PRELIMINARIES

Our terminology is standard. Unless otherwise stated X and Y are
topological spaces and \( \mathbb{R} \) and \( \mathbb{N} \) denote the real line and the set of natural numbers
respectively and \( \emptyset \) denotes the null set. We denote the closure and interior of a
subset \( A \) of \( X \) respectively by \( \text{Cl}(A) \) and \( \text{Int}(A) \) and the difference of two sets \( A \)
and \( B \) is denoted by \( A \setminus B \). The following definitions and results are used here.

DEFINITION 0.1:

Let \((X, \rho)\) be a metric space. If \( A \) and \( B \) are subsets of \( X \) then we say that
\( A \) is dense in \( B \) if \( B \subseteq \text{Cl}(A) \). In particular, \( A \) is said to be everywhere dense if
\( \text{Cl}(A) = X \).

DEFINITION 0.2:

Let \((X, \rho)\) be a metric space. A set \( A \) subsets of \( X \) is said to be nowhere
dense if its closure \( \text{Cl}(A) \) contains no sphere, or equivalently if \( \text{Cl}(A) \) has no
interior point.

DEFINITION 0.3:

If a set \( A \) is the union of countable number of nowhere dense sets then \( A \) is
said to be of first category.
DEFINITION 0.4:
A set, which is not of first category is said to be a set of second category.

DEFINITION 0.5:
The complement of a set of first category is called a residual set.

DEFINITION 0.6:
If a set is obtained as the sum of denumerable number of closed sets then the set is said to be a set of type $F_\sigma$.

DEFINITION 0.7:
If a set is obtained as the intersection of denumerable number of open sets then the set is said to be a set of type $G_\delta$.

PROPOSITION 0.7.1:
The complement of a set of type $F_\sigma$ is a set of type $G_\delta$ and the complement of a set of type $G_\delta$ is an $F_\sigma$ set.

DEFINITION 0.8:
The functions, which are pointwise limits of convergent sequences of continuous functions are said to be of Baire class one, denoted by $B_1$. 
DEFINITION 0.9: [71]
A subset $A$ of $X$ is said to have the property of Baire if it can be represented in the form $A = (G \setminus P) \cup Q$, where $G$ is open and $P, Q$ are sets of first category.

DEFINITION 0.10: [34]
A topological space $X$ is said to be a Baire space if every nonempty open set of $X$ is a set of second Baire category.

DEFINITION 0.11:
A set $A \subseteq X$ is said to be $B^*$ set if it is not nowhere dense in $X$ and have the property of Baire.

DEFINITION 0.12: [13]
A perfect road of a function $f$ at a point $x$ is a perfect set $P$ such that,

i) $x$ is a bi-lateral point of accumulation of $P$, and

ii) $f|_P$ is continuous at $x$.

DEFINITION 0.13:
A function $f: X \to Y$ is continuous at a point $x \in X$ if, for each open neighbourhood $V$ of $f(x)$, there is an open neighbourhood $U$ of $x$ such that $f(U) \subseteq V$. 
DEFINITION 0.14: [24]
A function $f: X \to Y$ is quasi-continuous at a point $x \in X$ if, for each open
neighbourhood $U$ of $x$ and each open neighbourhood $V$ of $f(x)$, there is a nonempty
open set $G \subseteq U$ such that $f(G) \subseteq V$.

If $f$ is quasi-continuous at each point it is said to be quasi-continuous on $X$.

DEFINITION 0.15: [50]
A set $A \subseteq X$ is said to be semi-open if and only if there exists an open set $O$
such that $O \subseteq A \subseteq \text{Cl}(O)$. In other words, $A$ is said to be semi-open if $A \subseteq \text{Cl}(%20\text{Int}(A))$.

We will write $U \in \text{S.O.}(X, x)$ whenever $U$ is semi-open in $X$ and contains $x$.

DEFINITION 0.16: [26]
A subset $A$ of a topological space $X$ is said to be pre-open if $A \subseteq \text{Int}(\text{Cl}(A))$.

DEFINITION 0.17: [50]
A function $f: X \to Y$ is semi-continuous in the sense of Levine if for any
open set $V$ in $Y$, $f^{-1}(V)$ is semi-open in $X$.

DEFINITION 0.18: [5]
A function $f: X \to Y$ is simply continuous if for each open set $V$ in $Y$, the
set $f^{-1}(V)$ is the union of an open set and a nowhere dense set in $X$. 
DEFINITION 0.19: [62]
A function $f: X \to Y$ is called almost continuous at a point $p \in X$, if for any
neighbourhood $V$ of the point $f(p)$ in $Y$ we have $p \in \text{Int}(\text{Cl}(f^{-1}(V)))$.
If $f$ is almost continuous at each point then it is said to be so on $X$.

DEFINITION 0.20: [85]
A function $f: X \to Y$ (where $Y$ is a metric space with metric $d$) is cliquish at a
point $x \in X$ if for each $\epsilon > 0$ and each open neighbourhood $U$ of $x$, there is a
nonempty open set $G \subset U$ such that, $d(f(x_1), f(x_2)) < \epsilon$ for any $x_1, x_2 \in G$.
If $f$ is cliquish at each point, it is said to be so over the whole of $X$.

DEFINITION 0.21: [55]
A function $f: X \to Y$ is said to be B-continuous at $x \in X$, if for any open
neighbourhood $U$ of $x$ and $V$ of $f(x)$, there is a set $B$ which is either open or of
second category having the Baire property such that $B \subset U \cap f^{-1}(V)$.
If $f$ is B-continuous at each point of $X$ it is said to be B-continuous over $X$.

