REPRINTS
Explosive instabilities in nonlinear perturbation

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Abstract. We present a general theoretical investigation of three-wave interactions by the method of nonlinear perturbation, with special emphasis on nonlinear explosive instabilities in the presence of linear damping or growth.

1. Introduction

There is a growing interest in the possibility of unbounded solutions to the equations of nonlinear interactions in plasma. The corresponding explosive instabilities may cause some astrophysical phenomena (Sturrock 1966) as well as enhanced losses in laboratory plasma (Kadomtsev et al 1965, Dikasov et al 1965, Coppi et al 1969). The effects of phase are of considerable importance in nonlinear wave interactions in plasma (Engelmann and Wilhelmsson 1969), and the coupled equations for three-wave nonlinear equations can be integrated analytically when the coupling constants are real. However, in the general case of an explosive instability where one takes the coupling coefficients to be complex, with phases not equal to 0, π, the problem becomes considerably more complicated (Wilhelmsson and Stenflo 1970). It has been pointed out that in the nonlinear perturbation developed by Coffey and Ford (1969) and others, Case (1966) has the distinct advantage of separating a given motion into a secular motion plus a rapidly fluctuating motion of small amplitude. In the present paper we discuss, within the framework of nonlinear perturbation theory, how a nonlinear three-wave interaction becomes explosive in the presence of linear damping of the waves. We note that the explosive instability studied by the well-defined phase approach is a first-order phenomenon in the order of nonlinear perturbation, and explosive instability may be developed to higher orders.

2. A brief review of the perturbation method

Coffey and Ford (1969) have presented a form of the method of averaging called the method of rapidly rotating phase. We consider the following set of coupled differential equations:

\[ \frac{dx_i}{dt} = \varepsilon A_i(X, \Psi), \quad i = 1, 2, \ldots, \gamma \]  \hspace{1cm} (1a)

\[ \frac{d\psi_j}{dt} = \omega_j(X) + \varepsilon B_j(X, \Psi), \quad j = 1, 2, \ldots, s \]  \hspace{1cm} (1b)

\[ X = (x_1, x_2, \ldots, x_\gamma), \]

\[ \Psi = (\psi_1, \psi_2, \ldots, \psi_s), \]

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where $\epsilon$ is a small parameter and $X$, $\Psi$ and the $A_i$'s and $B_j$'s are periodic functions of the $\phi_k$'s with period $2\pi$. When $\epsilon = 0$, the $x_i$'s are constant and the $\psi_j$'s are linear functions of time.

When $\epsilon$ is small the $x_i$'s will experience a slow secular growth with a small-amplitude rapid fluctuation superimposed on it. Similarly the $\psi_j$'s will experience a rapid secular growth on which is superimposed a small-amplitude rapid fluctuation. The method is utilised to separate this secular motion from the rapidly fluctuating motion. To do this we seek a solution of the form

$$x_i = y_i + \sum_{n=1}^{\infty} e^n F_i^{(n)}(\phi), \quad i = 1, 2, \ldots, \gamma,$$

$$\psi_j = \phi_j + \sum_{n=1}^{\infty} e^n G_j^{(n)}(\phi), \quad j = 1, 2, \ldots, s,$$

where $F_i^{(n)}$ and $G_j^{(n)}$ are periodic functions of each of the $\phi_k$'s, with period $2\pi$.

We further require that $y_i$ and $\phi_j$ should satisfy the following differential equations:

$$\frac{dy_i}{dt} = \sum_{n=1}^{\infty} e^n a_i^{(n)}(y), \quad i = 1, 2, \ldots, \gamma,$$

$$\frac{d\phi_j}{dt} = \omega_j(y) + \sum_{n=1}^{\infty} e^n b_j^{(n)}(y), \quad j = 1, 2, \ldots, s,$$

where the right-hand sides of equations (3) are required to be independent of the $\phi_k$'s and the $y_i$'s and $\phi_j$'s describe only secular motions since they are solutions of a system of differential equations which are independent of the rapidly changing phases $\phi_k$. The rapid fluctuations of $x_i$ and $\psi_j$ about $y_i$ and $\phi_j$ are given by the terms in the series in equation (2). We shall illustrate the working of the general method in the general system of three interacting waves in the presence of dissipation.

3. Basic coupled mode equations

A unified description of the nonlinear interaction of the waves can be made by the system of equations

$$\frac{da_0}{dt} - i\omega_0 a_0 = c_{12}^1 a_1 a_2,$$

$$\frac{da_1}{dt} - i\omega_1 a_1 = c_{02}^1 a_0 a_2^*,$$

$$\frac{da_2}{dt} - i\omega_2 a_2 = c_{01}^1 a_0 a_1^*.$$

Following Wilhelmsson and Stenflo (1970) we can write

$$a_i = eA_i(t) \exp(iRe \omega_i), \quad A_i = \hat{u}_i \exp(i\phi_i), \quad e_{\eta} = v_{\eta} \exp(i\theta_{\eta}),$$

$$\nu_i = \text{Im}(\omega_i), \quad \hat{u}_i = |A_i|, \quad v_{\eta} = |e_{\eta}|,$$

$$\Delta \omega = \text{Re}(\omega_0) - \text{Re}(\omega_1) - \text{Re}(\omega_2),$$

$$\phi = \phi_0 - \phi_1 - \phi_2 + \Delta \omega t.$$
We obtain the real system

\[
\begin{align*}
\frac{d\tilde{u}_0}{dt} + \nu_0 \tilde{u}_0 &= \epsilon v_{12} \tilde{u}_1 \tilde{u}_2 \cos(\phi + \theta_{12}), \\
\frac{d\tilde{u}_1}{dt} + \nu_1 \tilde{u}_1 &= \epsilon v_{02} \tilde{u}_0 \tilde{u}_2 \cos(\phi + \theta_{02}), \\
\frac{d\tilde{u}_2}{dt} + \nu_2 \tilde{u}_2 &= \epsilon v_{01} \tilde{u}_0 \tilde{u}_1 \cos(\phi + \theta_{01}), \\
\end{align*}
\]

(6)

\[
\frac{d\phi}{dt} = \Delta \omega - \epsilon v_{12} \frac{\tilde{u}_1 \tilde{u}_2}{\tilde{u}_0} \sin(\phi + \theta_{12}) - \epsilon v_{02} \frac{\tilde{u}_0 \tilde{u}_2}{\tilde{u}_1} \sin(\phi + \theta_{02}) - \epsilon v_{01} \frac{\tilde{u}_0 \tilde{u}_1}{\tilde{u}_2} \sin(\phi + \theta_{01}).
\]

Using further renormalisations,

\[
\tilde{u}_0 \to (v_{01}v_{02})^{1/2} u_0, \quad \tilde{u}_1 \to (v_{01}v_{12})^{1/2} u_1, \quad \tilde{u}_2 \to (v_{02}v_{12})^{1/2} u_2,
\]

we obtain

\[
\begin{align*}
\frac{d\tilde{u}_0}{dt} + \nu_0 \tilde{u}_0 &= \epsilon u_1 u_2 \cos(\phi + \theta_{12}), \\
\frac{d\tilde{u}_1}{dt} + \nu_1 \tilde{u}_1 &= \epsilon u_0 u_2 \cos(\phi + \theta_{02}), \\
\frac{d\tilde{u}_2}{dt} + \nu_2 \tilde{u}_2 &= \epsilon u_0 u_1 \cos(\phi + \theta_{01}), \\
\end{align*}
\]

(7a)

\[
\frac{d\phi}{dt} = \Delta \omega - \epsilon \frac{u_1 u_2}{u_0} \sin(\phi + \theta_{12}) - \epsilon \frac{u_0 u_2}{u_1} \sin(\phi + \theta_{02}) - \epsilon \frac{u_0 u_1}{u_2} \sin(\phi + \theta_{01}).
\]

(7b)

One may have explosively unstable solutions to equation (7a) when both amplitudes on the right-hand side of equation (7a) grow. This is possible only if all three amplitudes grow at the same time. In the next sections we shall discuss how the nonlinear three-wave interactions become explosive in the presence of linear damping and dissipation.

4. Effect of linear damping and dissipation on explosive instabilities of three interacting waves

To solve the set of equations (7a) and (7b) we find that the method of perturbation due to Coffey and Ford (1969) is the most suitable when \(\Delta \omega \neq 0\), although it has limitations when \(\Delta \omega = 0\).

