( 1 + \frac{\pi^2 T^2}{3 p_f^2} \right). \] (36)

Note that the $T$-independent terms of these expressions are identical to those in Eqs. (14) and (15). Thus, the susceptibility at nonzero temperature is given by

\[ \chi_{\text{ex}} = \chi_{\text{kin}} \left[ 1 - \frac{g^2}{2m^2} \left( 1 + \frac{\pi^2 T^2}{3 p_f^2} \right) \right]^{-1}. \] (37)

In the NR limit, the interaction parameter takes the following form [1,11]:

\[ f_{\text{pp'}} = \frac{-2g^2}{9} \left( 1 + s \cdot s' \right). \] (38)

For the spin antiparallel interaction $s = -s'$, then $f_{\text{pp'}} = 0$. Thus, the contribution due to the scattering of quarks with unlike spin states vanishes and the dominant contribution to energy density comes from the parallel spin states ($s = s'$). By performing the angular integration of Eq. (4), the exchange energy density up to term $O(T^2)$ becomes

\[ E_{\text{ex}} = -\frac{g^2}{8\pi^2} \sum \int \frac{d^3 p}{(2\pi)^3} \int \frac{d^3 p'}{(2\pi)^3} \ln \left| \frac{p + p'}{p - p'} \right| \]

\[ \simeq -\frac{g^2}{8\pi^2} \int \frac{d^3 p}{(2\pi)^3} \int \frac{d^3 p'}{(2\pi)^3} \left[ 1 + \xi \right] \cdot \frac{1 + \xi}{1 - \xi} \cdot \frac{1}{3} \cdot \frac{1}{3}. \] (39)

The kinetic energy density is found to be

\[ E_{\text{kin}} = \frac{3p_f^2}{20\pi^2 m_q} \left( 1 + \xi \right) \cdot \frac{1 + \xi}{1 - \xi} \cdot \frac{1}{3} \cdot \frac{1}{3}. \] (40)

A separate contribution from kinetic and exchange energies to susceptibility becomes

\[ \chi_{\text{kin}}^{-1} = f_p \left[ 1 - \frac{2 m^2 T^2}{3 p_f^2} \right], \]

\[ \chi_{\text{ex}}^{-1} = -\frac{g^2}{18\pi^2} \left( 1 - \frac{m^2 T^2}{6 p_f^2} \right). \] (41)

Thus, at low temperature the susceptibility turns out to be

\[ \chi_{\text{ex}} = \chi_{\text{kin}} \left[ 1 - \frac{g^2}{3\pi^2} \left( 1 + \frac{m^2 T^2}{6 p_f^2} \right) \right]^{-1}. \] (42)

IV. SUMMARY AND CONCLUSION

In this work we derive the spin susceptibility for degenerate quark matter with corrections due to correlation contributions. Analytic expressions for susceptibility are also derived in both the UR and NR limits. It is observed that at low density susceptibility changes sign and becomes negative, thus suggesting the possibility of a ferromagnetic phase transition. In addition, we also derive single-particle energy, sound velocity, and incompressibility up to $O(g^4 \ln g^2)$. As far as the equation of state is concerned, in the present model, we find that the equation of state for polarized matter is stiffer than that of unpolarized matter. We also determine the exchange energy and susceptibility at nonzero temperature of spin-polarized quark matter.

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APPENDIX

To calculate the correlation contribution to the spin susceptibility we have, from Eq. (12),

\[
A_{1L} = -\frac{g^4 p_f^2 \sec^4 \theta_E \csc^6 \theta_E}{1152 \pi^2 \varepsilon_f^2 (m_f^2 + p_f^2 \sec^2 \theta_E)^2} \times \ln \left( \frac{\csc^2 \theta_E [p_f - e_f \cot \theta_E \tan^{-1}(v_f \tan \theta_E)]}{2 \pi \varepsilon_f} \right)
\]

\[
\times \left\{ \begin{array}{l}
64 e_f^2 \cos^2 \theta_E \tan^{-1}(v_f \tan \theta_E)(p_f^2 + m_f^2 \cos^2 \theta_E) \\
2 p_f e_f \sin 2 \theta_E \tan^{-1}(v_f \tan \theta_E) \left[ 12 m_f^4 + 51 m_f^2 p_f^2 \\
+ m_f^4 (4 m_f^2 + 5 p_f^2) \cos 4 \theta_E + 68 m_f^2 p_f^2 \\
+ 4 p_f e_f \sin \theta_E \left[ 6 m_f^6 + 29 m_f^4 p_f^2 \\
+ m_f^4 (2 m_f^2 + 3 p_f^2) \cos 4 \theta_E + 36 m_f^2 p_f^2 \\
+ 4 m_f^2 (2 m_f^4 + 4 m_f^2 p_f^2 + 3 p_f^2) \cos 2 \theta_E + 16 p_f^4 \right] \right],
\end{array} \right.
\]

(A1)

\[
B_{1T} = \frac{g^4 p_f^2 \cos^2 \theta_E \csc^4 \theta_E}{1152 \pi^2 \varepsilon_f^2 (m_f^2 \cos^2 \theta_E + p_f^2)} \times \ln \left( \frac{g^2 \cot \theta_E \csc^2 \theta_E}{8 \pi \varepsilon_f} \right) \left[ 2 \tan^{-1}(v_f \tan \theta_E) \right]
\]

\[
\times \left\{ \begin{array}{l}
(m_f^2 \cos^2 \theta_E + p_f^2) - p_f e_f \sin 2 \theta_E \\
(32 e_f^2 \tan^{-1}(v_f \tan \theta_E))(m_f^2 \cos^2 \theta_E + p_f^2)^2 \\
- 8 p_f e_f \left[ m_f^4 + p_f^4 + m_f^2 p_f^2 (1 + \cos^2 \theta_E) \right] \sin^2 2 \theta_E \\
+ 2 p_f \tan^{-1}(v_f \tan \theta_E) \sin 2 \theta_E \\
\times \left[ 8 m_f^6 + 3 m_f^4 p_f^2 + m_f^2 p_f^4 \cos 4 \theta_E + 36 m_f^2 p_f^2 \\
+ 4 m_f^2 (2 m_f^4 + 4 m_f^2 p_f^2 + 3 p_f^2) \cos 2 \theta_E + 16 p_f^4 \right].
\end{array} \right.
\]

(A2)

with \( v_f = p_f / e_f \).
Symmetric and anti-symmetric Landau parameters and magnetic properties of dense quark matter

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We calculate the dimensionless Fermi liquid parameters (FLPs), \( f_{\text{sym}}^{(\pm)} \) and \( f_{\text{asym}}^{(\pm)} \), for spin asymmetric dense quark matter. In general, the FLPs are infrared divergent due to the exchange of massless gluons. To remove such divergences, the hard density loop (HDL) corrected gluon propagator is used. The FLPs so determined are then invoked to calculate magnetic properties such as magnetization \( (M) \) and magnetic susceptibility \( \chi_M \) of spin polarized quark matter. Finally, we investigate the possibility of magnetic instability by studying the density dependence of \( (M) \) and \( \chi_M \).

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I. INTRODUCTION

The study of strongly interacting matter has been an area of contemporary research for quite some time now. Such studies are usually made in the extreme condition of temperature and/or density. The high temperature (\( T \)) studies are more relevant to the ultrarelativistic heavy ion collisions while the investigations involving high chemical potential (\( \mu \)) or extreme case of cold matter are more appropriate to astrophysics [1]. It has been recently shown that the degenerate quark matter can show para-ferro phase transition below a critical density [3]. To examine this possibility, in Ref. [3], a variational calculation was performed. Subsequently, various other calculations were also performed in different formalisms to investigate such a possibility with varied conclusions [4–11].

The issue of spontaneous phase transition in dense quark system at zero temperature was also examined in [12] by invoking relativistic Fermi liquid theory (RFLT). In particular, this was accomplished by calculating the chemical potential (\( \mu \)) and energy density of degenerate quark matter in terms of the Landau parameters (LPs). The RFLT was first developed by Baym and Chin [13,14] to study the properties of high density nuclear matter. However, the formalism developed in Ref. [13] is valid for unpolarized matter and LPs calculated there are spin averaged. Here, on the other hand, we deal with polarized quark matter which requires evaluation of the LPs with explicit spin dependencies.

Recently, in [15,16] the authors have studied the magnetic properties of degenerate quark matter in presence of weak uniform external magnetic field \( B \). Similar investigation was also made in Ref. [11] by evaluating the effective potential and employing quark magnetic moment as an order parameter. These calculations were, however, restricted to the case of unpolarized matter. On the contrary, our concern here is the magnetic properties of polarized quark system. Consequently, we first determine various spin combination of LPs such as spin symmetric \( (f_{\text{sym}}^{(\pm)}) \) and spin anti-symmetric \( (f_{\text{asym}}^{(\pm)}) \) parameters and express quantities such as magnetization and magnetic susceptibility in terms of these parameters. It is needless to mention that unlike [11,15,16], the expressions for \( (M) \) and \( \chi_M \), as presented here, depend on the spin polarization parameter \( \xi = (n^+ - n^-)/(n^+ + n^-) \), where \( n^+ \) and \( n^- \) correspond to densities of spin-up and spin-down quarks, respectively.

It is well known that the calculations of LPs require evaluation of the forward scattering amplitudes which are plagued with infrared divergences arising out of the exchange of massless gluons. Formally, such divergences can be removed by using HDL corrected gluon propagator. This can also be achieved by introducing screening mass for the gluons. Such regularizations are necessary for the evaluation of individual LPs. On the other hand, in various physical quantities like the ones we calculate here, the LPs appear in particular combinations where such divergences cancel at least to the order with which we are presently concerned.

The plan of the article is as follows. In Sec. II we derive the expressions of LPs for polarized quark matter. In Sec. III, we calculate magnetic susceptibility in terms of LPs with explicit spin dependencies both with bare and HDL corrected gluon propagator. In Sec. IV we summarize and conclude.

