CHAPTER - 4

ON B*-CONTINUOUS AND B*-CLUSTER CONTINUOUS MULTIFUNCTIONS

4.1 INTRODUCTION:

In this chapter we introduce new notions of upper and lower B*- continuous multifunctions as well as upper and lower B*- cluster continuous multifunctions defined on a topological space X, and obtain some characterizations along with some properties of such functions in connection with B*- closed or B*-open sets.

The multifunctions considered here are defined on X and assume their values in P(Y) \ φ, where P(Y) is the power set of Y. Multifunctions are denoted by capital letters F, G, H etc.

In case of multifunctions we write simply F: X → Y instead of F: X → P(Y) \ φ.

If F: X → Y is a multifunction then for A ⊆ Y we denote F+(A) = {x ∈ X: F(x) ⊆ A} and F−(A) = {x ∈ X: F(x) ∩ A ≠ φ}. It is clear that F+(Y \ A) = X \ F+(A) and F−(Y \ A) = X \ F−(A).
4.2 THE B*-CONTINUOUS MULTIFUNCTION:

**Definition 4.2.1:** A multifunction \( F: X \rightarrow Y \) is lower (upper) B*-continuous at a point \( x \) if for every open sets \( U, V \) with \( x \in U \), \( F(x) \cap V \neq \emptyset \) (\( F(x) \subseteq V \)), there exists a B*-set \( B \) such that \( B \subseteq F(V) \cap U \) (\( B \subseteq F^*(V) \cap U \)).

\( F \) is B* Continuous at \( x \) if it is both lower and upper B* - continuous at \( x \).

\( F \) is lower B* Continuous, upper B* Continuous and B* Continuous over \( X \) if it is respectively so at any point \( x \).

It is clear that upper (lower) quasi-continuity implies upper (lower) B*-continuity. But the converse is not true which follows from the example below.

**Example 4.2.1:** Let \( X = Y = [0, 1] \) with the usual topology.

Define \( F: X \rightarrow Y \) by,

\[
F(x) = \begin{cases} 
0 & \text{if } x \text{ is irrational} \\
[0, 1] & \text{if } x \text{ is rational}
\end{cases}
\]

\( F \) is not upper quasi-continuous at \( x \) if \( x \) is irrational. But \( F \) is upper B*-continuous at any irrational point \( x \).

We now give some characterisation of upper B*-continuous multifunction:

**Theorem 4.2.1:** For a multifunction \( F: X \rightarrow Y \) the following conditions are equivalent:

1. \( F \) is upper B*-continuous at a point \( x \in X \).
(2) For each open neighbourhood $U$ of $x$ and any open set $V$ of $Y$ with $x \in F^{*}(V)$, $F^{*}(V) \cap U$ is not nowhere dense.

(3) For each open set $U$ containing $x$ and each open set $V$ of $Y$ with $x \in F^{*}(V)$, there exists a nonempty open set $W$ of $X$ with $W \subset U$ such that $W \subset \operatorname{Cl}(F^{*}(V))$.

(4) For each open set $V$ of $Y$ with $x \in F^{*}(V)$, there exists $U_{0} \in S.O.(X, x)$ such that $U_{0} \subset \operatorname{Cl}(F^{*}(V))$.

(5) $F^{*}(V) \subset \operatorname{Cl}(\operatorname{Int}(\operatorname{Cl}(F^{*}(V))))$, for each open set $V$ of $Y$.

(6) $\operatorname{Int}(\operatorname{Cl}(\operatorname{Int}(F^{*}(A)))) \subset F^{*}(A)$, for every closed set $A$ in $Y$.

**Proof:** (1) $\Rightarrow$ (2) Obvious.

(2) $\Rightarrow$ (3): From (2) it follows that for every open neighbourhood $U$ of $x$ and any open set $V$ of $Y$ with $x \in F^{*}(V)$, $F^{*}(V) \cap U$ is not nowhere dense. Hence, there exists an open set $W \subset U$, such that $W' \cap F^{*}(V) \cap U \neq \emptyset$ for every open subset $W'$ of $W$. This implies that $W \subset \operatorname{Cl}(F^{*}(V))$.