Following Coffey and Ford (1969) we seek a solution in the form

\[
\begin{align*}
u_i &= y_i + \epsilon F_i^{(1)}(\phi) + \epsilon^2 F_i^{(2)}(\phi) + \ldots, \\
\psi &= \phi + \epsilon G^{(1)}(\phi) + \epsilon^2 G^{(2)}(\phi) + \ldots,
\end{align*}
\]

(8a)

(8b)

where

\[
\begin{align*}
\phi &= \Delta \omega + \epsilon \delta^{(1)}(y) + \epsilon^2 \delta^{(2)}(y) + \ldots, \\
y_i &= a_i^{(0)} + \epsilon a_i^{(1)}(y) + \epsilon^2 a_i^{(2)}(y) + \ldots
\end{align*}
\]

(9a)

(9b)

The \(a_i^{(0)}\) term in equation (9b) appears because of the presence of the term with coefficient \(\nu_1\) in equation (7a). Inserting equation (8) in equation (7), using equation (9)
and equating powers of $e$ we obtain the following sequence of equations:

\[ a_i^{(0)} + v_j y_i = 0, \]
\[ a_0^{(1)} + \left( \frac{\partial F_0^{(1)}}{\partial \phi} \right) \Delta \omega + \nu_0 F_0^{(1)} = y_1 y_2 \cos(\phi + \theta_{12}), \]
\[ a_1^{(1)} + \left( \frac{\partial F_1^{(1)}}{\partial \phi} \right) \Delta \omega + \nu_1 F_1^{(1)} = y_0 y_2 \cos(\phi + \theta_{02}), \]
\[ a_2^{(1)} + \left( \frac{\partial F_2^{(1)}}{\partial \phi} \right) \Delta \omega + \nu_2 F_2^{(1)} = y_0 y_1 \cos(\phi + \theta_{01}), \]

\[
\frac{b^{(1)}}{\partial \phi} \Delta \omega = - \left( \frac{y_1 y_2}{y_0} \sin(\phi + \theta_{12}) + \frac{y_0 y_2}{y_1} \sin(\phi + \theta_{02}) + \frac{y_0 y_1}{y_2} \sin(\phi + \theta_{01}) \right).
\]

From the next power of $e$ we obtain

\[ a_0^{(2)} + \frac{\partial F_0^{(2)}}{\partial \phi} \Delta \omega + \nu_0 F_0^{(2)} + b^{(1)} \frac{\partial F_0^{(1)}}{\partial \phi} = -y_1 y_2 G^{(1)}(\phi) \sin(\phi + \theta_{12}) + (y_1 F_1^{(1)} + y_2 F_1^{(1)}) \cos(\phi + \theta_{12}), \]
\[ a_1^{(2)} + \frac{\partial F_1^{(2)}}{\partial \phi} \Delta \omega + \nu_1 F_1^{(2)} + b^{(1)} \frac{\partial F_1^{(1)}}{\partial \phi} = -y_0 y_2 G^{(1)}(\phi) \sin(\phi + \theta_{02}) + (y_2 F_0^{(1)} + y_0 F_0^{(1)}) \cos(\phi + \theta_{02}), \]
\[ a_2^{(2)} + \frac{\partial F_2^{(2)}}{\partial \phi} \Delta \omega + \nu_2 F_2^{(2)} + b^{(1)} \frac{\partial F_2^{(1)}}{\partial \phi} = -y_0 y_1 G^{(1)}(\phi) \sin(\phi + \theta_{01}) + (y_1 F_0^{(1)} + y_0 F_0^{(1)}) \cos(\phi + \theta_{01}), \]

\[
\frac{b^{(2)}}{\partial \phi} \Delta \omega = -[(y_2 F_1^{(1)} + y_1 F_2^{(1)})/y_0 - (y_1 y_2 / y_0) F_0^{(1)}] \sin(\phi + \theta_{12})
-[(y_2 F_0^{(1)} + y_0 F_2^{(1)})/y_1 - (y_0 y_2 / y_1) F_1^{(1)}] \sin(\phi + \theta_{02})
-[(y_1 F_0^{(1)} + y_0 F_1^{(1)})/y_2 - (y_0 y_1 / y_2) F_2^{(1)}] \sin(\phi + \theta_{01})
-G^{(1)}[(y_1 y_2 / y_0) \cos(\phi + \theta_{12}) + (y_0 y_1 / y_2) \cos(\phi + \theta_{01})
+(y_0 y_2 / y_1) \cos(\phi + \theta_{02})].
\]

We solve the sequence of equations (7) and find

\[ a_i^{(1)} = 0, \quad b^{(1)} = 0, \]
\[ F_0^{(1)} = (y_2 y_1 / \Delta \omega_0) \sin(\phi + \theta_{12} + \eta_0), \]
\[ F_1^{(1)} = (y_0 y_2 / \Delta \omega_1) \sin(\phi + \theta_{02} + \eta_1), \]
\[ F_2^{(1)} = (y_0 y_1 / \Delta \omega_2) \sin(\phi + \theta_{01} + \eta_2), \]

\[ G^{(1)} = \frac{1}{\Delta \omega} \left( \frac{y_1 y_2}{y_0} \cos(\phi + \theta_{12}) + \frac{y_0 y_2}{y_1} \cos(\phi + \theta_{02}) + \frac{y_0 y_1}{y_2} \cos(\phi + \theta_{01}) \right), \]

where

\[ \tan \eta_j = v_j / \Delta \omega_j, \quad 1/\Delta \omega_j = (v_j^2 + \Delta \omega^2)^{-1/2}. \]
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One can obtain from equations (11)

\[ a_0 = \frac{y_0 y_0}{2 \Delta \omega_2} \sin(\theta_{01} - \theta_{12} + \eta_2) + \frac{y_1 y_2}{2 \Delta \omega_1} \sin(\theta_{02} - \theta_{12} + \eta_1) \]

\[ - \frac{1}{2 \Delta \omega} \left[ y_0 y_0 \sin(\theta_{12} - \theta_{01}) + y_0 y_2 \sin(\theta_{12} - \theta_{02}) \right], \]

\[ a_1 = \frac{y_1 y_0^2}{2 \Delta \omega_2} \sin(\theta_{01} - \theta_{02} + \eta_2) + \frac{y_1 y_2^2}{2 \Delta \omega_0} \sin(\theta_{12} - \theta_{02} + \eta_0) \]

\[ - \frac{1}{2 \Delta \omega} \left[ y_1 y_0^2 \sin(\theta_{02} - \theta_{12}) + y_1 y_2 \sin(\theta_{02} - \theta_{01}) \right], \]

\[ a_2 = \frac{y_2 y_0^2}{2 \Delta \omega_1} \sin(\theta_{02} - \theta_{01} + \eta_1) + \frac{y_2 y_1^2}{2 \Delta \omega_0} \sin(\theta_{12} - \theta_{01} + \eta_0) \]

\[ - \frac{1}{2 \Delta \omega} \left[ y_2 y_0^2 \sin(\theta_{01} - \theta_{12}) + y_2 y_1 \sin(\theta_{01} - \theta_{02}) \right], \]

\[ \frac{\partial F^{(2)}}{\partial \phi} = \Delta \omega + \nu_0 F^{(2)} \]

\[ = \frac{y_0 y_1^2}{2 \Delta \omega_2} \sin(2 \phi + \theta_{01} + \theta_{12} + \eta_2) + \frac{y_0 y_2}{2 \Delta \omega_1} \sin(2 \phi + \theta_{02} + \theta_{12} + \eta_1) \]

\[ + \frac{1}{2 \Delta \omega} \left( y_0 y_0^2 \sin(2 \phi + \theta_{12} + \theta_{02}) \right), \]

\[ \frac{\partial F^{(2)}}{\partial \phi} = \Delta \omega + \nu_1 F^{(2)} \]

\[ = \frac{y_1 y_0}{2 \Delta \omega_2} \sin(2 \phi + \theta_{01} + \theta_{02} + \eta_2) + \frac{y_1 y_2}{2 \Delta \omega_0} \sin(2 \phi + \theta_{12} + \theta_{02} + \eta_0) \]

\[ - \frac{1}{2 \Delta \omega} \left( y_1 y_0^2 \sin(2 \phi + \theta_{02} + \theta_{01}) + y_1 y_2 \sin(2 \phi + \theta_{02} + \theta_{12}) \right), \]

\[ \frac{\partial F^{(2)}}{\partial \phi} = \Delta \omega + \nu_2 F^{(2)} \]

\[ = \frac{y_2 y_0^2}{2 \Delta \omega_1} \sin(2 \phi + \theta_{02} + \theta_{01} + \eta_1) + \frac{y_2 y_1^2}{2 \Delta \omega_0} \sin(2 \phi + \theta_{12} + \theta_{01} + \eta_0) \]

\[ + \frac{1}{2 \Delta \omega} \left( y_2 y_0^2 \sin(2 \phi + \theta_{02} + \theta_{01}) + y_2 y_1^2 \sin(2 \phi + \theta_{01} + \theta_{12}) \right), \]
4.1. The dissipation-free case