II. SYMMETRIC AND ANTI-SYMMETRIC LANDAU PARAMETERS

In this section we calculate LPs for spin polarized quark matter. We are dealing with quasiparticles whose spins are all eigenstates of the spin along a given direction, viz. \( z \). The quasiparticle interaction can be written as the sum of two parts, viz. spin symmetric \( (f_{\text{sym}}^{(\pm)}) \) and anti-symmetric \( (f_{\text{asym}}^{(\pm)}) \) parameters [14,16]:

\[
f_{\text{sym}}^{(\pm)} = f_{\text{sym}}^{(\pm)} + (s \cdot s') f_{\text{asym}}^{(\pm)},
\]

(1)
Assuming that the spins are randomly oriented with respect to the momentum, we take average over the angles $\theta_1$ and $\theta_2$ corresponding to spins $s$ and $s'$. The angular averaged interaction parameter is given by \[ f'_{pp'}^{s,s'} = \int \frac{d\Omega_1}{4\pi} \int \frac{d\Omega_2}{4\pi} f_{pp'}^{s,s'} |_{p=p',s'=s'}. \] (2)

Here the spin may be either parallel ($s = s'$) or antiparallel ($s = -s'$) [3,12]. Thus the scattering possibilities are denoted by $(+, +), (+, -), (-, -), \ldots$, etc. The interaction parameters can now be redefined as

\[ f_{pp'}^{++} = f_{pp'}^{sym} + f_{pp'}^{asym} \]
\[ f_{pp'}^{+-} = f_{pp'}^{sym} - f_{pp'}^{asym} \] (3)

Once these interaction parameters are known, the FLPs can be determined by expanding $f_{pp'}^{sym(asym)}$ into the Legendre polynomial:

\[ f_{pp'}^{l'} = (2l + 1) \int d\Omega P_l(\cos \theta) f_{pp'}^{l'} \] (4)

where $\cos \theta = \vec{p} \cdot \vec{p}'$. We define symmetric and antisymmetric part of LPs $f_{pp'}^{sym(asym)}$ what one does to dealing with the isospins in nuclear matter [12,14]:

\[ f_{pp'}^{+(--)asym} = \frac{1}{2} f_{pp'}^{++} + f_{pp'}^{--}, \]
\[ f_{pp'}^{+(--)sym} = \frac{1}{2} f_{pp'}^{++} - f_{pp'}^{--}. \] (5)

It should be noted here that, $f_{pp'}^{++} = f_{pp'}^{--}$. The dimensionless LPs are defined as $F_{l'}^{sym(asym)} = N'_{l'} f_{ll'}^{sym(asym)}$ [12], where $N'_{l'}$ is the density of states at the Fermi surface, which can be written as

\[ N'_{l'} = \int \frac{d^3p}{(2\pi)^3} \delta(p_{Sz} - \mu^s) \]
\[ = N_{l'} \frac{\partial p}{2\pi^2} \frac{S_{l'} p_{sz}}{\delta p_{sz}} |_{p=p'}, \] (6)

Here, $N_{l'}$ is the color factor, $p_{sz}$ and $\mu^s$ are the spin dependent quasiparticle energy and chemical potential, respectively. It is evident from Eq. (6) that for spin polarized matter, the density of states is spin dependent. This, as we shall see, makes the calculation cumbersome. In the above expression $(\partial p/\delta p_{sz})|_{p=p'}$ is the inverse Fermi velocity ($1/v_f$), where $v_f$ is given by [12,15]

\[ v_f = \frac{p_f}{\mu^s} = \frac{N_{l'} p_{sz} f_{ll'}^{sym}}{2\pi^2} \frac{3}{3}. \] (7)

With the bare propagator, the angular averaged spin dependent interaction parameter yields [12]

\[ f_{pp'}^{++} |_{p=p',s'=s'} = \frac{g^2}{9e_f^2 p_f^2 (1 - \cos \theta)} \times \left[ 2m^2 - p_f^2 (1 - \cos \theta) + \frac{2m^2 p_f^2}{3(e_f^2 + m^2)} \right], \]
\[ f_{pp'}^{+-} |_{p=p',s'=s'} = \frac{g^2}{9e_f^2 e_f^2} \left\{ 1 - \frac{m_q^2 p_f^2}{3(e_f^2 + m_e)} + \frac{m_q^2 p_f^2}{3(e_f^2 + m_e)} \times \frac{1}{(m_0^2 - e_f^2 e_f^2 + p_f^2 e_f^2 \cos \theta)} \right\}. \] (8)

Here, $m_q$ is the quark mass, $p_f^2 = p_f^2 (1 \pm \xi)^{1/3}$, $e_f^2 = (p_f^2 + m^2)^{1/2}$, and $p_f$ is the Fermi momentum of the unpolarized matter ($\xi = 0$). Similarly, $f_{pp'}^{+-}$ can be obtained by replacing $p_f^+ \leftrightarrow p_{f'}^+$ and $e_f^2 \leftrightarrow e_{f'}^2$ in Eq. (8). One can find dimensionless LPs, $F_{l'}^{sym}$ and $F_{l'}^{asym}$ (suppressing spin indices) by considering the OGE interaction. But both of these $f_{ll'}^{sym(asym)}$ exhibit infrared divergences because of the term $(1 - \cos \theta)$ that appears in the denominator of the interaction parameter [see Eq. (8)]. This divergence disappears if one uses HDL corrected gluon propagator to evaluate the scattering amplitudes [17].

To construct HDL corrected gluon propagator with explicit spin dependence one needs to evaluate the expressions for longitudinal ($\Pi_L$) and transverse ($\Pi_T$) polarization which have been derived in [9]. We borrow the results directly:

\[ \Pi_L(k_0, k) = \frac{g^2}{4\pi^2} (C_0 - 1) \sum_{n=\pm} p_f n_f \]
\[ \times \ln \left[ \frac{C_0 + p_f}{C_0 - p_f} \right], \] (10)
\[ \Pi_T(k_0, k) = \frac{g^2}{16\pi^2} C_0 \sum_{n=\pm} n_f^2 \left[ \frac{2C_0}{v_f^2} + \left( 1 - \frac{C_0^2}{v_f^2} \right) \right] \times \ln \left[ \frac{C_0 + p_f}{C_0 - p_f} \right]. \] (11)

Here, $C_0 = k_f/k_L$ is the dimensionless variable and $v_f^2 = p_f^2 / C_0 - p_f^2$. It might be noted here, that the expressions for $\Pi_L$ and $\Pi_T$ look rather similar to what one obtains in the case of unpolarized matter ($\xi = 0$) [18] with only a difference in $v_f^2$. In the static limit, i.e., $C_0 \to 0$, the spin dependent Debye mass $(m_D)$ is given by

\[ m_D = m_D^2 = \frac{C_0^2}{4\pi^2} \sum_{n=\pm} n_f e_f^2. \] (12)

It is to be mentioned here, that the screening mass of the gluon is spin dependent and the transverse gluons are screened only dynamically [15,16]. With these, the symmetric combination
of dimensionless LPs are found to be

\[ P_{0}^{+\text{sym}} = \frac{g^{2} p^{+}_{f}}{4 \pi^{2}} \left[ \frac{1}{\epsilon_{f}} \left( \frac{12}{p^{+}_{f}} + \frac{12 m_{q}^{2} e_{f}^{2}}{p^{+}_{f}^{2} (m_{q} + e_{f})} \right) \right] - \frac{m_{q}^{2}}{p^{+}_{f}^{2}} \epsilon_{f} \text{ln} \left[ \frac{2 m_{q}^{2} - m_{D}^{2} - 2 p^{+}_{f} \epsilon_{f} - 2 s \epsilon_{f}}{2 m_{q}^{2} - m_{D}^{2} - 2 p^{+}_{f} \epsilon_{f} - 2 s \epsilon_{f}} \right] \] 

\[ P_{1}^{+\text{sym}} = \frac{g^{2} p^{+}_{f}}{16 \pi^{2}} \left[ \frac{1}{\epsilon_{f}} \left( \frac{12}{p^{+}_{f}} + \frac{12 m_{q}^{2} e_{f}^{2}}{p^{+}_{f}^{2} (m_{q} + e_{f})} \right) \right] \frac{m_{q}^{2}}{p^{+}_{f}^{2}} \epsilon_{f} \text{ln} \left[ \frac{2 m_{q}^{2} - m_{D}^{2} - 2 p^{+}_{f} \epsilon_{f} - 2 s \epsilon_{f}}{2 m_{q}^{2} - m_{D}^{2} - 2 p^{+}_{f} \epsilon_{f} - 2 s \epsilon_{f}} \right] \] 

where \( (M) \) is the magnetization for each flavor. Here, the magnetic field is considered to be significantly weak for which the spinors remain unaffected and only modification enters through the single particle energy. Here, we consider one flavor quark matter and suppress the flavor indices.

In deriving Eqs. (13), we consider exchange of longitudinal gluons only. In Eqs. (13) and (14), the term in the first square bracket arises due to the scattering of like-spin states \((\pm\pm)\), while the latter comes from the scattering of unlike-spin states \((\pm\mp)\).

In Fig. 1, the density dependence of symmetric and anti-symmetric combination of dimensionless LPs is shown. Similar plots for the LPs in isospin asymmetric nuclear matter can be found in [19]. There, however, the calculated LPs are finite, as the nucleon-nucleon interactions involve exchanges of massive mesons such as \(a, \omega, \delta, \) and \(\rho, \) etc. It is interesting to note that the results of isospin asymmetric nuclear matter for the LPs are qualitatively same as those of dense quark system.