(3) $\Rightarrow$ (4): From (3) it follows that for every open neighbourhood $U$ of $x$ and any open set $V$ of $Y$ with $x \in F^{*}(V)$, there exists a nonempty open set $G$ of $X$ such that $G \subset U$ and $G \subset \operatorname{Cl}(F^{*}(V))$. Let $V$ be an open set of $Y$ containing $F(x)$. Let $U(x)$ be a family of open neighbourhoods of $x$. For each $U \in U(x)$, there exists a nonempty open set $G(U)$ of $X$ such that $G(U) \subset U$ and $G(U) \subset \operatorname{Cl}(F^{*}(V))$. Set $W = \bigcup_{U} G(U)$. Then $W$ is an open set of $X$, $x \in \operatorname{Cl}(W)$ and $W \subset \operatorname{Cl}(F^{*}(V))$. Now take $U_{0} = W \cup \{x\}$. Then $W \subset U_{0} \subset \operatorname{Cl}(W)$ and $U_{0} \in S.O.(X, x)$ and also $U_{0} \subset \operatorname{Cl}(F^{*}(V))$. 

(4) \Rightarrow (5): Let \( V \) be any open set in \( Y \) and \( x \in F^+(V) \). Then there exists \( U \in S.O.(X,x) \) such that \( U \subseteq \text{Cl}(F^+(V)) \). Again \( x \in U \subseteq \text{Cl}(\text{Int}(U)) \), as \( U \) is semi open.

Again, \( \text{Cl}(\text{Int}(U)) \subseteq \text{Cl}(\text{Int}(\text{Cl}(F^+(V)))) \).

i.e. \( x \in \text{Cl}(\text{Int}(\text{Cl}(F^+(V)))) \). Therefore, \( F^+(V) \subseteq \text{Cl}(\text{Int}(\text{Cl}(F^+(V)))) \).

(5) \Rightarrow (6): Let \( A \) be any closed set in \( Y \). Then \( Y \setminus A \) is open.

Then, \( F'(Y \setminus A) \subseteq \text{Cl}(\text{Int}(\text{Cl}(Y \setminus F(A)))) \).

This implies, \( X \setminus F'(A) \subseteq \text{Cl}(\text{Int}(X \setminus F(A))) \)

\[ = \text{Cl}(\text{Int}(X \setminus \text{Int}(F^+(A)))) \]
\[ = \text{Cl}(X \setminus \text{Cl}(\text{Int}(F'(A)))) \]
\[ = X \setminus \text{Int}(\text{Cl}(\text{Int}(F^+(A)))) \].

Therefore, \( \text{Int}(\text{Cl}(\text{Int}(F(A)))) \subseteq F(A) \).

(6) \Rightarrow (5): Similar.

(5) \Rightarrow (1): Let \( x \in X \) and \( U \) be any open set of \( X \) containing \( x \) and \( V \) be any open set of \( Y \) such that \( F(x) \subseteq V \). Then, \( x \in F^+(V) \subseteq \text{Cl}(\text{Int}(\text{Cl}(F(V)))) \).

Hence, \( \varnothing \neq U \cap \text{Int}(\text{Cl}(F^+(V))) = \text{Int}(U \cap \text{Cl}(F^+(V))) \subseteq \text{Cl}(U \cap F^+(V)) \).

This implies, \( \text{Cl}(U \cap F^+(V)) \neq \varnothing \) and \( U \cap F^+(V) \) is not nowhere dense.

Thus, \( (U \cap F^+(V)) \cap (U \cap \text{Int}(\text{Cl}(F^+(V)))) \neq \varnothing \). \( \Rightarrow U \cap F^+(V) \cap \text{Int}(\text{Cl}(F^+(V))) \neq \varnothing \).

i.e. \( U \cap H \neq \varnothing \), where \( H = F^+(V) \cap \text{Int}(\text{Cl}(F^+(V))) \) is a pre open set [26].

Let \( B = U \cap H \). Then \( B \) is a nonempty pre open set [26] and hence is a \( B^* \)-set.