For the dissipation-free case the angles $\theta_i$ will either be the same (explosively unstable case) or differ by $\pi$ (stable case). When all $\theta_i$ are the same and we assume further that $\nu_0 = \nu_1 = \nu$ and $\nu_2 = 0$, then from equations (9b) and equations (13a) we obtain

$$y_0 + \nu y_0 = (y_0 y^2 \sin \eta)/2\Delta \omega_k$$

$$y_1 + \nu y_1 = (y_1 y^2 \sin \eta)/2\Delta \omega_k,$$

$$y_2 = (y_1 y_0^2 \sin \eta)/2\Delta \omega_k + (y_2 y^3 \sin \eta)/2\Delta \omega_k.$$  

We can write

$$y = Y e^{-\lambda t}, \quad x = Y^2, \quad \lambda = \sin \eta/\Delta \omega_k$$

to obtain

$$d(x_2 - x_0 - x_1)/dt = 0,$$

$$d(\log x_2)/dt = \lambda (x_0 + x_1).$$
Equations (14b) and (14c) can be integrated to obtain
\[ x_2 = P \frac{\mu_+ + \mu_- e^{\lambda t}}{\mu_+ - \mu_- e^{\lambda t}}, \]
\[ x_0 = x_1 = A e^{\left(\mu_0 - t\right)} / (\mu_+ - \mu_- e^{\lambda t})^{1/\lambda}, \]
where \( \mu_\pm = x_2(0) \pm P, P = x_2(0) - x_1(0) - x_0(0) \) and \( A \) is a constant. All \( x_i \) will go to infinity and the time of explosion will be
\[ t_\infty = \frac{1}{\lambda \rho} \log \frac{\mu_+}{\mu_-}. \]

If we choose the initial amplitudes such that \( P = 0 \) we obtain the simplest possible form
\[ x_2(t) = \frac{1}{1/x_2(0) - \lambda t}, \]
and thus the time of explosion \( t_\infty = 1/\lambda x_2(0) \). In the presence of damping \( \lambda < 1 \), which shows that the time of explosion will be delayed. In an exactly similar manner one can obtain that the \( x_i \) have a stable solution when the \( \theta_i \) differ by \( \pi \).

4.2. Influence of dissipation when \( \theta_i \) are present

Using (12a) the differential equation (9a) becomes, to second order in \( \epsilon \) with \( x_i = y_i^2 \),
\[
\frac{dx_0}{dt} + 2 \nu_0 x_0 = \epsilon^2 \left[ x_0 x_1 \left( \frac{1}{\Delta \omega_2^2} \sin(\theta_{01} - \theta_{12} + \eta_2) + \frac{1}{\Delta \omega_3} \sin(\theta_{01} - \theta_{12}) \right) \right.
\]
\[ + x_0 x_2 \left( \frac{1}{\Delta \omega_0} \sin(\theta_{12} - \theta_{02} + \eta_1) + \frac{1}{\Delta \omega_3} \sin(\theta_{12} - \theta_{02}) \right) \],
\[
\frac{dx_1}{dt} + 2 \nu_1 x_1 = \epsilon^2 \left[ x_0 x_1 \left( \frac{1}{\Delta \omega_2} \sin(\theta_{01} - \theta_{02} + \eta_2) + \frac{1}{\Delta \omega} \sin(\theta_{01} - \theta_{02}) \right) \right.
\]
\[ + x_1 x_2 \left( \frac{1}{\Delta \omega_0} \sin(\theta_{12} - \theta_{01} + \eta_1) + \frac{1}{\Delta \omega} \sin(\theta_{12} - \theta_{01}) \right) \],
\[
\frac{dx_2}{dt} + 2 \nu_2 x_2 = \epsilon^2 \left[ x_0 x_2 \left( \frac{1}{\Delta \omega_1} \sin(\theta_{02} - \theta_{01} + \eta_1) + \frac{1}{\Delta \omega} \sin(\theta_{02} - \theta_{01}) \right) \right.
\]
\[ + x_1 x_2 \left( \frac{1}{\Delta \omega_0} \sin(\theta_{12} - \theta_{01} + \eta_0) + \frac{1}{\Delta \omega} \sin(\theta_{12} - \theta_{01}) \right) \].

Assuming all \( \nu_i \) are the same, we have \( \eta_0 = \eta_1 = \eta_2 = \eta \), also assuming \( \Delta \omega \gg \nu \), and \( \eta \) much less than the difference of the \( \theta_i \). One can derive the following constants of motion from equation (15a):
\[
\frac{d}{d\tau} \left( \log x_0 + \log x_1 + \log x_2 \right) = 0, \quad (15b)
\]
\[
\frac{d}{d\tau} \left[ x_0 \sin(\theta_{01} - \theta_{02}) + x_1 \sin(\theta_{12} - \theta_{01}) + x_2 \sin(\theta_{02} - \theta_{12}) \right] = 0, \quad (15c)
\]
where
\[ x_i = x_i e^{2\nu \tau}, \quad \tau = (1 - e^{-2\nu \tau})/2\nu. \]
Equations (15a) can be called the generalised Volterra equations (Hirota 1976). It is interesting to note that the constants of motion (15b) are very similar to the condition of equilibrium obtained from the entropy function (Dikasov et al 1965) and that equation (15c) is identical to that of Wilhelmsson and Stenflo (1970).

The method of nonlinear perturbation (Coffey and Ford 1969) does not in general lead to an explicit solution of the original set of equations. It is a method very suitable for separation of the secular motion from the rapid periodic fluctuation and for reducing the problem to that of solving the differential equations for secular motion alone. We proceed to solve the differential equations for secular motion (equations (9a) and (9b)) for different orders of perturbation. We write

\[ P = x_0(0) \sin(\theta_{01} - \theta_{02}) + x_1(0) \sin(\theta_{12} - \theta_{01}) + x_2(0) \sin(\theta_{02} - \theta_{12}), \]

\[ Q = x_0(0)x_1(0)x_2(0). \]

Equations (15a) can be written as

\[
\int_{x_0(0)}^{x_1(0)} \frac{dx_0}{2[(x_0^2/\Delta\omega^2)[P - x_0 \sin(\theta_{01} - \theta_{02})]^2 - (4Qx_0/\Delta\omega^2) \sin(\theta_{12} - \theta_{01}) \sin(\theta_{02} - \theta_{12})]^{1/2}} = \int_0^\tau \varepsilon^2 d\tau, \\
\int_{x_1(0)}^{x_2(0)} \frac{dx_1}{2[(x_1^2/\Delta\omega^2)[P - x_1 \sin(\theta_{12} - \theta_{01})]^2 - (4Qx_1/\Delta\omega^2) \sin(\theta_{02} - \theta_{12}) \sin(\theta_{01} - \theta_{02})]^{1/2}} = \int_0^\tau \varepsilon^2 d\tau, \\
\int_{x_2(0)}^{x_3(0)} \frac{dx_2}{2[(x_2^2/\Delta\omega^2)[P - x_2 \sin(\theta_{02} - \theta_{12})]^2 - (4Qx_2/\Delta\omega^2) \sin(\theta_{02} - \theta_{12}) \sin(\theta_{12} - \theta_{01})]^{1/2}} = \int_0^\tau \varepsilon^2 d\tau. \tag{16}
\]

Equations (16) are in general elliptic integrals and the solution can be expressed in terms of elliptic functions, depending on whether the solution of \( \pi(x_i) = 0 \) has either four real roots or two real and two complex roots (Weiland and Wilhelmsson 1977). We notice that the solution of \( \pi(x_i) = 0 \) (\( \pi(x_i) \) stands for the denominator of equation (16)) has (a) two real roots and two complex roots when \( \theta_{02} > \theta_{12} > \theta_{01} \); (b) four real roots when any two of three \( \theta_{ij} \) are equal.