DEFINITION 0.22:
A function $f: X \to Y$ is said to be $B^*$-continuous at $x$ if for each open set $U$
containing $x$ in $X$ and each open set $V$ containing $f(x)$ in $Y$, there is a $B^*$-set such
that $B \subset U \cap f^{-1}(V)$.
If $f$ be $B^*$-continuous at each point of $X$, then $f$ is said to be $B^*$-continuous on $X$. 
DEFINITION 0.23:

A multifunction $F: X \to Y$ is said to be upper (lower) continuous at a point $p \in X$ if for any open set $V$, $F(p) \subseteq V (F(p) \cap V \neq \emptyset)$, there exists a neighbourhood $U$ of $p$ such that $F(x) \subseteq V (F(x) \cap V \neq \emptyset)$ for any $x \in U$. It is called continuous at $p$ if it is both upper and lower continuous.

DEFINITION 0.24: [60]

A multifunction $F: X \to Y$ is said to be upper (lower) quasi-continuous at a point $p \in X$ if for any open set $V \subseteq Y$, such that $F(p) \subseteq V (F(p) \cap V \neq \emptyset)$, and for any open set $U$ containing $p$, there exists a nonempty open set $G \subseteq U$ such that $F(x) \subseteq V (F(x) \cap V \neq \emptyset)$ for any $x \in G$.

It is said to be upper (lower) quasi-continuous if it is so at any point $x \in X$.

DEFINITION 0.25:

Let $X$ be a topological space. Then a family $U$ of subsets of $X$ is said to be locally finite if for each $x \in X$, there exists a neighbourhood $N$ of $x$ which intersects only finitely many members of $U$.

DEFINITION 0.26: [47]

A space is called paracompact if it is regular and if every open cover of it has an open locally finite refinement, which is also a cover.
DEFINITION 0.27: [83]

Let $\Omega$ be the first uncountable ordinal number; let $X$ be a set and $Y$ be a metric space. A transfinite sequence $\{f_\zeta\}_{\zeta \in \Omega}$ of functions from $X$ into $Y$ converges pointwise to a function $f: X \rightarrow Y$ if for every $x \in X$ and every neighbourhood $U$ of $f(x)$ there exists an ordinal number $\eta < \Omega$ such that $f_\zeta(x) \in U$ for every $\eta < \zeta < \Omega$.

We shall denote this convergence by $f_\zeta \rightarrow f$ or more precisely by $\lim_{\zeta \in \Omega} f_\zeta = f$.

DEFINITION 0.28: [21]

Let $X$ be a set and $Y$ be a metric space (with metric $d$). Then the sequence $\{f_n\}_{n=1}^\infty$ of functions $f_n: X \rightarrow Y$ ($n = 1, 2, \ldots$) is said to converge quasi-uniformly to a limit function $f: X \rightarrow Y$, if

(i) $f_n$ converges pointwise to $f$ and

(ii) for every $\varepsilon > 0$, for each $n \in \mathbb{N}$, there exists $r(n) \in \mathbb{N}$ such that $\sup_{x} \min_{0 < i < r(n)} d(f_{n+i}(x), f(x)) < \varepsilon$.

DEFINITION 0.29: [15]

Let $f_n, f: X \rightarrow \mathbb{R}, n = 1, 2, \ldots$. We shall say that the sequence $\{f_n\}_{n=1}^\infty$ converges quasi-normally to $f$ on $X$, if there is a sequence $\{\varepsilon_n\}_{n=0}^\infty$ of nonnegative real numbers converging to zero such that for every $x \in X$ there is an index $k_x$ such that $|f_n(x) - f(x)| \leq \varepsilon_n$ for every $n \geq k_x$. 
DEFINITION 0.30:

Let $A = (a_{mn})$ be an infinite matrix with elements in the real line $\mathbb{R}$ and $\{s_n\}$ be a sequence of real numbers. Let us consider $t_m = \sum_{n=1}^{m} a_{mn} s_n$ where it is assumed that the right hand series is convergent for all $m = 0, 1, 2, \ldots$ Then $\{t_m\}$ represents A-transform of $\{s_n\}$ generated by the matrix $A = (a_{mn})$. This is a linear transformation.

If the sequence $\{t_m\}$ converges to the limit $s$ then the sequence $\{s_n\}$ is said to be A-summable to $s$ and it is written as $\text{A-lim } s_n = s$.

DEFINITION 0.31:

Let $A=(a_{mn})$ be an infinite matrix with elements in the real line $\mathbb{R}$ and $\{s_n\}$ be a sequence of real numbers. If $t_m \to s$ as $m \to \infty$ whenever $s_n \to s$, then the matrix transformation is called regular transformation and the infinite matrix $A = (a_{mn})$ is said to be regular. Symbolically we shall write $s_n \xrightarrow{A} s$.

RESULT 0.31.1: [86]

A necessary and sufficient condition for an infinite matrix $A = (a_{mn})$ to be regular or Toeplitz matrix or simply T-matrix (according to O. Toeplitz) are

\begin{align*}
\text{i) } \sup_{m} \sum_{n=1}^{m} |a_{mn}| < \infty
\end{align*}
ii) \( \lim_{m \to \infty} a_{mn} = 0 \) for all \( n \)

iii) \( \lim_{m \to \infty} \sum_{n=1}^{m} a_{mn} = 1 \).

**DEFINITION 0.32:**

An infinite matrix \( A = (a_{mn}) \) is called triangular matrix if \( a_{mn} = 0 \) for \( n > m \).

**DEFINITION 0.33:**

The triangular matrix \( A = (a_{mn}) \) is called Cesaro matrix of order 1 and is designated by \( (C, 1) \), if \( a_{mn} = \frac{1}{m} \), \( n \leq m \); and \( = 0 \), \( n > m \).