III. MAGNETIC SUSCEPTIBILITY

Now, we proceed to calculate the magnetic susceptibility for which an uniform magnetic field is applied along the \(z\) axis. The magnetic susceptibility is defined as [15,16]

\[ \chi_{M} = \sum_{f} \frac{\partial (M)_{f}}{\partial B} \bigg|_{B=0} , \]

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and $\delta n^\text{sym}$ is therefore given by

$$\delta n^\text{sym} = 1 - \frac{2}{2} \ell_D(p) e_B \left( \frac{\partial n^+}{\partial e_p} + \frac{\partial n^-}{\partial e_p} \right)$$

$$+ N_c \frac{\partial n^+}{\partial e_p} (f_0^- + \delta n^+ + f_0^+ - \delta n^-)$$

$$- N_c \frac{\partial n^-}{\partial e_p} (f_0^+ + \delta n^+ + f_0^- - \delta n^-).$$  \(19\)

With the help of Eqs. (17) and (19) the average magnetization becomes

$$(\langle M \rangle) = \frac{1}{2} \delta_D \mu^2 B \frac{(N^+(0) + N^-(0))}{1 + [N^+(0) + N^-(0)] \delta^\text{sym}},$$  \(20\)

where we have suppressed the spin indices for $L^\text{sym}$. The expression of $\langle M \rangle$ may be compared with the one presented in [15,16] to see the difference between the unpolarized and polarized matter. Likewise, the magnetic susceptibility is found to be

$$\chi_M = \left( \frac{\delta_D \mu^2}{2} \right)^2 \frac{(N^+(0) + N^-(0))}{1 + [N^+(0) + N^-(0)] \delta^\text{sym}},$$  \(21\)

where $\delta_D$ is the angular averaged gyromagnetic ratio [15,16].

A. Susceptibility with bare propagator

We have already mentioned that the individual LPs are infrared divergent when evaluated with the bare gluon propagator. But the combination $(L^\text{sym} - L^\text{asym})$ is always finite and turns out to be

$$f_0^\text{sym} - \frac{1}{2} f_1^\text{sym} = \int_{-1}^{+1} d(\cos \theta)(1 - \cos \theta)(f_{pp}^+ + f_{pp}^-)$$

$$- \int_{-1}^{+1} d(\cos \theta)(1 + \cos \theta)(f_{pp}^+ - f_{pp}^-) = I_1 - I_2.$$  \(24\)

Using Eqs. (8), (9), and (24) we have

$$I_1 = -\frac{g^2}{36} \left[ \frac{1}{p^2 e_f e_f} \left( 2m^2 - p^2 e_f + 3(e_f e_f + m_q) \right) + (p^2 - p e_f e_f - e_f e_f) \right],$$  \(25\)

$$I_2 = \frac{g^2}{36 e_f e_f} \times \frac{1}{3 p^2} \frac{p^2}{(m_q + e_f e_f)^2}$$

$$\times \left[ -2p^2 p^2 (p^2 e_f e_f + m_q p^2 - 3 p^2 e_f) + m^2 (p^2 - 3 p^2 p^2 + p^2) \right] + m^2 e_f (m^2 p^2 - p^2 p^2 - p^2 e_f - m^2 e_f)$$

$$- (p^2 - p^2 p^2 + m^2 e_f) (e_f e_f + m_q[p^2 + p^2]),$$

$$\frac{1}{36 e_f e_f} \ln \left( \frac{m^2 p^2}{m^2 p^2 - p^2 e_f - e_f e_f} \right).$$  \(26\)

To determine $\chi_M$ for various $\xi$, we insert Eq. (24) in Eq. (22), where $I_1$ and $I_2$ are given by Eqs. (25) and (26).

B. Susceptibility with HDL corrected propagator

In this section we consider the screening effects due to HDL corrected propagator of the gauge field [17]. The scattering amplitude can be written as [15]

$$\mathcal{M}_{P,\nu P'P'} = -2 g^2 [T^{\alpha\beta}(P_s, P' s) D_{\alpha\nu} + T^{\nu}(P_s, P' s) D_{\alpha\beta}].$$  \(27\)

In the Coulomb gauge, we have $D_{\alpha\beta} = \Delta_\alpha + q\partial_\beta/q^2$, $\Delta_\alpha$, where $q = p - p'$. $\Delta_\alpha$ and $\Delta_\beta$ denote the longitudinal and transverse gluon propagators given by [20]

$$\Delta_\alpha = \frac{1}{q^2 + m^2_D}, \quad \Delta_\beta = \frac{1}{q^2 - q^2}.$$  \(28\)
FIG. 2. Density dependence of magnetic susceptibility. Screening effects (solid line) are compared with the simple OGE case (dashed line) for unpolarized quark matter.

For spin parallel \((s = s')\) and antiparallel \((s = -s')\) interactions, \(\Delta_i\) and \(\Delta_s\) have the following form:

\[
\begin{align*}
\Delta_i(s = s')_{p'\neq p} &= \frac{1}{2p_f^2(1 - \cos \theta) + m_D^2}, \\
\Delta_i(s = -s')_{p'\neq p} &= \frac{1}{2p_f^2(1 - \cos \theta) + m_D^2}, \\
\Delta_s(s = s')_{p'\neq p} &= \frac{1}{2p_f^2(1 - \cos \theta) + m_D^2}, \\
\Delta_s(s = -s')_{p'\neq p} &= \frac{1}{2p_f^2(1 - \cos \theta) + m_D^2},
\end{align*}
\]

and

\[
\begin{align*}
\text{and} \quad \Delta_i(s = s')_{p' = p} &= \frac{1}{2p_f^2(1 - \cos \theta) + m_D^2}, \\
\Delta_s(s = s')_{p' = p} &= \frac{1}{2p_f^2(1 - \cos \theta) + m_D^2}.
\end{align*}
\]

The matrix element given by Eq. (27) can be calculated easily with OGE. We find that [16]

\[
T^{0\theta}(P_s, P'_s') = \text{Tr}[\gamma^0\rho(P, s)\gamma^0\rho(P', s')] = \frac{1}{6m_q^2(m_q + 4\epsilon')} \left[ 12m_q^2 + 12m_s^2\epsilon' \right]
\]

\[
+ 6m_q^2 P_f^2(1 + \cos \theta) + 6m_s^2 P_f^2(2 + \cos \theta)
\]

\[
+ P_f^2(2 + 3\cos \theta),
\]

(35)

Similarly, the coefficient of \(\Delta_s\) turns out to be

\[
\left[ T_{ij} \times \left( \frac{3p^j}{q^2} - \frac{d'(q)}{q^2} \right) \right]^{+} = \frac{1}{6m_q^2(m_q + 4\epsilon') \left( m_q + 4\epsilon' \right)} \left[ 12m_q^2 + 12m_s^2\epsilon' \right]
\]

\[
\times \left[ -p_f^2 \left[ m_q(p_f^2 + P_f^2) + P_f^2\epsilon' + p_f^2\epsilon \right] \right] \left[ 2p_f^2 + m_q(m_q + 4\epsilon') \right]
\]

\[
+ \left[ m_q(m_q + 4\epsilon') \right] \left[ -p_f^2 \left[ m_q(p_f^2 + P_f^2) + P_f^2\epsilon' + p_f^2\epsilon \right] \right]
\]

\[
+ 2\left[ p_f^2 - 3p_f^2 P_f + p_f^2 \right] \left( m_q + 4\epsilon' \right) \left( m_q^2 - 4\epsilon' \right) \]}

\[
+ p_f^2 P_f \cos \theta \left[ 2p_f^2 + m_q \left[ p_f^2 + P_f^2 \right] + P_f^2\epsilon' + p_f^2\epsilon \right]
\]

\[
+ m_q \left[ 2p_f^2 - 3p_f^2 P_f + p_f^2 \right] \left( m_q + 4\epsilon' \right) \left( m_q^2 - 4\epsilon' \right) \]}

\[
+ m_q \left[ 5p_f^2 - 6p_f^2 P_f + 3p_f^2 \epsilon' \right] + m_q \epsilon' \left[ 3p_f^2 - 6p_f^2 P_f + 5p_f^2 \right]
\]

\[
+ 3\epsilon' \epsilon' \left[ p_f^2 + p_f^2 - 2p_f^2 P_f \cos \theta \right] \].
\]

(38)
FIG. 3. Magnetic susceptibility vs Fermi momentum using screened gluon mass for unpolarized and complete polarized quark matter.

Using Eqs. (27)–(30) and (35)–(38) one can easily calculate the required combination \((f_0^\text{sym} - f_1^\text{sym})\) to evaluate the magnetic susceptibility. Inserting this particular combination of \(f_0\) and \(f_1\) in Eq. (22) we get \(\chi_M\). To determine \(\chi_M\), we need to evaluate first \(\mu^+\) and \(\mu^0\). This can be done by adopting the procedure outlined in Ref. [12]. With these, we can estimate \(\chi_M\) numerically for the polarized and unpolarized matter at various densities. The corresponding results are discussed below.

In Fig. 2 we plot the magnetic susceptibility of cold and dense unpolarized quark matter as a function of Fermi momentum. It is observed that, upon inclusion of the screening effects, the divergence move toward lower densities. This is consistent with what one obtains for unpolarized matter [15,16]. Such shifts toward lower density are expected, as we know, that the screening effect weakens the Fock exchange interaction (see Refs. [15,16]). Moreover, this divergence is related to the magnetic phase transition of quark matter which shows up when the square bracketed term in Eq. (22) vanishes. As noted in [3] and also in [12], this density approximately corresponds to the critical density for paramagnetic phase transition. For the numerical estimation, we take \(a_c = 2.2\) and \(m_q = 300\text{ MeV} [3,12,15,16]\).

In Fig. 3, the density dependence of magnetic susceptibility both for unpolarized and polarized matter has been shown. We see that the magnetic susceptibility diverges at some critical density which increases with increasing \(\xi\). It is apparent from the figure that, if the critical density for para-ferro phase transition becomes lower than the critical density for the magnetic transition, the latter cannot take place. Thus, we conclude, that the magnetic transition depends on the critical density of para-ferro phase transition.

FIG. 4. Variation of magnetization with \(\xi\) for a magnetic field \(B = 10^7\text{ G}\). Solid, dashed and dash-dotted lines represent 0.1 fm\(^{-3}\), 0.15 fm\(^{-3}\), and 0.25 fm\(^{-3}\), respectively.

In Fig. 4, we show \(\xi\) dependence of the magnetization for various densities. Note that the divergences appear at higher \(\xi\) for larger density. Here the magnetic dipole moments of the quarks are taken to be \(\mu_0 = 1.852\mu_N\), \(\mu_\pi = -0.972\mu_N\), and \(\mu_\Sigma = -0.581\mu_N\), where the nuclear magneton \(\mu_N = 3.152 \times 10^{-14}\text{ MeV/Tesla} [21]\).