The significant change one can note is that \( \pi(x_i) \) is in general biquadratic, while in the dissipation-free case in the absence of linear damping it is cubic. The general solution has been discussed in the Appendix.

For simplicity let us take the initial values of \( x_i \) and \( \theta_{ij} \) such that when \( P = 0 \) all the \( \pi(x_i) \) of equations (16) have two real and two complex roots. The integrals of equations (16) can be written in the form (see Appendix)

\[
\int_{s(0)}^{s_i(0)} \frac{dx_i}{(\pi(x_i))^{1/2}} = \frac{g_i - f_i}{\sqrt{A_i}} \int_{u_i(0)}^{u_i(0)} \frac{du_i}{[(1 + m_j\mu_i^2)(1 + n_j\mu_i^2)]^{1/2}}. \tag{17a}
\]
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with

\[ x_j = \frac{f_j + g_j \mu_j}{1 + \mu_j}, \quad f_j = -\frac{(\sqrt{3} + 1)}{2} \alpha_j, \quad g_j = \frac{(\sqrt{3} - 1)}{2} \alpha_j. \]

\[ \alpha_j = (\sin \psi_j)^{-1} (4Q \sin \psi_0 \sin \psi_1 \sin \psi_2)^{1/3}, \quad j = 0, 1, 2, \]

\[ \psi_0 = \theta_{01} - \theta_{02}, \quad \psi_1 = \theta_{12} - \theta_{01}, \quad \psi_2 = \theta_{02} - \theta_{12}, \]

\[ n_j = 1, \quad A_j = (3\sqrt{3}/4) \alpha_j^4 (\sqrt{3} + 2), \]

\[ m_j = -((\sqrt{3} + 2)^{-2} = -\cot^2 \theta. \]

Then equations (16) reduce to

\[ \int_{u_j(0)}^{u_j(\tau)} \frac{du_j}{[(1 + m_j \mu_j)^2(1 + n_j \mu_j^2)]^{1/2}} = \frac{\sqrt{A_j}}{g_j - f_j} \Delta \omega \int_0^r \frac{2 \epsilon^2}{dr}. \]  

(18)

Integrating equation (18) (Abramowitz and Stegun 1965) one obtains

\[ u_j = \sin \theta \text{sd} \left[ \left( \frac{\sqrt{A_j}}{g_j - f_j} \frac{2 \epsilon^2}{\Delta \omega} \sin \psi_j \right) \cosec \theta (\tau + \tau_j) \right] \sin^2 \theta \]  

(19)

(where sd is Jacobi's elliptic function) where \( \tau_j \) is defined as

\[ \tau_j = \frac{g_j - f_j}{\sqrt{A_j}} \frac{\sin \theta}{(2 \epsilon^2 / \Delta \omega) \sin \psi_j} \text{sd}^{-1} \left( \frac{u_j(0)}{\sin \theta} \right) \sin^2 \theta. \]

When \( u_j(\tau) = -1 \), \( x_j \) tends to infinity and one can obtain from equation (19) the time of explosion given by

\[ \sin \theta \text{sd} \left[ \left( \frac{\sqrt{A_j}}{g_j - f_j} \frac{2 \epsilon^2}{\Delta \omega} \sin \psi_j \right) \cosec \theta (\tau + \tau_j) \right] \sin^2 \theta + 1 = 0. \]  

(20)

When, in addition to \( P = 0 \), the initial conditions of \( x_j(0) \) and \( \theta_j \) are such that the \( u_j(0) \) are given by \( u_j(0) = \sin(\theta + \tau) \text{sd}(l \sin^2 \theta) \) with \( l \) a constant, all the \( x_j \) will grow to infinity at the same time and the time of explosion will be obtained as

\[ \tau_{\infty} = 3^{-1/4} \frac{\Delta \omega}{2 \epsilon^2} \left[ l - \text{sd}^{-1}(\cosec \theta \sin^2 \theta) \right]. \]

5. Discussion

In this section we discuss some current literature on explosive instabilities (see Fukai et al (1970, 1971), Aamodt and Sloan (1967, 1968)) relevant to our present work. They have investigated this phenomenon by deriving the equations for the time evolution of the complex wave amplitude, retaining only the second-order nonlinear interaction term. In an attempt to obtain more physically acceptable results, several authors (Fukai et al (1970), Dysthe (1970), Oraevskii et al (1973a, b), Weiland and Wilhelmsson (1977)) retained the third-order nonlinear terms in the equations for the complex amplitudes to obtain new coupled mode equations which in particular cases were amenable to analysis.

The insufficiency of the first-order approximation for a resonant wave interaction and in a multistream plasma has been well discussed by Sedlacek (1975a, b, 1976), who
used the theory of nearly multiple periodic Hamiltonian systems (Coffey 1969, Sedlacek 1975a, b).

In our paper we have used the nonlinear perturbation technique developed by Coffey and Ford (1969) to separate the fast oscillations from the slow evolution of the whole system. This is a significant advantage over the time-averaging scheme of Bogolyubov and Krylov and Mitropolski, and over the method of averaged Lagrangians as elaborated by Dougherty (1970) and applied by Galloway and Kim (1971) and Boyd and Turner (1972, 1973).

Another main advantage of our method lies in the fact that analysis of the explosive and stabilised nature of three-wave interactions is possible from the explicit solutions of the secular motion.

It is interesting to note that the nonlinear coupled mode equation (equation (7)) is reproduced from the first-order equation (equation (10)) of our analysis when all \( F \) and \( G \) are taken to be zero. Because of this coincidence, one may conclude that our first-order approximation is exactly equivalent to the usual time-averaging scheme.

A new set of constants of motion is obtained in the second-order approximation. If one wishes, one can study the character of motion by the phase plane analysis (Minorsky 1962). In such a description, when a curve is drawn with the amplitude as independent variable and the amplitude derivative as dependent variable (see equation (15a)), one obtains the phase portraits of explosive instability. Again the effect of inclusion of the phases \( \theta \) and \( \Delta \omega \) may introduce a large negative root of \( \pi(x) = 0 \) (equation (16)). In such cases, in the second-order approximation, the motion will be stabilised. The significant influence of the third-order nonlinear term, having a similar effect, has been discussed by Weiland and Wilhelmsson (1977) and Byers \textit{et al} (1971).

In our analysis, it has been shown how in the second-order approximation the explosive character of three-wave nonlinear interactions is greatly influenced by the presence of linear damping and dissipation. The same order of nonlinearity that causes explosive instability in the first-order approximation may stabilise the waves in the second-order approximation.

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Appendix

An integral of the form \( \int \frac{dz}{(\pi(z))^{1/2}} \) can be expressed in terms of an elliptic integral when \( \pi(z) \) can be written as

\[
\pi(z) = (z^2 + pz + q)(z^2 + rz + s)
\]

where \( p, q, r, s \) are real. In the transformation \( z = (f + gu)/(1 + u) \) let \( f, g \) be so chosen that the coefficient of \( u \) in each quadratic is zero; then \( \pi(z) \) will take the form

\[
\pi(z) = A(1 + mu)^2(1 + nu^2)/(1 + u)^4,
\]
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with

\[ A = (f^2 + pf + q)(f^2 + rf + s), \]
\[ m = \frac{g^2 + pg + q}{f^2 + pf + q}, \quad n = \frac{g^2 + rg + s}{f^2 + rf + s}, \] (A3)
\[ f + g = \frac{2(q - s)}{r - p}, \quad fg = \frac{ps - qr}{r - p}. \]

Now let the two equations

\[ z^2 + pz + q = 0 \quad \text{and} \quad z^2 + rz + s = 0 \] (A4)

have roots \( x_1, x_2 \) and \( x_3, x_4 \) respectively so that

\[ x_1 + x_2 = -p, \quad x_1x_2 = q, \]
\[ x_3 + x_4 = -r, \quad x_3x_4 = s. \]

Further, \( f \) and \( g \) are the roots of the equation

\[ (r - p)f^2 + 2(s - q)f + (ps - qr) = 0. \]

Accordingly the roots will be real when

\[ (s - q)^2 - (r - p)(ps - qr) > 0. \] (A5)

The inequality can be written as

\[ (x_1 - x_3)(x_1 - x_4)(x_2 - x_3)(x_2 - x_4) > 0. \] (A6)

This inequality holds when at least one of equations (A4) has imaginary roots. If both equations have two real roots the factors of \( \pi(z) \) can always be written so that \( x_1 > x_2 > x_3 > x_4 \).