Also, \( B \subseteq (U \cap F^+(V)) \). Hence, \( F \) is upper \( B^* \)-continuous at \( x \).
Theorem 4.2.2: For a multifunction $F: X \to Y$ the following conditions are equivalent:

1) $F$ is lower $B^*$-continuous at a point $x \in X$.

2) For each open neighbourhood $U$ of $x$ and any open set $V$ of $Y$ with $x \in F(V)$, $F'(V) \cap U$ is not nowhere dense.

3) For each open set $U$ containing $x$ and each open set $V$ of $Y$ with $x \in F'(V)$, there exists a nonempty open set $W \subset U$ such that $W \subset \text{Cl}(F'(V))$.

4) For each open set $V$ of $Y$ with $x \in F'(V)$, there exists $O \in \text{S.O.}(X, x)$ such that $O \subset \text{Cl}(F'(V))$.

5) $F'(V) \subset \text{Cl}(\text{Int}(\text{Cl}(F'(V))))$, for each open set $V$ of $Y$.

6) $\text{Int}(\text{Cl}(\text{Int}(F'(A)))) \subset F'(A)$, for every closed set $A$ in $Y$.

The proof is similar to theorem 4.2.1.

4.3 UPPER (LOWER) $B^*$-CLUSTER CONTINUOUS MULTIFUNCTION AND ITS CONVERGENCE

In this section we introduce the concept of $B^*$-open and $B^*$-closed set; and $B^*$-cluster continuous multifunction is defined with the help of these sets. Also we introduced the notion of upper and lower semi-uniform convergence and study some related properties.
**Definition 4.3.1:** Let $X$ be a topological space and $P$ be a subset of $X$. $x \in X$ is said to be $B^*$-cluster point of the set $P$ if for every $B^*$-set $B$ including $x$, $P \cap B \neq \emptyset$.

The set of all $B^*$-cluster points of $P$ is called the cluster derived set of $P$ and denoted by $\text{cls-d-} P$.

A set $P$ is said to be a $B^*$-closed set, if $P = \text{cls-d-} P$.

The complement of a $B^*$-closed set is $B^*$-open.

A multifunction $F$ is said to be lower (upper) $B^*$-cluster continuous if $F^-(V)$ ($F^+(V)$) is $B^*$-closed for every closed set $V$ in $Y$.

**Example 4.3.1:** In the set $\mathbb{R}$ of real numbers with usual topology, the set $\mathbb{Q}$ of rational numbers and $\mathbb{R}\setminus \mathbb{Q}$ of irrational numbers are $B^*$-closed as well as $B^*$-open.

**Example 4.3.2:** The set $\mathbb{R} \setminus \{1, 2, \ldots, n\}$ is not $B^*$-closed as well as not $B^*$-open.

In what follows $(\Lambda, \geq)$ is a directed set, $\{F_a\}$ is a net of multifunctions $F_a : X \rightarrow Y$, $a \in \Lambda$ and $F$ is a multifunction on $X$ into $Y$.

**Definition 4.3.2:** $\{F_a : a \in \Lambda\}$ is said to be upper semi-uniformly convergent to $F$ on $X$ if

(i) For every open set $U$ of $Y$ with $F^+(U) \neq \emptyset$, and for every $a \in \Lambda$ there exists $a_0 \in \Lambda$ with $a_0 \geq a$ such that $x \in F_{a_0}^+(U)$ for all $x \in F^+(U)$. 

(ii) For every $x \in X$ and every open set $U$ of $Y$ such that $x \in F^*(U)$ there exists $a_0 \in \Lambda$ such that $x \in F_{a_0}^-(U)$ for all $a \geq a_0$.

**Definition 4.3.3:** $\{F_a : a \in \Lambda\}$ is said to be lower semi-uniformly convergent to $F$ on $X$ if

(i) For every open set $U$ of $Y$ with $F^-(U) \neq \emptyset$, and for every $a \in \Lambda$ there exists $a_0 \in \Lambda$ with $a_0 \geq a$ such that $x \in F_{a_0}^-(U)$ for all $x \in F^*(U)$.