IV. SUMMARY AND CONCLUSION

In this work, we calculate dimensionless LPs \(f_{0,1}^{\text{sym}}\) and \(f_{0,1}^{\text{asym}}\) for dense quark matter. These are then used to calculate magnetic susceptibility and magnetization of degenerate quark matter and the results are found to be consistent with previous calculations in the appropriate limits. The qualitative behavior of the LPs as a function of density is also found to be very similar to those of nuclear matter having isospin asymmetry.

We observe that \(\chi_M\) is free of all the infrared divergences even in the massless gluon limit. It is, however, numerically sensitive to the Debye mass. It is shown that the critical density for the magnetic transition in polarized matter is higher than that of the unpolarized one. The divergence and sign change of the magnetic susceptibility signal the magnetic instability of the ferromagnetic phase.

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Pionic contribution to relativistic Fermi liquid parameters

Kausik Pal

Abstract: We calculate pionic contribution to the relativistic Fermi liquid parameters (RFLPs) using a chiral effective Lagrangian. The RFLPs so determined are then used to calculate chemical potential, exchange and nuclear symmetry energies due to nN interaction. We also evaluate two loop ring diagrams involving $\sigma$, $\omega$, and $\pi$ meson exchanges and compare results with what one obtains from the relativistic Fermi liquid theory (RFLT).

PACS Nos: 21.65.-f, 13.75.Cs, 13.75.Gx, 21.30.Fe

Résumé : Nous calculons la contribution des pions aux paramètres du liquide de Fermi relativiste (RFLPs), en utilisant un Lagrangien effectif chiral. Les paramètres ainsi obtenus sont alors utilisés pour calculer le potentiel chimique, les énergies d’échange et de symétrie nucléaire dues à l’interaction nN. Nous évaluons également deux diagrammes en boucle impliquant l’échange de mésons $\sigma$, $\omega$ et $\pi$ et nous comparons nos résultats avec ceux obtenus de la théorie du liquide de Fermi relativiste (RFLT).

[Traduit par la Rédaction]

. Introduction

Fermi liquid theory (FLT) provides us with one of the most important theoretical schemes to study the properties of strongly interacting Fermi systems involving low-lying excitations near the Fermi surface [1]. Although developed initially in the context of studying the properties of $^3$He, it is widespread applications in other disciplines of many-body physics such as superconductivity, super fluidity, nuclear and neutron star matter, etc. [2, 3].

In nuclear physics, FLT was first extended and used by Migdal [4] to study the properties of both unbound nuclear matter and finite nuclei [5]. FLT also provides a theoretical foundation for the nuclear shell model [5] as well as nuclear dynamics of low-energy excitations [2, 6]. In particular, eq. 7 reveals the connection between Landau, Brueckner-Hartee-Fock (BHF) model. Some of the recent work that examines Fermi liquid properties of hadronic matter also incorporates a Rho (BR) scaling, which is very important for the study of the properties of hadrons in dense nuclear matter (DNM) [10, 11].

Most of the earlier nuclear matter calculations that involved Landau theory were done in a nonrelativistic framework. The relativistic extension of FLT was first developed by Baym and Chin [3] in the context of studying the properties of DNM. In [3], the authors invoked the Walecka model (WM) to calculate various interaction parameters ($f_{\rho\pi}$) but did not consider mean fields (MFs) for the $\sigma$ and $\omega$ mesons, for which the FLPs are calculated perturbatively.

Later, Matsui revisited the problem in [12] and starts from the expression of energy density in the presence of scalar and vector meson MFs and uses the functional derivatives to determine the FLPs. The results are qualitatively different than the perturbative results (see [3, 12]). A comparison of relativistic and nonrelativistic calculations were performed in [8, 9], which also discuss how the FLPs are modified in the presence of the $\sigma$ and $\omega$ MFs and contrast those with perturbative results.

Besides $\sigma$ and $\omega$ mesons, ref. 12 also includes the $\rho$ and $\pi$ mesons, and the model adopted was originally proposed by Serot to incorporate pions into the WM. It is to be noted, however, that the FLPs presented in [12] are independent of the $\pi$ meson. This is because $\pi$, being a pseudoscalar, fails to contribute to the MF level. Hence to estimate the pionic contribution to FLPs, it is necessary to go beyond MF formalism.

It is to be noted that, to our knowledge, such relativistic calculations that include $\pi$ exchange do not exist, despite the fact that the pion has a special status in nuclear physics, as it is responsible for the spin–isospin dependent long-range nuclear force. Furthermore, there are various nonrelativistic calculations, including the celebrated work of Migdal, which show that the pionic contribution to FLPs are important [10, 11, 13] and most dominant for low-energy excitations. We mention here that ref. 11 discusses how to incorporate relativistic corrections to $F_F^\pi$ in the static potential model calculation. We, however, take the approach of [3], in which all fields are treated relativistically.

The other major departure of the present work from refs. 3 and 12 resides in the choice of model for the description of the many-body nuclear system. Unlike previous calculations [3, 12], here we use the recently developed chiral

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In particular, at the MF level it gives rise to a tachyonic mode in matter at densities as low as 0.1\rho_0, where \rho_0 is the nuclear saturation density [15]. Inclusion of exchange diagrams removes such unphysical modes but makes the effective mass unrealistically large. If one simply replaces the PS coupling of ref. 14 with the pseudovector (PV) interaction, these difficulties can be avoided. This, however, makes the theory non-renormalizable [16].

In [16] it was shown how, starting from a PS coupling one can arrive at a PV representation that preserves renormalizability of the theory and at the same time yields realistic results for the pion dispersion relations in matter. But this model was also not found to be trouble free, particularly it had several shortcomings in describing \pi N dynamics in matter (not discussed here but see [17–22]).

Furthermore, the WM itself has several problems in relation to convergence that prevent the use of a systematic expansion scheme to perform perturbative calculations. This was first exposed in ref. 23.

The most recent model that provides us with a systematic scheme to study the dense nuclear system is provided by chEFT [21, 23–25], where the criterion of renormalizability is given up in favor of a successful relativistic description of DNM and the properties of finite nuclei. The chEFT, apart from \sigma and \omega mesons, also includes pions and therefore is best suited for the present purpose.

In this paper we estimate the contributions of pion exchange to the FLPs within the framework of RFLT, and subsequently use the parameters to calculate various quantities such as pionic contribution to the chemical potential, energy density, symmetry energy (\delta), etc. For completeness and direct comparison with the two-loop results, we also calculate the exchange energy due to the interaction mediated by the \sigma, \omega, and \pi mesons.

2. Formalism

We will quickly outline the formalism. In FLT the energy density \(E\) of an interacting system is the functional of occupation number \(n_p\) of the quasi-particle states of momentum \(p\). The excitation of the system is equivalent to the change of occupation number by an amount \(\delta n_p\). The corresponding energy of the system is given by,

\[
E = E^0 + \int d^3p \left\{ \frac{1}{2} \varepsilon^2 \delta n_p + \sum_{\alpha} \sum_{\alpha'} \frac{1}{2} \frac{d^3p}{(2\pi)^3} \frac{d^3p'}{(2\pi)^3} \left( f_{p\alpha'\alpha} - f_{p\alpha'\alpha} \right) \delta n_{p'} \delta n_{p'} \right\}
\]

Here, \(\varepsilon\) is the spin index. The quasi-particle energy can be written as,

\[
\varepsilon_{p\alpha} = \varepsilon^0_{p\alpha} + \int \frac{d^3p'}{(2\pi)^3} f_{p\alpha'\alpha} \delta n_{p'} + \int \frac{d^3p'}{(2\pi)^3} f_{p\alpha'\alpha} \delta n_{p'}
\]

where superscript 0 denotes the ground state [2]. It is to be remembered that although the interaction \(f_{p\alpha'\alpha}\) between the quasiparticles is not small, the problem is greatly simplified because it is sufficient to consider only pair collisions between the quasiparticles [5].

Table 1. Dimensionless Landau parameters and chemical potential at \(\rho = \rho_0\).

<table>
<thead>
<tr>
<th>Meson</th>
<th>(F_0)</th>
<th>(F_1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>\sigma</td>
<td>-5.04</td>
<td>0.875</td>
</tr>
<tr>
<td>\omega</td>
<td>5.44</td>
<td>-0.93</td>
</tr>
<tr>
<td>\pi</td>
<td>0.68</td>
<td>-0.20</td>
</tr>
</tbody>
</table>

Table 2. Exchange energy in MeV from FLT at \(\rho = \rho_0\).

<table>
<thead>
<tr>
<th>Meson</th>
<th>(E^m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>\sigma</td>
<td>40.48</td>
</tr>
<tr>
<td>\omega</td>
<td>-23.41</td>
</tr>
<tr>
<td>\pi</td>
<td>12.49</td>
</tr>
</tbody>
</table>
Since quasi-particles are well defined only near the Fermi surface, one assumes
\[ \sigma_p = \mu + \gamma(p - p_f) \]
\[ p \approx p' \approx p_f \]
\[ \text{(3)} \]

Then the LPs \( f_{\pi} \) are defined by the Legendre expansion of \( f_{\pi \gamma} \text{ with}\)
\[ f_{\pi} = \frac{2i + 1}{4} \sum_{l} \frac{d\Omega}{4\pi} P_l(\cos \theta) f_{\pi \gamma} \]
\[ \text{(4)} \]

where \( \theta \) is the angle between \( p \) and \( p' \), both taken to be on
the Fermi surface, and the integration is over all directions of \( p \). We restrict ourselves to \( l \leq 1 \), i.e., \( f_0 \) and \( f_1 \), as contributions decrease rapidly at higher \( l \) [3, 12].