Thus the inequality holds in this case also. Then the integral can be reduced to an elliptic integral (Abramowitz and Stegun 1965):

\[ \int \frac{dz}{(\pi(z))^{1/2}} = \frac{g - f}{\sqrt{A}} \int_0^u \frac{du}{\sqrt{[1 + m u^2](1 + n u^2)^{1/2}}}. \] (A7)

For the case \( m < 0, n > 0 \)

\[ \int \frac{dz}{(\pi(z))^{1/2}} = \frac{g - f}{(A m n)^{1/2}} \left[ \frac{du}{(1/m - u^2)(1/n + u^2)^{1/2}} \right]_{u_0}^u - \frac{du}{(1/m - u^2)(1/n + u^2)^{1/2}} \]

\[ = \frac{g - f}{(A m n)^{1/2}} \left[ sd^{-1}\left( \frac{u(m + n)^{1/2}}{mn} \right) \frac{m}{m + n} - sd^{-1}\left( \frac{u_0(m + n)^{1/2}}{mn} \right) \frac{m}{m + n} \right], \]

where \( sd^{-1} \) is Jacobi's inverse elliptic function.

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The stabilisation of explosive instabilities in the presence of a third-order nonlinear effect

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Abstract. The interaction of three waves in the presence of a third-order nonlinear interaction term is investigated by the method of nonlinear perturbation. The analytical solutions obtained here are found to be in agreement with the numerical solutions already given by Weiland and Wilhelmsson, and thus complement their work.

1. Introduction

This is a sequel to our previous paper (Khan et al 1980), where we discussed within the framework of nonlinear perturbation theory (Coffey and Ford 1969) how the nonlinear three-wave interaction becomes explosive in the presence of linear damping of the waves. It was shown that in the general case when the coupling coefficients become complex with their phases not equal to zero or $\pi$, the problem of the occurrence of an explosive instability becomes considerably more complicated. It has been noted that the explosive instability studied by a well defined phase approach is a first-order phenomenon, and this type of instability may be developed in higher orders of nonlinear perturbation.

Recently, Fukai et al (1969, 1970), Byers et al (1971) and Oraevskii et al (1973a,b) have studied the influence of third-order nonlinear terms on explosive instabilities in the coherent phase description for real second-order coupling factors. An upper limit for the amplitude of a nonlinear instability as a result of an amplitude-dependent frequency shift was also pointed out. Nonlinear stabilisation of explosive flute instabilities of mirror confined plasma has been discussed by Dum and Sudan (1969). Weiland and Wilhelmsson (1973) and Weiland (1974) have extended the investigation to include a linear dissipation and also an imaginary part of the third-order frequency shift. In these papers the saturation of the explosive instability by third-order terms is studied analytically and by means of computers. Nonlinear instabilities arising from the interaction of positive and negative energy waves have recently been observed on a computer model (Byers et al 1971, Shchinov et al 1973). The influence of an explosive instability on the plasma distribution has also been studied (Hamasaki and Krall 1971). Wilhelmsson (1970, 1972) has shown that the imaginary part of the complex third-order coupling factors might have a decisive influence for the stabilisation of explosive instabilities.

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In our present paper the effect of nonlinear dissipation on explosive instabilities has been considered in the presence of second-order coupling factors. It is observed that the amplitudes of the waves lead to stabilisation in the presence of such dissipation proportional to the square of the amplitudes, but in the absence of any linear dissipation the amplitudes do exhibit unlimited growth.

2. Basic equations

The basic equations of the waves interacting nonlinearly to higher orders are

\[
\begin{align*}
\frac{\partial a_0}{\partial t} - i\omega a_0 &= c_{02} a_1 a_2 - ia_0 \sum_{k=0}^{2} \alpha_{0k} |a_k|^2, \\
\frac{\partial a_1}{\partial t} - i\omega a_1 &= c_{02} a_0 a_2 - ia_1 \sum_{k=0}^{2} \alpha_{1k} |a_k|^2, \\
\frac{\partial a_2}{\partial t} - i\omega a_2 &= c_{01} a_0 a_2 - ia_2 \sum_{k=0}^{2} \alpha_{2k} |a_k|^2.
\end{align*}
\]

Following our previous work (Khan et al 1980), taking the second-order coupling factors to be equal to one, we obtain from (1) the corresponding real and imaginary parts as

\[
\begin{align*}
\dot{u}_0/\dot{t} + v_0 u_0 &= \varepsilon u_1 u_2 \cos (\phi + \theta_{12}) - \varepsilon^2 u_0 \delta \nu_0, \\
\dot{u}_1/\dot{t} + v_1 u_1 &= \varepsilon u_0 u_2 \cos (\phi + \theta_{02}) - \varepsilon^2 u_1 \delta \nu_1, \\
\dot{u}_2/\dot{t} + v_2 u_2 &= \varepsilon u_0 u_1 \cos (\phi + \theta_{01}) - \varepsilon^2 u_2 \delta \nu_2, \\
\dot{\phi} &= \Delta \omega - \varepsilon \frac{u_1 u_2}{u_0} \sin (\phi + \theta_{12}) - \varepsilon \frac{u_0 u_2}{u_1} \sin (\phi + \theta_{02}) - \varepsilon \frac{u_0 u_1}{u_2} \sin (\phi + \theta_{01}) - \varepsilon^2 \delta \omega',
\end{align*}
\]

where

\[
\begin{align*}
\delta \omega' &= \sum_{j=0}^{2} \beta_j u_j^2 \quad \text{(the nonlinear frequency shift)}, \\
\delta \nu_k' &= -\sum_{k=0}^{2} \text{Im} (\alpha_{jk}) u_k^2 \quad \text{(the effective nonlinear dissipation)}, \\
\beta_j &= \text{Re} \alpha_{j0} - \text{Re} \alpha_{j1} - \text{Re} \alpha_{j2},
\end{align*}
\]

are the matrix elements of the coupling factors for the third-order terms taking normalisation of the amplitudes.

In order to solve the set of equations (2), we use the method of perturbation due to Coffey and Ford (1969), which is suitable for \( \Delta \omega \neq 0 \) and has limitations for \( \Delta \omega = 0 \). To separate the secular motion from the rapidly fluctuating motion, a solution is introduced of the form

\[
\begin{align*}
u_i &= y_i + \sum_{n=1}^{\infty} \varepsilon^n F_{i}^{(n)}(\psi), \quad i = 1, 2, \ldots, r, \\
\phi &= \psi + \sum_{n=1}^{\infty} \varepsilon^n G_{j}^{(n)}(\psi), \quad j = 1, 2, \ldots, s,
\end{align*}
\]
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\[
y_i = \sum_{n=0}^{\infty} \epsilon^n a_i^{(n)}(y), \quad i = 1, 2, \ldots, r,
\]

\[
\psi = \Delta \omega + \sum_{n=1}^{\infty} \epsilon^n b^{(n)}(y), \quad j = 1, 2, \ldots, s.
\]

Inserting (3), (4) in (2) and using (5) and (6), we finally obtain, after equating the powers of \(\epsilon\), the following set of equations (for details see Khan et al (1980)):

\[
a_f^{(0)} + \nu_j \psi = 0, \quad a_f^{(1)} = 0, \quad b^{(1)} = 0,
\]

\[
F_0^{(1)} = \frac{y_2 y_1}{\Delta \omega_0} \sin (\psi + \theta_{12} + \eta_0), \quad F_1^{(1)} = \frac{y_2 y_0}{\Delta \omega_1} \sin (\psi + \theta_{02} + \eta_1),
\]

\[
G^{(1)} = \frac{1}{\Delta \omega} \left( \frac{y_1 y_2}{y_0} \cos (\psi + \theta_{12}) + \frac{y_0 y_2}{y_1} \cos (\psi + \theta_{02}) + \frac{y_0 y_1}{y_2} \cos (\psi + \theta_{01}) \right),
\]

\[
\tan \eta_j = \frac{\nu_j}{\Delta \omega_j}, \quad \frac{1}{\Delta \omega_j} = \frac{1}{(\nu_j^2 + \Delta \omega^2)^{1/2}}.
\]