(ii) For every $x \in X$ and every open set $U$ of $Y$ such that $x \in F^*(U)$ there exists $a_0 \in \Lambda$ such that $x \in F_{a_0}^+(U)$ for all $a \geq a_0$.

**Definition 4.3.4:** $\{F_a : a \in \Lambda\}$ is said to be semi-uniformly convergent to $F$ on $X$ if it is upper as well as lower semi-uniformly convergent to $F$ on $X$.

**Theorem 4.3.1:** The following conditions are equivalent:

(1) $F$ is lower $B^*$-cluster continuous.

(2) For each open set $V$ of $Y$, $F^*(V)$ is $B^*$-open in $X$.

(3) For each $x \in X$ and for every open set $V$ of $Y$, such that $x \in F^*(V)$ there is a $B^*$-set $B$ in $X$ containing $x$ such that $B \subseteq F^*(V)$.

**Proof:** (1) $\iff$ (2): $F$ is lower $B^*$-cluster continuous. Let $V$ be any open set in $Y$. Then $(Y \setminus V)$ is closed in $Y$. Then $F^-(Y \setminus V)$ is $B^*$-closed.
But \( F^*(V) = X \setminus F^-(Y \setminus V) \), i.e. \( F^*(V) \) is \( B^* \)-open in \( X \); and conversely.

(2) \( \Rightarrow \) (3) : Let \( x \in X \) and \( V \) be an open set in \( Y \) containing \( F(x) \). By hypothesis, \( F^*(V) \) is \( B^* \)-open in \( X \). But \( F^*(V) = X \setminus F^-(Y \setminus V) \). So, \( F^-(Y \setminus V) \) is \( B^* \)-closed in \( X \).

Obviously \( x \notin F^-(Y \setminus V) \). Therefore \( x \) is not a \( B^* \)-cluster point of \( F^-(Y \setminus V) \).

Then, there exists a \( B^* \)-set \( B \) containing \( x \), \( B \cap F^-(Y \setminus V) = \emptyset \).

\[ \Rightarrow \quad F(B) \cap (Y \setminus V) = \emptyset. \Rightarrow \quad F(B) \subseteq V \Rightarrow \quad B \subseteq F^*(V). \]

(3) \( \Rightarrow \) (2) : Let \( V \) be an open set in \( Y \) and let \( x \notin F^*(Y \setminus V) \), i.e. \( F(x) \subseteq V \). Then, \( x \in F^*(V) \).

By hypothesis there exists a \( B^* \)-set \( B \) containing \( x \) such that \( B \subseteq F^*(V) \).

So, \( F(B) \cap (Y \setminus V) = \emptyset \). Hence, \( B \cap F^*(Y \setminus V) = \emptyset \).

Consequently, \( F^*(Y \setminus V) \) is \( B^* \)-closed.

But \( F^*(V) = X \setminus F^-(Y \setminus V) \) and hence \( F^*(V) \) is \( B^* \)-open.

**Theorem 4.3.2:** The following conditions are equivalent:

1) \( F \) is upper \( B^* \)-cluster continuous.

2) For each open set \( V \) of \( Y \), \( F^*(V) \) is \( B^* \)-open in \( X \).

3) For each \( x \in X \) and for every open set \( V \) of \( Y \), such that \( x \in F^*(V) \) there is a \( B^* \)-set \( B \) such that \( B \subseteq F^*(V) \).

Proof is similar to theorem 4.3.1.

**Theorem 4.3.3:** Let \( \{F_a\}_{a \in \Lambda} \) be a net of lower \( B^* \)-cluster continuous multifunctions from \( X \) to a normal space \( Y \). If \( \{F_a\}_{a \in \Lambda} \) is lower semi-uniformly convergent to a
multifunction $F: X \to Y$ such that $F(x)$ is closed for each $x \in X$, then $F$ is lower $B^*$-cluster continuous.