Now the Landau Fermi liquid interaction \( f_{\pi \gamma} \) is related to the two particle forward scattering amplitude via [3],
\[ f_{\pi \gamma}(p, p') = \frac{M}{2\pi^2} \frac{M_{p \pi}}{m^2} \pi \sum_{l} \frac{d\Omega}{4\pi} P_l(\cos \theta) f_{\pi \gamma} \text{ with}\]
\[ \text{(5)} \]

where the Lorentz invariant matrix \( M_{p \pi} \) consists of the usual direct and exchange amplitude, which may be evaluated directly from the relevant Feynman diagrams. The spin-averaged scattering amplitude \( f_{\pi \gamma} \) is given by [3],
\[ f_{\gamma}(p, p') = \frac{1}{4\pi^2} \sum_{l} \frac{d\Omega}{4\pi} P_l(\cos \theta) f_{\pi \gamma} \text{ with}\]
\[ \text{(6)} \]

The dimensionless LPs are \( F_1 = N(0)f_0 \), where \( N(0) \) is the density of states at the Fermi surface defined as [12]
\[ N(0) = \frac{8\pi B_{\pi \gamma} \pi}{2\pi^2} \]
\[ \text{(7)} \]

Here \( g_s \) and \( g_t \) are the spin and isospin degeneracy factors, respectively.

3. Landau parameters

By retaining only the lowest order terms in the pion fields, one obtains the following Lagrangian from the chirally invariant Lagrangian [24, 25]
\[ \mathcal{L} = \psi \left[ \gamma^\mu (\partial_\mu - g_\omega \omega_\mu) - i \frac{e}{f_\pi} \gamma^\mu \gamma_5 \partial_\mu \right] \psi - (M - g_\omega \Phi_\omega) \psi \]
+ \[ \frac{i}{2} \gamma^\mu \psi \partial_\mu \Phi_\omega - \frac{1}{2} m_\omega^2 \Phi_\omega^2 - \frac{1}{4} \omega^{\mu \nu} \omega_{\mu \nu} + \frac{1}{2} m_\omega^2 \omega \cdot \omega \]
+ \[ \frac{1}{2} \gamma^\mu \Phi_\omega \cdot \partial_\mu \Phi_\omega - \frac{1}{2} m_\omega^2 \Phi_\omega^2 + 2 \omega \cdot \omega + \Delta \Sigma \]
\[ \text{(8)} \]

where
\[ \omega_\mu = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu, \quad \Sigma = \frac{1}{2}(\tau \cdot \Phi_\omega) \]

and \( \tau \) is the isospin index. Here \( \psi \) is the nucleon field, and \( \Phi_\omega \) is the meson fields (isospin-scalar, isoscalar-vector, and isovector-pseudoscalar, respectively). The terms \( L_{\omega \omega} \) and \( L_{\Delta} \) contain the nonlinear and counterterms, respectively (for explicit expressions see [25]).

Because of the presence of pion fields in the chiral Lagrangian, we have a component in the interaction that acts on the isospin fluctuation. One can derive the isospin-dependent: quasi-particle interaction along the lines of ref. 3. For pions, as mentioned earlier, the direct term vanishes, and it is only the exchange diagram that contributes to the interaction parameter,
\[ f_{\pi \gamma}(p, p') = \frac{3 g_\omega^2 M^2}{2 4 f_\pi^2 f_\gamma^2} \left( \frac{p_f^2 (1 - \cos \theta)}{2 p_f^2 (1 - \cos \theta) + m_\omega^2} \right) \]
\[ \text{(9)} \]

where \( \tau_f = (p_f^2 + M^2)^{1/2} \) and \( g_\omega^2 = 1.5876, f_\pi = 93 \text{ MeV}, \) \( m_\omega = 139 \text{ MeV} \) [25]. The factor \( 3/2 \) arises because the isospin factor is \( 3/2 \) in the isoscalar channel [11].

The effective nucleon mass \( M^* \) is determined self-consistently from the following equation [12],
\[ M^* = M - \frac{g_\omega^2}{m_\omega^2} \sum_{r} n_r \left( \frac{M^*}{(p_f^2 + M^*)^{1/2}} \right) \]
\[ \text{(10)} \]

Using (4) and (9), we can derive isoscalar LPs \( f_{0 x, x} \) and \( f_{1 x, x} \),
\[ f_{0 x, x} = \frac{3 g_\omega^2 M^2}{2 m_\omega^2} \left( \frac{m_\omega^2}{2 p_f^2} \right) \left( 1 + \frac{4 p_f^2}{m_\omega^2} \right) \]
\[ \text{(11)} \]
and
\[ \frac{1}{3} f_{1 x, x} = \frac{3 g_\omega^2 M^2}{2 m_\omega^2} \left( \frac{m_\omega^2}{2 p_f^2} \right) \left( 1 + \frac{4 p_f^2}{m_\omega^2} \right) \]
\[ \times \left[ -2 + \frac{m_\omega^2}{2 p_f^2} + \frac{1}{2} \ln \left( 1 + \frac{4 p_f^2}{m_\omega^2} \right) \right] \]
\[ \text{(12)} \]

Using (11) and (12) we find that
\[ f_{0 x, x} = \frac{3 g_\omega^2 M^2}{8 p_f^2} \left( \frac{m_\omega^2}{2 p_f^2} \right) \left( \frac{1}{2} \ln \left( 1 + \frac{4 p_f^2}{m_\omega^2} \right) \right) \]
\[ \times \left[ -2 + \frac{m_\omega^2}{2 p_f^2} + \frac{1}{2} \ln \left( 1 + \frac{4 p_f^2}{m_\omega^2} \right) \right] \]
\[ \text{(13)} \]

It is this combination, i.e., \( f_0 - (1/3)f_1 \), which appears in the calculation of chemical potential and other relevant quantities. For the massless pion, (13) turns out to be finite
\[ f_{0 x, x} = \frac{3 g_\omega^2 M^2}{16 p_f^2} \left( \frac{m_\omega^2}{2 p_f^2} \right) \left( \frac{1}{2} \ln \left( 1 + \frac{4 p_f^2}{m_\omega^2} \right) \right) \]
\[ \text{(14)} \]

It is to be noted that in the massless limit for the \( \sigma \) and \( \omega \) mesons, \( f_{0 x, x} \) and \( f_{1 x, x} \) diverge as shown in [3], in contrast to the pion, for which even in the massless limit these are finite. This is due to the presence of \( (1 - \cos \theta) \) in the numerator of (9), unlike for \( \sigma \) and \( \omega \) mesons.

The dimensionless LPs can be determined by equating the equation \( F_0 = N(0)f_0 \) and \( F_1 = N(0)f_1 \), where \( N(0) \) is the density of states at the Fermi surface defined in (7). Thus the dimensionless parameters are
\[ F_{0 x, x} = \frac{3 g_\omega^2 M^2}{2 m_\omega^2} \left( \frac{m_\omega^2}{2 p_f^2} \right) \left( -2 + \frac{m_\omega^2}{2 p_f^2} + \frac{1}{2} \ln \left( 1 + \frac{4 p_f^2}{m_\omega^2} \right) \right) \]
\[ \text{(15)} \]
and
\[
\frac{1}{3} F^{\pi,\pi}_{1} = -8\pi i \frac{3g_{\pi}^2 m_{\pi}^2 M^2}{128\pi^2 p_T^2 p_T f_{\pi}} \times \left[ -2 + \left( \frac{m_{\pi}^2}{2p_T^2} + 1 \right) \ln \left( 1 + \frac{4p_T^2}{m_{\pi}^2} \right) \right] \tag{16}
\]

In Fig. 1, we show the density dependence of \( F_0 \) and \( F_1 \) on pionic interaction. Numerically, at nuclear matter density \( \rho_0 = 0.148 \text{ fm}^{-3} \), \( F_0^{\pi,\pi} = 0.68 \) and \( F_1^{\pi,\pi} = -0.2 \). In the non-relativistic limit \( p_T \to M^* \), one obtains the same expression for \( F_1^\pi \) that was reported in [10, 13].

4. Chemical potential

We will now proceed to calculate the chemical potential, which in principle will be different for the neutron and proton in asymmetric nuclear matter.

Although in the present work we deal with symmetric nuclear matter (SNM), for which \( p_n = p_p \), to calculate the general expression for chemical potential with arbitrary asymmetry \( (\alpha = (n_n - n_p)/(n_n + n_p)) \), we use the distribution function with explicit isospin index \( (a \text{ or } b) \), so that the variation of distribution function gives [2],

\[
\delta n_a = -N_a(0) \sum_b f_b^a \delta n_b - \delta \mu_a \tag{17}
\]

where \( N_a(0) \) is the isospin dependent density of states at the corresponding Fermi surface. Equation (17) yields

\[
\frac{\partial \mu_a}{\partial n_b} = \frac{1}{N_a(0)} + \sum_b \delta f_b^a \frac{\partial \mu_a}{\partial n_b} \tag{18}
\]

In our case \( b \) (or \( b' \)) = \( n, p \). Separately for neutron and proton we have,

\[
\frac{\partial \mu_n}{\partial p_T} \left( \frac{1}{N_n(0) + f_n^{n}} \right) \left( -\frac{f_n^{n}}{N_n(0)} + f_n^{p} \right) \left( \frac{\partial n_n}{\partial p_T} \right) \tag{19}
\]

where the superscripts \( nn \) and \( np \) denote neutron-neutron and neutron-proton quasi-particle interaction. \( N_n(0) \) and \( N_p(0) \) denote the density of states for the neutron and proton Fermi surfaces, respectively [26, 27]. For pion exchange in SNM, \( f_{\pi}^{n}(\rho_0) = 2g_{\pi}^{n} \eta \) [13], and from (19), we have

\[
\frac{\partial \mu_n}{\partial n_b} = \frac{1}{N_n(0)} + \frac{f_n^{n}}{N_n(0)} \frac{\partial n_n}{\partial n_b} \tag{20}
\]

where \( F_{\pi}^{n} = N_n(0)f_{\pi}^{n} \). Similarly, one can determine \( \delta \mu_p \).