Also,

\[
a_0^{(2)} = y_0 y_1^2 \left( \frac{1}{2 \Delta \omega_2} \sin (\theta_{01} - \theta_{12} + \eta_2) + \frac{1}{2 \Delta \omega} \sin (\theta_{01} - \theta_{12}) \right)
\]

\[+ y_0 y_2^2 \left( \frac{1}{2 \Delta \omega_1} \sin (\theta_{02} - \theta_{12} + \eta_1) + \frac{1}{2 \Delta \omega} \sin (\theta_{02} - \theta_{12}) \right) - y_0 \delta \nu_0,
\]

\[
a_1^{(2)} = y_1 y_0^2 \left( \frac{1}{2 \Delta \omega_2} \sin (\theta_{01} - \theta_{02} + \eta_2) + \frac{1}{2 \Delta \omega} \sin (\theta_{01} - \theta_{02}) \right)
\]

\[+ y_1 y_2^2 \left( \frac{1}{2 \Delta \omega_0} \sin (\theta_{12} - \theta_{02} + \eta_0) + \frac{1}{2 \Delta \omega} \sin (\theta_{12} - \theta_{02}) \right) - y_1 \delta \nu_1,
\]

\[
a_2^{(2)} = y_2 y_0^2 \left( \frac{1}{2 \Delta \omega_1} \sin (\theta_{02} - \theta_{01} + \eta_1) + \frac{1}{2 \Delta \omega} \sin (\theta_{02} - \theta_{01}) \right)
\]

\[+ y_2 y_1^2 \left( \frac{1}{2 \Delta \omega_0} \sin (\theta_{12} - \theta_{01} + \eta_0) + \frac{1}{2 \Delta \omega} \sin (\theta_{12} - \theta_{01}) \right) - y_2 \delta \nu_2,
\]

where

\[
\delta \nu_j = - \frac{2}{k=0} \text{Im} (\alpha_k) y_k^2 \quad (j = 0, 1, 2),
\]

\[
p^{(2)} = \frac{y_2 y_1^2}{2 y_0} \left( \frac{\cos \eta_0}{\Delta \omega_0} \frac{1}{\Delta \omega} + \frac{y_0 y_2^2}{2 y_1^2} \left( \frac{\cos \eta_1}{\Delta \omega_1} \frac{1}{\Delta \omega} + \frac{y_0 y_1^2}{2 y_2^2} \left( \frac{\cos \eta_2}{\Delta \omega_2} \frac{1}{\Delta \omega} \right. \right. \right.
\]

\[- y_0^2 \left( \frac{\cos (\theta_{02} - \theta_{01} + \eta_1)}{2 \Delta \omega_1} + \frac{\cos (\theta_{01} - \theta_{02} + \eta_2)}{2 \Delta \omega_2} \right) \Delta \omega
\]

\[- y_1^2 \left( \frac{\cos (\theta_{12} + \eta_2)}{2 \Delta \omega_2} + \frac{\cos (\theta_{12} - \theta_{01} + \eta_0)}{2 \Delta \omega_0} + \frac{\cos (\theta_{12} - \theta_{01})}{\Delta \omega} \right)
\]
\[-y_2^2 \left( \frac{\cos (\theta_2 - \theta_1 + \eta_1)}{2\Delta \omega_1} + \frac{\cos (\theta_2 - \theta_2 + \eta_0)}{2\Delta \omega_0} + \frac{\cos (\theta_2 - \theta_0)}{\Delta \omega} \right) - \delta \omega \]

where \( \delta \omega = \sum_{i=0}^{3} \beta_i \gamma_i \).

\[ G^{(2)} = \frac{1}{4\Delta \omega} \left( \frac{y_1^2 \sin (2\phi + \theta_1 + \theta_2)}{\Delta \omega_2} + \frac{y_2^2 \sin (2\phi + \theta_1 + \theta_2)}{\Delta \omega_0} + \frac{y_1^2 \sin (2\phi + \theta_0 + \theta_2 + \eta_1)}{\Delta \omega_1} \right) \]

\[ + \frac{y_2^2 \sin (2\phi + \theta_0 + \theta_1 + \theta_2)}{\Delta \omega_0} + \frac{y_1^2 \sin (2\phi + \theta_0 + \theta_1 + \theta_2 + \eta_0)}{\Delta \omega_0} \]

\[ - \left( \frac{y_1^2 y_2^2}{y_0 \Delta \omega_0} \sin (2\phi + 2\theta_1 + \eta_0) + \frac{y^2 y_2^2}{y_1 \Delta \omega_1} \sin (2\phi + 2\theta_2 + \eta_1) \right) \]

\[ + \frac{y_2^2 \sin (2\phi + 2\theta_0 + \theta_2)}{\Delta \omega_2} \]

\[ - \frac{1}{\Delta \omega} \left( \frac{y_1^2 y_2^2}{y_0^2} \sin 2(\phi + \theta_1) + \frac{y_2^2 y_2^2}{y_1^2} \sin 2(\phi + \theta_0 + \theta_2) + \frac{y_2^2 y_2^2}{y_2^2} \sin 2(\phi + \theta_0) \right) \]

\[ - \frac{2}{\Delta \omega} \left( \frac{y_0^2 \sin (2\phi + \theta_0 + \theta_1)}{\Delta \omega_0} + \frac{y_1^2 \sin (2\phi + \theta_1 + \theta_0)}{\Delta \omega_1} \right) \]

\[ + \frac{y_2^2 \sin (2\phi + \theta_2 + \theta_0)}{\Delta \omega_2} \right). \]

3. Solution of the secular motion in second order

From equations (5) and (8) we can obtain the second-order equations as

\[ \dot{y}_0 + \nu_0 y_0 = e^2 \left[ y_0 y_1^2 \left( \frac{\sin (\theta_0 - \theta_2 + \eta_2)}{2\Delta \omega_2} + \frac{\sin (\theta_0 - \theta_1)}{2\Delta \omega_1} \right) \right. \]

\[ + \left. y_0 y_2^2 \left( \frac{\sin (\theta_0 - \theta_2 + \eta_1)}{2\Delta \omega_2} + \frac{\sin (\theta_0 - \theta_2)}{2\Delta \omega} \right) - y_0 \delta \nu_0 \right], \]

\[ \dot{y}_1 + \nu_1 y_1 = e^2 \left[ y_1 y_0^2 \left( \frac{\sin (\theta_0 - \theta_0 - \eta_2)}{2\Delta \omega_0} + \frac{\sin (\theta_0 - \theta_0)}{2\Delta \omega} \right) \right. \]

\[ + \left. y_1 y_2^2 \left( \frac{\sin (\theta_0 - \theta_0 + \eta_1)}{2\Delta \omega_0} + \frac{\sin (\theta_0 - \theta_0)}{2\Delta \omega} \right) - y_1 \delta \nu_1 \right], \]

\[ \dot{y}_2 + \nu_2 y_2 = e^2 \left[ y_2 y_0^2 \left( \frac{\sin (\theta_0 - \theta_0 + \eta_1)}{2\Delta \omega_0} + \frac{\sin (\theta_0 - \theta_0)}{2\Delta \omega} \right) \right. \]

\[ + \left. y_2 y_2^2 \left( \frac{\sin (\theta_0 - \theta_0 + \eta_0)}{2\Delta \omega_0} + \frac{\sin (\theta_0 - \theta_0)}{2\Delta \omega} \right) - y_2 \delta \nu_2 \right]. \]

3.1. Case of equal amplitudes

We consider the dissipative case when all \( \nu = \nu \) and \( y_i = y \) to obtain from equation (9)

\[ \dot{y} + \nu y = e^2 y^3 k \] (10)
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where

\[
k = \frac{\sin(\theta_{01} - \theta_{12} + \eta)}{2\Delta\omega_k} + \frac{\sin(\theta_{02} - \theta_{12} + \eta)}{2\Delta\omega_k} + \frac{\sin(\theta_{01} - \theta_{12})}{2\Delta\omega} + \frac{\sin(\theta_{02} - \theta_{12})}{2\Delta\omega} \\
+ \left(\text{Im } \alpha_{00} + \text{Im } \alpha_{01} + \text{Im } \alpha_{02}\right) \\
\frac{\sin(\theta_{01} - \theta_{02} + \eta) + \sin(\theta_{12} - \theta_{02} + \eta)}{2\Delta\omega_k} + \frac{\sin(\theta_{01} - \theta_{02}) + \sin(\theta_{12} - \theta_{02})}{2\Delta\omega} \\
+ \text{Im } (\alpha_{10}) + \text{Im } (\alpha_{11}) + \text{Im } (\alpha_{12}) \\
\frac{\sin(\theta_{02} - \theta_{01} + \eta) + \sin(\theta_{12} - \theta_{01} + \eta)}{2\Delta\omega_k} + \frac{\sin(\theta_{02} - \theta_{01}) + \sin(\theta_{12} - \theta_{01})}{2\Delta\omega} \\
+ \text{Im } (\alpha_{20}) + \text{Im } (\alpha_{21}) + \text{Im } (\alpha_{22}).
\]