**Proof:** We suppose that $F$ be not lower $B^*$-cluster continuous but all $F_a$ are lower $B^*$-cluster continuous. Then there exists a point $x_0 \in X$ and an open set $U$ of $Y$ containing $F(x_0)$ such that for every $B^*$-set $B$ containing $x_0$, there exists $x \in B$ so that $F(x) \not\subset U$. It is evident that $x \not\in F^*(U)$.

Since $F(x)$ is closed in $Y$, then by the normality of $Y$ there exists an open set $V$ of $Y$ such that $F(x_0) \subset V \subset \text{Cl}(V) \subset U$. Let $V_1 = Y \setminus \text{Cl}(V)$. Then, $Y \setminus U \subset V_1$.

As ${F_a}$ is a lower semi-uniformly convergent to $F$, there exists $a_0 \in \Lambda$ such that $x_0 \in F_a(V)$ for all $a \geq a_0$. Since $F(x) \not\subset U$ then $F(x) \cap (Y \setminus U) \neq \emptyset \Rightarrow F(x) \cap V_1 \neq \emptyset$.

Therefore, $x \in F^*(V_1)$ i.e. $F^*(V_1) \neq \emptyset$. As ${F_a}$ is lower semi-uniformly convergent to $F$, there exists $a_1 \in \Lambda$ with $a_1 \geq a_0$ we have $y \in F_{a_1}^{-1}(V_1)$ for all $y \in F^*(V_1)$. Hence, $x \in F_{a_1}^{-1}(V_1)$. Since $V \cap V_1 = \emptyset$, hence $F_{a_1}(x) \not\subset V$.

According to theorem 4.3.1 it follows that $F_{a_1}$ is not lower $B^*$-cluster continuous, which is a contradiction. Hence, the theorem.

**Theorem 4.3.4:** Let ${F_a}$ be a net of upper $B^*$-cluster continuous multifunctions from $X$ to a regular space $Y$. If ${F_a}$ is upper semi-uniformly convergent to a multifunction $F: X \to Y$, then $F$ is upper $B^*$-cluster continuous.
Proof: We suppose that \( F \) be not upper \( B^* \)-cluster continuous but all \( F_a \) are upper \( B^* \)-cluster continuous. Then by theorem 4.3.2, there exists a point \( x_0 \in X \) and an open set \( U \) of \( Y \) intersecting \( F(x_0) \) such that for every \( B^* \)-set \( B \) containing \( x_0 \), there exists a point \( x \in B \) with \( F(x) \cap U = \emptyset \).

Then for each \( B^* \)-set \( B \) there exists a point \( x \in B \) so that \( x \not\in F(U) \).

Since \( F(x_0) \cap U \neq \emptyset \) let us take an arbitrary point \( z \) of \( F(x_0) \cap U \). Then by the regularity of \( Y \) there exists an open set \( V \) of \( Y \) such that \( z \in V \subset \text{Cl}(V) \subset U \). Let \( V_1 = Y \setminus \text{Cl}(V) \). Then, evidently \( x_0 \in F'(V) \).

Since \( \{F_a\}_{a \in A} \) is upper semi-uniformly convergent to \( F \), there exists \( a_0 \in A \) such that \( x_0 \in F_{a_0}^-(V) \) for all \( a \geq a_0 \). Since \( x \not\in F'(V) \), then \( x \in F'(Y \setminus V) \).

Therefore, \( x \in F'(V_1) \). Since \( \{F_a\}_{a \in A} \) is upper semi-uniformly convergent to \( F \), there exists \( a_1 \in A \) with \( a_1 \geq a_0 \) such that \( F_{a_1}^{-}(y) \subset V_1 \) for each \( y \in F'(V_1) \).

Hence, \( F_{a_1}(x) \subset V_1 \). Therefore \( x \in F_{a_1}^+(V_1) \). Since \( V \cap V_1 = \emptyset \), hence \( x \not\in F_{a_1}^{-}(V_1) \).

According to theorem 4.3.2 it follows that \( F_{a_1} \) is not upper \( B^* \)-cluster continuous, which is a contradiction. Hence the theorem.

The paper (revised according to referee’s suggestion) containing the contents of this chapter has been sent for publication in SOOCHOW JOURNAL OF MATHEMATICS.