Motivated by [5], we define

\[
f_b^a = \frac{1}{2} \sum bf_b^a \tag{21}
\]

Evidently in SNM, \( f_{\pi}^{n} = f_{\pi}^{p} \), and therefore we write \( \mu_n = \mu_p = \mu [3] \)

\[
\mu = \left[ \mu_0 + g_{\pi}^n \frac{f_{\pi}^{n}}{2\pi} \left( f_0 - \frac{1}{3} f_1 \right) \right] \eta \tag{22}
\]

Here \( f_0 \) and \( f_1 \) are given by (11) and (12).

To calculate \( \mu \), it is sufficient to let \( x = f_1 \) on the right-hand side of (22). With the constant of integration adjusted so that at high density \( f_1 \approx f_1 \), (22) upon integration, together with (13) yields

\[
\mu = f_1 - 8\pi i \frac{3g_{\pi}^2 M^4}{128\pi^2 p_T^2 f_{\pi}} \times \left[ -2y_1^2 \sqrt{-y_1^2 + 4 \tan^{-1} \left( \frac{x}{\sqrt{-y_1^2}} \right)} \ln \left( 1 + \frac{4x^2}{y_1^2} \right) \right.

\left. + \frac{y_1^2}{x} \ln \left( 1 + \frac{4x^2}{y_1^2} \right) \right] \tag{23}
\]

where \( x = p_T/M^* \) and \( y_1 = m_{\pi}/M^* \).

The calculations of \( \mu \) for other mesons is straightforward. However, for brevity, we do not present the corresponding expressions for \( \sigma \) and \( \omega \) mesons but quote their numerical values in Table 1. The numbers cited above are relevant for normal nuclear matter density, \( \rho_0 = 0.148 \text{ fm}^{-3} \).

For the coupling constants, we adopt the same parameter set as designated by M0A in [25].

Interestingly, the individual contribution to \( \mu \) of \( \sigma \) and \( \omega \) mesons are large, while the sum of their contributions to \( \mu \) is small because of the sensitive cancellation of \( \mu_{\sigma} \) and \( \mu_{\omega} \), as can be seen in Table 1. Such a cancellation is responsible for nuclear saturation dynamics [8, 9]. Numerically, \( F_{\pi}^{n+n} \) is approximately 3/2 times smaller than \( F_{\pi}^{n} \).

5. Exchange energy

Once \( \mu \) is determined, one can readily calculate the energy density due to \( n - N \) interaction in SNM as [3, 28]

\[
E_{\pi}^{n} = \int d\rho(\mu - (g - \sigma)) = -8\pi i \frac{3g_{\pi}^2 M^4}{128p_T^2 \pi^4} \times \left[ I_1 + \frac{I_2}{4} \right] \tag{24}
\]

where

\[
\eta = \sqrt{1 + x^2}
\]

and

\[
I_1 = -2y_1^2 \sqrt{-y_1^2 + 4 \tan^{-1} \left( \frac{x}{\sqrt{-y_1^2}} \right)} \left( \frac{1}{x} \ln \left( x^2 + \frac{1}{x} \right) \right) dx \tag{25}
\]

For the massless pion this reads as

\[
E_{\pi}^{n} |_{m_{\pi} = 0} = -8\pi i \frac{3g_{\pi}^2 M^4}{128p_T^2 \pi^4} \left( \frac{1}{x} \ln \left( x^2 + \frac{1}{x} \right) \right) dx \tag{26}
\]
The contribution arising from pion exchange from the direct evaluation of Fig. 2c reads as [25]

\[ E^{\pi}_{\text{ex}} = \frac{3\hbar^2}{32\pi^2m_0^3} \int \int \frac{d(p)dp'}{\epsilon_p \epsilon_{p'}} \left\{ \frac{2\epsilon_p \epsilon_{p'}}{2\epsilon_p \epsilon_{p'}} - \frac{1}{p^2} \right\} \frac{1}{\sin\theta} \left( \phi_{\pi} \phi_{\pi'} \right) \left( \phi_{\pi} \phi_{\pi'} \right) \cos\theta - M^2 \right\}

(27)

Similarly, from Figs. 2a and 2b one can determine exchange energy due to \( \sigma \) and \( \omega \) meson interaction [25]. It might be mentioned here that in the limit \( m_{\pi} \to 0 \), (27) can be evaluated analytically, which reproduces (26) derived in the FLT approach.

6. Symmetry energy

Knowing the "isovector" combination of the LPs, one can determine nuclear symmetry energy. The symmetry energy is defined as the energy difference between the neutron matter and symmetric nuclear matter and is given by the following expression [12, 29]

\[ \beta = \frac{1}{2} \frac{\partial E}{\partial\rho} \mid_{\rho=\rho_0} \]

(28)

In terms of LPs, the symmetry energy can be expressed as [5]

\[ \beta = \frac{F'_0}{6\rho_0} \left( 1 + 2F'_0 \right) \]

(29)

where

\[ F'_0 = \frac{1}{2} \left( F_{\text{iso}} - F_{\text{mp}} \right) \]

is the isovector combination of the dimensionless Landau parameters \( F_0 \) [5, 29, 1]. Deriving \( F'_0 \), one can find symmetry energy. Numerically at saturation density (\( \rho = \rho_0 \)), we obtain \( \beta = 14.57 \text{ MeV} \). So, a relatively small contribution to \( \beta \) comes from one pion exchange diagram [30].

7. Summary and conclusion

In this paper, we calculate RFLPs within the framework of RFLT. For the description of a dense nuclear system, chEFT is invoked. Although our main focus was to estimate the contribution of pions to RFLPs, for comparison and completeness we also present results for the \( \sigma \) and \( \omega \) mesons. It is seen that the pionic contribution to the FLPs are significantly larger compared with the combined contributions of \( \sigma \) and \( \omega \) mesons. Thus, any realistic relativistic calculation for the FLPs should include \( \pi \) mesons, which necessarily implies going beyond the MF calculations. The LPs that we determine here are subsequently used to calculate exchange and symmetry energy of the system. Finally, we evaluate two-loop ring diagrams with the same set of interaction parameters and show that the numerical results are consistent with those obtained from the FLT. It might be mentioned here that in the present calculation we have ignored nucleon-nucleon correlations, which might be worthwhile to investigate. It should, however, be noted that the inclusion of correlation energy would require a readjustment of the coupling parameter to reproduce the saturation properties of nuclear matter.

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Non-Fermi liquid corrections to the neutrino mean free path in dense quark matter

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We calculate the neutrino mean free path with non-Fermi liquid corrections in quark matter from scattering and absorption processes for both degenerate and nondegenerate neutrinos. We show that the mean free path decreases due to the non-Fermi liquid corrections, leading to $l_{\text{MFP}} \sim [\ldots + \ldots C^2 \alpha^2 \ln(m_D/T)^2]$. This reduction results in a higher rate of scattering.

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1. INTRODUCTION

Recently, there has been a substantial effort to study the properties of cold and warm quark matter. Such studies are important to understanding the properties of the astrophysical compact objects like neutron stars and pulsars. There have been a lot of experimental efforts, like the Einstein Laboratory, ROSAT, CHANDRA, and XMM, where various measurements are performed to understand the properties of neutron stars [1–3].

There is a possibility that, at the core of neutron stars, the density may go up to 5 ~ 6 times the normal nuclear matter density where the matter is not expected to be in the hadronic phase [4,5]. In fact, under such a scenario, one expects that it would be more appropriate to describe the core of such dense stars as degenerate quark matter [6], which is our main interest in the present work.

It is known that the newly born neutron stars cool via the emissions of neutrinos and antineutrinos within a few minutes, involving the direct [7–9] or the modified URCA processes [8,10]. The direct URCA reactions can proceed in neutron-rich matter if the ratio of the proton number density to the total baryon number density exceeds a critical value which follows from the energy and momentum conservation properties [11]. In the modified URCA process, another nucleon catalyzes the reaction to occur under situations where the direct URCA reaction is forbidden.

For quark matter, the dominant contribution to the emission of neutrinos is given by the quark analogous of $\beta$ decay and the electron capture [9]. These reactions are named as “quark direct URCA” processes which have been studied in detail by Iwamoto [7,8]. Our main focus here is to calculate the neutrino mean free path (MFP). This reduction results in a higher rate of scattering.

We expect from the usual Fermi liquid theory (FLT) [12,13]. Similar non-Fermi liquid behavior of various thermodynamical quantities has recently drawn significant attention [4,5,9,14–16].

For example, in [4,14], the authors have computed the leading contribution to the interaction part of specific heat and entropy when the temperature is much smaller than the chemical potential of quark matter. It has been shown, in the case of specific heat, for ideal gas, the leading-order term goes as $C_{\text{S}} \mu^2 T^2$, while the correction term involves $C_{\text{S}} T^2$. In the interacting case, as shown in Ref. [14], the correction to the leading-order contribution involves nonanalytic terms, known as non-Fermi liquid (NFL) corrections, which have been discussed extensively in [4,5,9,14–16]. Ref. [15] examined NFL effects in the normal phase of high density QCD matter both using the Dyson-Schwinger equation and the renormalization group theory. Like specific heat, the magnetic susceptibility shows that similar non-Fermi liquid behavior has been shown in [16]. Being motivated by this series of works, we undertake the present investigation to estimate and see the consequences of such effects in the case of the neutrino MFP.

In FLT, quarks are treated as quasiparticles, and their energy ($E$) is regarded as a functional of the distribution function [12,13,17–20]. FLT is restricted to the low-lying excitations near the Fermi surface, where the lifetime of quasiparticles is long enough. Therefore, it is an important tool to study the properties of nuclear (or quark) matter. In Ref. [21], it has been argued that the exchange of dynamically screened transverse gluons introduces infrared divergences in the quark self-energies that lead to the breakdown of the Fermi liquid description of cold and dense QCD in perturbation theory. A detailed study of non-Fermi liquid aspects of the normal state was presented in Ref. [5]. There, the spectral density, dispersion relation, and width of quasiparticles with momenta near the Fermi surface were derived at $T = 0$ by implementing a renormalization group resummation of the leading logarithmic infrared divergences associated with the emission of soft dynamically screened transverse gluons [4,5].