(11)

The solution of the equation (10) is of the form

\[
y(t) = \gamma^2 + (t_1 - Bt)^2\]

\[\times \gamma^2 e^{-2k/\nu},
\]

(12)

where

\[
y = e^{2k/\nu},
\]

\[t_1 = (1/y(0))(1 - \gamma^2 y^2(0))^{1/2}, \quad B = (\nu/y(0))(1 - \gamma^2 y^2(0))^{1/2}.
\]

The solution (12) is a soliton and the amplitudes tend to zero for large times. The amplitude is limited to a maximum value

\[
y_{\text{max}} = 1/\gamma = \nu^{1/2}/\delta k^{1/2}.
\]

(13)

For \(\gamma = 0\) one has the expression for the amplitude

\[
y(t) = y(0)/(1 - \nu t)
\]

(14)

and the time of explosion

\[t_\infty = 1/\nu.
\]

(15)

In the limit \(\nu = y(0)\) this is exactly the same as that obtained by Weiland and Wilhelmsson (1977). It is interesting to note that if \(\gamma\) is not equal to zero, \(y(t)\) stays finite. The amplitudes are generally limited for physically realistic situations and the singular solution corresponds to the limiting case \(\gamma = 0\).

3.2. Case of unequal amplitudes

We assume the effective nonlinear dissipations (Davydova et al 1975) are such that

\[
\text{Im } (\alpha_{10}) = -\text{Im } (\alpha_{01}) = \mu, \quad \text{Im } (\alpha_{21}) = -\text{Im } (\alpha_{12}) = \mu,
\]

(16)

\[
\text{Im } (\alpha_{20}) = -\text{Im } (\alpha_{02}) = \mu, \quad \text{Im } (\alpha_{ii}) = 0, \quad i = 0, 1, 2.
\]

With these assumptions, and further with \(x_i = y_i^2, \nu \ll \Delta \omega\) (Fukai et al 1969), the constants of motion are obtained as

\[
X_0 + X_1 + X_2 = P, \quad X_0 X_1 X_2 = Q,
\]

(17)

where

\[x_i = X_i e^{-2\nu}, \quad \tau = (1/2\nu)(1 - e^{-2\nu}).\]
Following our previous work (Khan et al 1980), we obtain
\[
\frac{K}{f} = \frac{\sin \theta \text{sd} \left[ \frac{\sqrt{A}}{g - f} 2e^2 \left( \frac{\delta}{\Delta \omega} + \mu \right) \cosec \theta (\tau + \tau_\iota) \right]}{\sin^2 \theta}
\]
and \(\tau_\iota\) is given by
\[
\tau_\iota = \frac{(g - f)}{\sqrt{A}} \frac{\sin \theta}{2e^2(\delta/\Delta \omega + \mu)} \text{sd}^{-1} \left( \frac{\mu(0)}{\sin^2 \theta} \right)
\]
where \(\text{sd}^{-1}\) is the Jacobian elliptic function.

When \(u_j(\tau) = -1\), \(X_j\) tends to infinity and one can obtain the time of explosion given by
\[
\sin \theta \text{sd} \left[ \frac{\sqrt{A}}{g - f} 2e^2 \left( \frac{\delta}{\Delta \omega} + \mu \right) \cosec \theta (\tau + \tau_\iota) \right] + 1 = 0.
\]

When the initial conditions are such that \(u_j(0) = \sin(\pi + \theta)\text{sd}(l|\sin^2 \theta)\) with \(l\) a constant, all the amplitudes will go to infinity at the same time and the time of explosion is given by
\[
t_\infty = 3^{-1/4} \left[ \frac{l - \text{sd}^{-1}(\cosec \theta|\sin^2 \theta)}{2e^2(\delta/\Delta \omega + \mu)} \right]
\]

4. Discussion

In this paper an attempt has been made to study analytically the saturation of explosive instabilities by means of third-order nonlinear effects. The nonlinear effects considerably change the phase dynamics when the amplitudes are large and the saturation occurs as found both numerically and analytically. Numerical solutions of these problems have already been given by Weiland and Wilhelmsson. Hence our work complements the earlier work in the sense that the analytical results obtained are in agreement with the numerical solutions. After the saturation point the amplitudes quickly decrease. This effect is shown in figure 1 when all amplitudes are assumed to be equal in the presence of linear and nonlinear dissipative terms. If, however, the amplitudes are different the saturation peaks will occur repeatedly as functions of time (see figure 2). Oraevskii et al (1973a, b) obtained a complete analytical solution to the nonlinear coupled mode equations including a third-order frequency shift. They found a soliton solution, and with more general initial conditions repeated explosions were obtained which could be expressed in terms of elliptic functions. Weiland and Wilhelmsson (1973) and Weiland (1974) extended this investigation to include a linear dissipation and also an imaginary part of third-order frequency shift. They gave the numerical solutions for the general nonlinear system of equations containing third-order nonlinear terms, and also some analytical results concerning linear dissipation and asymptotic behaviour. These numerical results have been compared with those
Figure 1. Curve I corresponds to the numerical solutions of Wilhelmsson, showing an unbounded solution ($y = 0$) and the stabilised solution ($y \neq 0$). Curve II corresponds to our analytical solution when $u(0) = 0.6$, $v = 0.6$, $\theta_{12} = 0$, $\theta_{02} = \pi/4$, $\theta_{01} = -\pi/4$, $\text{Im} \alpha = 0.05$, showing an unbounded solution for $\gamma = ek^{1/2}/\nu^{1/2} = 0$ and a stabilised solution for $\gamma \neq 0$. Curve III corresponds to our analytical solutions with only $v$ changed to 0.5.

Figure 2. (a) corresponds to the numerical solution of Wilhelmsson, showing repeated stabilised explosions. (b) corresponds to our analytical solution, showing repeated stabilised explosions with $\theta_{01} = \pi/3$, $\theta_{12} = \pi$, $\theta_{02} = -\pi/3$.

obtained by us analytically, and found to be in agreement at least qualitatively, as can be seen in figures 1 and 2.

As shown above, the effect of complex coupling coefficients of second-order nonlinear terms, the influence of linear dissipation as well as third-order effects were taken into account to obtain analytically a soliton solution. In the presence of dissipation, second- and third-order conductivities have a real part, which means that coupling factors are complex when one must include a linear dissipation in the system. It is well known (see for example Fukai et al (1969)) that the effect of cubic-order terms causes an amplitude-dependent frequency shift. As the waves grow they shift out of resonance and $\Delta \omega$ ultimately becomes large. In the asymptotic limit of saturation, the
values of the amplitudes are determined by introducing an effective coupling coefficient of the third-order coupling term, and the values are found to be independent of the initial conditions. It is shown that the nonlinear dissipation stabilises all amplitudes. Also the periodic solutions have been found in terms of the Jacobian elliptic function.

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EXPLOSIVE INSTABILITIES IN BEAM PLASMA SYSTEM

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Abstract—A formalism based on Lie Transform is used to study the explosive nature of the beam plasma instability. It is shown that the First order averaged Hamiltonian produces the explosive instability whilst the Second order averaged Hamiltonian limits the instability.

1. INTRODUCTION

The two stream instability occurs when there is a relative motion between the components of a plasma. The mechanism of two stream instability can be described as follows—a very small perturbation away from zero field at a given point causes a velocity modulation away from zero field at a given point. This, in time, produces a bunching of space charge in the direction of the beam which creates a much larger potential than that due to the original perturbation. The field due to one beam modulates the other one, which then influences the disturbance of the source in highly amplified form. Thus the perturbation builds up accumulatively and the instability results.

The nonlinear development of two stream instability has been studied extensively (Aamodt and Sloan, 1967, 1968; Fukai and Harris, 1971; Fukai et al., 1971; Dysthe, 1970; Oraevskii et al. 1973a, b) both analytically and numerically. Weiland and Wilhelmsen (1973), Jones et al. (1979) and Lisak (1978) included in the equations for the complex wave amplitudes the Third order nonlinear interaction term which saturates the instability of the beam plasma system.