We have already mentioned that such anomalous corrections are ultimately connected to the absence of
magnetic screening of gluons via Landau damping. One such calculation, which we find to be the most relevant for the present purpose, was performed in [9]. It was shown that the emissivity that receives logarithmic corrections is enhanced due to non-Fermi liquid effects. It might not be out-of-context here to recall two important works on the neutrino mean free path in QED plasma. One is due to Tubbs and Schramm [22], and the other is done by Lamb and Pethick [23]. In [22], the resultant mean free path was calculated in the neutronized core and just outside the core. It is concluded, in [23], that neutrino degeneracy reduces the neutrino mean free path, which suggests that neutrinos may flow out of the core rather slowly.

In our work, we show that the neutrino mean free path receives logarithmic corrections where the dressed gluon propagator is used instead of the bare propagator. In fact, we generalize the [8,22,23] results by incorporating the NFL corrections for the quark matter. The corrections to the MFP for degenerate and nondegenerate neutrinos, as we shall see, will involve different powers of $\alpha_s$.

In the interior of a neutron star, there are two distinct phenomena for which the neutrino mean free path is calculated: one is absorption, and the other one involves the scattering of neutrinos [8]. The corresponding mean free paths are denoted by $\lambda_{\text{mean}}$ and $\lambda_{\text{abs}}$. It is to be noted that one also defines another MFP, known as the transport mean free path, which enters into the calculation of the diffusion coefficient. The scattering MFP, on the other hand, is related to the relaxation time that characterizes the rate of change of the neutrino distribution function [8]. One can combine $\lambda_{\text{mean}}$ with $\lambda_{\text{abs}}$ to define the total mean free path [24].

$$\frac{1}{\lambda_{\text{total}}} = \frac{1}{\lambda_{\text{abs}}} + \frac{1}{\lambda_{\text{mean}}}. \quad (1)$$

II. MEAN FREE PATH

In this section, we calculate neutrino mean free paths in quark matter, including NFL corrections both for degenerate and nondegenerate neutrinos. When the neutrino chemical potential ($\mu_n$) is considered to be much larger than the temperature ($T$), the neutrinos become degenerate, and, for nondegenerate neutrinos, $\mu_n \ll T$. In our model, the Lagrangian density is described by [7]

$$\mathcal{L}_{\nu_r}(x) = \frac{G}{\sqrt{2}} l_\mu(x) \mathcal{J}_W^\mu(x) + \text{H.c.}, \quad (2)$$

where the weak coupling constant is $G = 1.166 \times 10^{-11}$ in MeV units, and $l_\mu$ and $\mathcal{J}_W^\mu$ are the lepton and hadron charged weak currents, respectively. The weak currents are

$$l_\mu(x) = \bar{\nu}_e \gamma_\mu (1 - \gamma_5) \nu_e + \bar{\mu} \gamma_\mu (1 - \gamma_5) \nu_\mu + \ldots, \quad (3)$$

$$\mathcal{J}_W^\mu(x) = \cos \theta_c \bar{\nu}_\mu \gamma^\mu (1 - \gamma_5) + \sin \theta_c \bar{\nu}_e \gamma^\mu (1 - \gamma_5) \nu_\mu + \ldots, \quad (4)$$

where $\theta_c$ is the Cabibbo angle ($\cos^2 \theta_c = 0.948$) [25].

The mean free path is determined by the quark-neutrino interaction in dense quark matter via weak processes. We consider the simplest $\beta$ decay reactions: the absorption process

$$d + \nu_e \rightarrow u + e^- \quad (5)$$

and also is its inverse relation

$$u + e^- \rightarrow d + \nu_e. \quad (6)$$

The neutrino mean free path is related to the total interaction rate due to neutrino emission averaged over the initial quark spins and summed over the final state phase space and spins. It is given by [8]

$$\frac{1}{\lambda_{\text{mean}}(E, \nu, T)} = \frac{g}{2E_\nu} \int \frac{d^3 p_d}{(2\pi)^3} \frac{1}{E_d} \int \frac{d^3 p_u}{(2\pi)^3} \frac{1}{E_u} \int \frac{d^3 p_e}{(2\pi)^3} \times \frac{1}{2E_\nu} (2\pi)^4 \delta^4(P_d + P_u - P_e) |M|^2 \times \langle n(p_d)|1 - n(p_d)|1 - n(p_u)|1 - n(p_e)|\rangle - \langle n(p_u)|1 - n(p_u)|1 - n(p_d)|\rangle. \quad (7)$$

where $g$ is the spin and color degeneracy, which, in the present case, is considered to be 6. Here, $E, p,$ and $n_p$ are the energy, momentum, and distribution function for the corresponding particle. $|M|^2$ is the squared invariant amplitude averaged over the initial $d$ quark spin and summed over final spins of the $u$ quark and electron as given by [8]

$$|M|^2 = \frac{1}{2} \sum_{\sigma_u, \sigma_d, \sigma_e} |M_{\sigma}|^2 = 64G^2\cos^2 \theta_c (P_d \cdot P_u)(P_u \cdot P_e). \quad (8)$$

Here, we work with the two-flavor system, as the interaction involving strange quarks is Cabibbo-suppressed [4,9].

A. Degenerate neutrinos

We now consider the case of degenerate neutrinos, i.e., when $\mu_n \gg T$; or, in other words, we consider trapped neutrino matter. So, in this case, both the direct [Eq. (5)] and inverse [Eq. (6)] processes can occur, and both the terms in Eq. (7) under curly brackets [8] are retained. Consequently, the $\beta$ equilibrium condition becomes $\mu_\mu + \mu_e = \mu_n + \mu_\nu$. Neglecting the quark-quark interactions and by using Eqs. (7) and (8), for the mean free path, one obtains
NON-FERMI LIQUID CORRECTIONS TO THE NEUTRINO...

\[
\frac{1}{\text{mean}} = \frac{3}{4\pi^2} G^2 \cos^2 \theta_c \int d^3 p_d \int d^3 p_u \int d^3 p_\nu (1 - \cos \theta_d) \times (1 - \cos \theta_u) \delta^4(P_d + P_\nu - P_u - P_\nu) \times [1 + e^{-\beta E_{\nu,d}}] \eta(p_d)[1 - n(p_u)]\frac{1}{2} n(p_\nu]
\]

(9)

In the square bracket, the second term \(e^{-\beta E_{\nu,d}}\) is due to the inverse process [Eq. (6)]. Since the masses of the \(u\) and \(d\) quarks and electrons are very small, one can neglect the mass effect on the mean free path. To carry out the momentum integration, we define \(p_d + p_\nu = |p_u + p_\nu|\) as a variable. Following the procedure described by Iwamoto [8], one has

\[
\sin \theta_d d\theta_d = \frac{\rho d\rho}{p f(d)p f(v)},
\]

(10)

\[
(1 - \cos \theta_u)(1 - \cos \theta_u) = \frac{p^4 - 2p^2 p_\nu^2 + p_\nu^4}{4 p f(d)p f(v)p f(u)p f(e)}
\]

(11)

and

\[
d^3 p_d = 2\pi \sin \theta_d d\theta_d d\nu d\omega = \frac{p f(d)}{p f(v)} dp d\omega dE_d
\]

\[
d^3 p_u = \frac{2\pi p f(u)}{p f(e)} dp d\omega dE_u
\]

\[
d^3 p_\nu = \frac{2\pi p f(\nu)}{p} dp d\omega dE_\nu
\]

(12)

(13)

where we denote the single particle energy \(E_p(\omega)\) as \(\omega\). For the free case, \(dp/d\omega\) is the inverse quark velocity. It is well-known that this slope of the dispersion relation changes in matter due to scattering from the Fermi surface and excitation of the Dirac vacuum. The modified dispersion relation can be obtained by computing the on-shell one-loop self-energy. For quasiparticles with momenta close to the Fermi momentum, the one-loop self-energy is dominated by the soft gluon exchanges [26]. The quasiparticle energy \(\omega\) satisfies the relation [26,27]

\[
\omega = E_p(\omega) + \text{Re} \Sigma(\omega, p(\omega)),
\]

(14)

where we have approximated only to the real part of self-energy, since the imaginary part of \(\Sigma\) turns out to be negligible compared to its real part [14,16]. The detailed analysis can be obtained in [27,28].

In the relativistic case, when the Fermi velocity \(v_f\) is close to the velocity of light \(c\), the exchange of magnetic gluons becomes important. In the nonrelativistic case, it is suppressed by a factor \((v/c)^2\) with respect to the exchange of electric gauge bosons and is usually neglected. The magnetic interaction, as stated before, is screened only dynamically, and the problem remains for the static gluons [26,29]. Therefore, to obtain a finite result, a suitable resummation has to be performed [30,31]. The analytical expressions for one-loop quark self-energy can be written as [14–16,21,26]

\[
\Sigma = \frac{g_2^{2}C_F}{12\pi^2} (\omega - \mu) \ln \left( \frac{m_D}{\omega - \mu} \right) + \frac{g_2^{2}C_F}{12\pi^2} (\omega - \mu).
\]

(15)

It exhibits a logarithmic singularity close to the Fermi surface, i.e., \(\omega \rightarrow \mu\). Thus, the long-ranged character of the magnetic interactions spoils the normal Fermi liquid behavior [12,13,17–20]. The breakdown of the Fermi liquid picture is associated with the vanishing of the discontinuity of the distribution function at the Fermi surface [5,21]. This nonperturbative nature of the self-energy gives rise to the non-Fermi liquid behavior. Here, \(m_D\) is a cutoff factor and should be an order of the Debye mass. Differentiating Eq. (14) with respect to \(p\), we obtain \(d\epsilon(\omega)\) at leading order in \(\frac{\omega}{m}\) as

\[
\frac{d\epsilon(\omega)}{d\omega} \propto \frac{1}{1 + C_F a_s} \ln \left( \frac{m_D}{\omega - \mu} \right) \frac{E_p(\omega)}{p(\omega)}.
\]

(16)

To derive the above, Eq. (17), we use the result of the integral [8,32]

\[
\int_0^\infty dE_d \int_0^\infty dE_u \int_0^\infty dE_\nu [1 + e^{-\beta E_{\nu,d}}] \eta(p_d)[1 - n(p_u)]\frac{1}{2} n(p_\nu]
\times [1 - n(p_u)] \delta(E_d + E_u - E_\nu - E_v)\]

\[
\approx \frac{1}{2} (E_\nu - \mu_\nu)^2 + \pi^2 T^2.
\]

(17)

Similarly, for \(p_f(d) - p_f(\nu)\) \(\approx [p_f(u) - p_f(e)]\), the corresponding expression for the mean free path can be obtained by replacing \(\mu_\nu \leftrightarrow \mu_d\) and \(\mu_\nu \leftrightarrow \mu_u\) in Eq. (17). Since quarks and electrons are assumed to be massless, the chemical equilibrium condition gives \(p_f(d) + p_f(\nu)\), which we use to derive Eq. (17).

The other major contribution to the mean free path arises from quark-neutrino scattering. The neutrino scattering process from degenerate quarks is given by

\[
q \rightarrow q + \nu_\nu(\bar{q}) \rightarrow q + \nu_\nu(\bar{q})
\]

(19)
for each quark component of the flavor \( i = u \) or \( d \). The scattering mean free path of the neutrinos in the degenerate case can be calculated similarly, as evaluated by Lamb and Pethick in [23] for electron-neutrino scattering. Assuming \( m_q / p_f \ll 1 \) and including the non-Fermi liquid correction through phase space, the mean free path is given by

\[
\frac{1}{l_{\text{mean}}} = \frac{3}{16} n_q \alpha_0 \left[ \frac{(E_u - \mu_u)^2 + \pi^2 T^2}{m_q^6} \right] \times \left[ 1 + \frac{C_F \alpha_s}{3\pi} \ln \left( \frac{m_D}{T} \right) \right]^2 \Lambda(x_i).
\]  

Here, \( m_q \) is the quark mass. \( C_V \) and \( C_A \) are the vector and axial vector coupling constants given in Table II of Ref. [8]. In Eq. (20), if we drop the color factor and the second square bracketed term, we obtain the results reported in [23] for dense and cold QED plasma with \( m_q \) replaced by \( m_t \) in Eq. (20), the constants \( \alpha_0 = 4G^2 m_q^6 / \pi^2 \) [22], and \( n_q \) is the number density of the quarks, given by

\[
n_q = 2 \int \frac{d^3 p}{(2\pi)^3} \frac{1}{e^{E/p} - 1},
\]

where \( 2 \) is the quark spin degeneracy factor. The explicit form of \( \Lambda(x_i) \) can be written as [8,23]

\[
\Lambda(x_i) = \frac{4}{3} \left\{ \frac{\mu_{q_i}}{\mu_q} \right\} \left[ \frac{C_V}{x_i} + \frac{C_A}{2x_i + 2} \right] + 2C_V C_A x_i.
\]

and \( x_i = \mu_{q_i} / \mu_q \) if \( \mu_{q_i} < \mu_q \), and \( x_i = \mu_q / \mu_{q_i} \) if \( \mu_{q_i} > \mu_q \).

### B. Nondegenerate neutrinos

We also derive the mean free path for nondegenerate neutrinos, i.e., when \( \mu_{q_i} \ll T \). For nondegenerate neutrinos, the inverse process (6) is dropped. Hence, we neglect the second term in the curly braces of Eq. (7). In this case, only those fermions whose momenta lie close to their respective Fermi surfaces can take part in a reaction. It is to be mentioned here, if quarks are treated as free, as discussed in [8,33,34], the matrix element vanishes, since \( u \) and \( d \) quarks and electrons are collinear in momenta. The inclusion of strong interactions between quarks relaxes these kinematic restrictions, resulting in a nonvanishing squared matrix amplitude. Since the neutrinos are produced thermally, we neglect the neutrino momentum in the energy-momentum conservation relation [8]. This is not the case for degenerate neutrinos where \( p_s \gg T \), and, therefore, such an approximation is not valid here. By doing an angular average over the direction of the outgoing neutrino, from Eq. (8), the squared matrix element is given by [9]

\[
|M|^2 = 64G^2 \cos^2 \theta p_f^2 (V_d \cdot p_f)(V_u \cdot p_u)
\]

\[
= 64G^2 \cos^2 \theta p_r^2 \frac{C_F \alpha_s}{\pi},
\]

where \( V = (1, v_f) \) is the four-velocity. To calculate Eq. (23), we have used the chemical equilibrium condition \( \mu_d = \mu_u + \mu_\nu \), and also, the relations derived from the Fermi liquid theory are given by [9]

\[
\mu_\nu = \frac{C_F \alpha_s}{2\pi} \delta \mu = \frac{C_F \alpha_s}{\pi} \mu_\nu.
\]

Putting \( |M|^2 \) in Eq. (7), we have

\[
\frac{1}{\Pi_{\text{ND}}^{\text{mean}}} = \frac{3C_F \alpha_s}{4\pi^2} \cos^2 \theta \int d^3 p_s \int d^3 p_u 
\]

\[
\times \int d^3 p_e \delta^4(P_d + P_e - p_u - p_s)n(p_d)
\]

\[
\times \left[ 1 - n(p_u)[1 - n(p_s)] \right].
\]

Neglecting the neutrino momentum in the neutrino momentum conserving \( \delta \) function, the integrals can be decoupled into two parts. Following the procedure of Iwamoto [8], the angular integral is given by

\[
\mathcal{A} = \int d\Omega_d \int d\Omega_u \int d\Omega_e \delta(p_d - p_u - p_e)
\]

\[
= \frac{8\pi^2}{\mu_d \mu_u \mu_\nu},
\]

and the other part by

\[
\mathcal{B} = \int_0^{\infty} p_s^2 dE_s dE_u \int_0^{\infty} p_u^2 dE_u dE_s
\]

\[
\times \int_0^{\infty} p_e^2 dE_e \delta(E_u + E_e - E_u - E_e) \delta(n(p_u)[1 - n(p_s)] - n(p_e)[1 - n(p_u)]).
\]

Changing the variables to \( x_d = (E_d - \mu_d) \beta \), \( x_u = (E_u - \mu_u) \beta \), and \( x_e = (E_e - \mu_e) \beta \), and denoting the single particle energy \( E_{s(d)} \) as \( \omega \), we have, from Eq. (27),

\[
\mathcal{B} = \int_{-\infty}^{\infty} dx_d dx_u dx_e \frac{d^2 p_s(\omega)}{d\omega} \frac{d^2 p_u(\omega)}{d\omega} - \frac{p_s^2 p_u^2}{p_e^2} \times \delta(x_d + x_u + x_e + \beta E_s)n(x_d)n(-x_u)n(-x_e).
\]

As the contribution dominates near the Fermi surfaces, the extension of the lower limit is a reasonable approximation [32,35].

Using Eq. (16) and performing the integration of Eq. (28) following the procedure defined in [16,32,35], we have

\[
\mathcal{B} = \mu_d^2 \mu_u^2 \mu_\nu^2 \left[ \frac{E_s^2 + \pi^2 T^2}{2(1 + e^{-\beta E_s})} \right] \left[ 1 - n(p_u)[1 - n(p_s)] \right].
\]

Using Eqs. (25), (26), and (29), the mean free path at leading order in \( T / \mu \) is given by
The first term is known from [8], and the additional terms are higher-order corrections to the previous results derived in the present work.

For the scattering of nondegenerate neutrinos in quark matter, the expression of the mean free path was given by Iwamoto [8]. We incorporate the anomalous effect which enters through phase space modification, giving rise to

\[
\frac{1}{\rho_{\text{scatt,ND}}^\text{mean}} = \frac{3C_F}{\pi^2} G^2 \cos^2 \theta \mu_d \mu_u \mu_e \frac{(E_v^2 + \pi^2 T^2)}{(1 + e^{-\beta E_v})} \times \left[ 1 + \frac{C_F \alpha_s}{3\pi} \ln \left( \frac{m_D}{T} \right) \right]^2.
\]

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For the scattering of nondegenerate neutrinos in quark matter, the expression of the mean free path was given by Iwamoto [8]. We incorporate the anomalous effect which enters through phase space modification, giving rise to

\[
\frac{1}{\rho_{\text{scatt,ND}}^\text{mean}} = \frac{C_F}{20} \frac{\alpha_s}{\pi^2} n_q \sigma_0 \left( \frac{E_v}{m_q} \right)^2 \left( \frac{E_v}{\mu_e} \right) \times \left[ 1 + \frac{C_F \alpha_s}{3\pi} \ln \left( \frac{m_D}{T} \right) \right]^2.
\]

Here, we have assumed \( m_q / p_f \ll 1 \) and the constants \( \sigma_0 \) and number density \( n_q \) defined earlier.

III. RESULTS AND CONCLUSION

Armed with the results of the previous sections, we now estimate the numerical values of the neutrino mean free paths. Here, \( E_v \) is set to be equal to 37, and \( m_q = 10 \text{ MeV} \) [8,24]. For the quark chemical potential, following Ref. [9], we take \( \mu_q \approx 500 \text{ MeV} \) corresponding to densities \( \rho_q = 6\rho_0 \), where \( \rho_0 \) is the nuclear matter saturation density. The electron chemical potential is determined by using the charge neutrality and beta equilibrium conditions, which yields \( \mu_e = 11 \text{ MeV} \). The other parameters used are the same as in [9].

From Fig. 1, we find that, for degenerate neutrinos, the anomalous logarithmic terms reduce the value of the mean free path appreciably both in the low and high temperature regime. Figure 2 shows that, for nondegenerate neutrinos, NFL corrections are quite large at low temperature, while, at higher temperature, they tend to merge. It is interesting to see, from these two plots, that NFL corrections to the MFP in degenerate neutrinos are less than those of non-degenerate neutrinos. This reduced mean free path is expected to influence the cooling of the compact stars. It is also to be noted that all the results presented above are restricted to the leading log approximation which can be extended to take the next-to-leading-order corrections into account [36]. We plan to undertake such investigations in a future publication [37].

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