These explosive instabilities, if they exist, are of practical interest as they give rise to a mechanism for rapid dissipation of coherent wave energy into thermal motion and hence may be effective for plasma heating. Recently, Pavlenko (1978), Kerst and Raether (1976), Escande (1979), Throop and Parker (1979), and Davydova and Sharmar (1978) have studied the evolution of explosive instability in beam plasma system also the stability-instability boundaries are established Golden et al. (1980). Current interest is also now focused on the experimental studies of explosive instability in beam plasma system (Nakamura, 1977, Sugaya et al., 1979; Nakamura and Mitani, 1976).

Here, we consider the nonlinear interaction which comes directly from the Hamiltonian. In order to discuss nonlinear instability Fukai and Harris (1971) considered a canonical transformation to diagonalize the linear part of the Hamiltonian. Sedlacek (1976) used a method which consists of first transforming the Hamiltonian into its diagonalized form. Then it was transformed into angle action variables corresponding to the resonance relation. One improper angle and its conjugate action was introduced. To separate the secular motion from the rapidly fluctuating motion a direct near identity canonical transformation was constructed perturbationally as a power series of the nonlinear
coupling constants. The method used in the above paper is lengthy and quite involved. Sedlacek's result can be obtained by the comparatively straightforward method of Lie transformation which are those generated by infinitesimal transformations in the region of identity transformation and allows transparent calculation of complicated results.

For canonical and also for noncanonical systems the Lie series and transform are simple and efficient algorithms which can be used to transform the variables and arbitrary functions to new variables. This method has the principal advantage over other existing methods, for example the K.B.M. method, the multiple time scale method, the Lagrangian method of averaging by Dougherty (1970), and applied by Galloway and kim (1971); Dewar (1973), Johnston and Kaufman (1978), Johnston (1976) and Boyd and Turner (1972) where the governing equations for the system had to be derived from a variational principle, is that the generating function is not a mixed function of the old and new variables, the theory is canonically invariant and it is possible to give a direct expansion of any function of old variables in terms of new variables. The method of averaged Lagrangian was used by Boyd and Turner (1971) to describe wave interaction in warm magnetized plasmas and also the nonlinear interaction between positive and negative energy waves which leads to explosive instabilities under certain conditions.

The purpose of the present paper is to investigate the possibilities of explosive instabilities in a beam plasma system and its stabilization in the second order calculation by the method of Lie transform. The constants of motion viz. the averaged Hamiltonians of the first and second orders are obtained. The first order averaged Hamiltonian shows an explosive behaviour whereas the second order averaged Hamiltonian bounds the unlimited growth of the amplitudes being independent of the initial conditions.

2. PROCEDURE OUTLINE

The Lie transformation procedure has been described in detail by Kamel (1969, 1970, 1971) and Nayfeh (1973). Still, for the sake of completeness we give here the essential steps which are used in our calculation.

To find the solution of the non-Hamiltonian system of differential equations

\[ \dot{y} = g(y, t, \epsilon) = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} g^{(n)}(y, t) \quad (2.1) \]

a Hamiltonian \( K \) is introduced of the form \( K = Y \cdot g(y, t) \) where \( Y \) is the linear adjoint vector and \( \epsilon \) is the small expansion parameter.

\[ K^{(n)}(y, Y, t) = Y \cdot g^{(n)}(y, t) \]

\[ \dot{y} = \frac{\partial K}{\partial Y} = g(y, t, \epsilon) \quad (2.2) \]

\[ \dot{Y} = - \frac{\partial K}{\partial y} = - Y \cdot \frac{\partial g}{\partial y} \quad (2.3) \]
After this Hamiltonization the perturbation method established by Kamel (1969) based on Lie Deprit transform and Lie Hori transform is applied so that the resulting Hamiltonian system

\[ H(x, X, t, \epsilon) = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} H_n(x, X, t) \]  

(2.4)

with

\[ \dot{x} = \frac{\partial H}{\partial X}, \dot{X} = -\frac{\partial H}{\partial x} \]  

(2.5)

will contain only long periodic terms and will be more easy to handle than the old system. Any function of the old variables \((y, Y)\) in the form of

\[ F(y, Y, t, \epsilon) = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} F^{(n)}(y, Y, t) \]  

(2.6)

can be expressed in terms of the new variables \((x, X)\) in the form of

\[ F[y(x, X, t, \epsilon), Y(x, X, t, \epsilon), t, \epsilon] = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} F_n(x, X, t) \]  

(2.7)

using the generating function \(W\).

3. DETAILS OF THE CALCULATION

Hamiltonian of a two stream plasma (Fukai and Harris, 1971; Sedlacek, 1976) is

\[ H(\xi_1, \xi_2, \xi_3) = H_0(\xi_1, \xi_2, \xi_3) + \mu H_1(\xi_1, \xi_2, \xi_3) \]  

(3.1)

\[ H_0 = \frac{1}{2mN} (\xi_1^2 + \xi_2^2) - V\pi_1 \frac{\partial \xi_1}{\partial x} + V\pi_2 \frac{\partial \xi_2}{\partial x} + \frac{1}{2} mN\omega_p^2 (\xi_1 + \xi_2)^2 \]  

(3.2)

where \(\mu\) is the expansion parameter. The canonical coordinates \(\xi_i(x, t), \xi_j(x, t)\) corresponds to the perturbation of the first and second beam being canonically conjugate to the momenta \(\pi_i(x, t), \pi_j(x, t)\) respectively. The canonical variables are related to the number density \(n_1(x, t), n_2(x, t)\) and the velocity \(v_1(x, t), v_2(x, t)\) by

\[ n_1 = -N \frac{\partial \xi_1}{\partial x}, \quad n_2 = -N \frac{\partial \xi_2}{\partial x} \]  

\[ v_1 = (mN)^{-1} \pi_1, \quad v_2 = (mN)^{-1} \pi_2. \]  

(3.3)
From the equations of motion

\[
\frac{\partial m_i}{\partial t} = - \frac{\partial H}{\partial \xi_i} + \frac{\partial}{\partial x} \left( \frac{\partial \xi_i}{\partial \xi_j} \frac{\partial H}{\partial \xi_j} \right)
\]

\[
\frac{\partial \xi_i}{\partial t} = \frac{\partial H}{\partial m_i} - \frac{\partial}{\partial x} \left( \frac{\partial \xi_i}{\partial \xi_j} \frac{\partial H}{\partial \xi_j} \right) \quad (i = 1, 2)
\]

One can have

\[
\frac{\partial \pi_1}{\partial t} = - mN\omega_p^2 (\xi_1 + \xi_2) - V \frac{\partial \pi_1}{\partial x} - \frac{\mu}{mN} \frac{\partial \pi_1}{\partial x}
\]

\[
\frac{\partial \pi_2}{\partial t} = - mN\omega_p^2 (\xi_1 + \xi_2) + V \frac{\partial \pi_2}{\partial x} - \frac{\mu}{mN} \frac{\partial \pi_2}{\partial x}
\]

\[
\frac{\partial \xi_1}{\partial t} = \frac{\pi_1}{mN} - V \frac{\partial \xi_1}{\partial x} - \frac{\mu}{mN} \frac{\partial \xi_1}{\partial x}
\]

\[
\frac{\partial \xi_2}{\partial t} = \frac{\pi_2}{mN} + V \frac{\partial \xi_2}{\partial x} - \frac{\mu}{mN} \frac{\partial \xi_2}{\partial x}
\]

where \( \omega_p \) is the electron plasma frequency; \( m \) is the electron mass; \( N \) is the unperturbed electron density.

Assuming \( (\partial/\partial x) \sim c^{-1}(\partial/\partial t) \) the system of equations (3.5) becomes

\[
(c + v) \dot{\xi}_1 = \frac{\pi_1 c}{mN} - \frac{\mu}{mN} \dot{\xi}_1
\]

\[
(c - v) \dot{\xi}_2 = \frac{\pi_2 c}{mN} - \frac{\mu}{mN} \dot{\xi}_2
\]

\[
(c + v) \dot{\pi}_1 = - mN\omega_p^2 (\xi_1 + \xi_2) - \frac{\mu \pi_1}{mN} \dot{\pi}_1
\]

\[
(c - v) \dot{\pi}_2 = - mN\omega_p^2 (\xi_1 + \xi_2) - \frac{\mu \pi_2}{mN} \dot{\pi}_2
\]

(dot donotes the differentiation w.r.t. \( t \)).

When \( \mu = 0 \) from the equation (3.6) one can have \( \dot{\xi}_i + \omega^2 \xi_i = 0 \) where,

\[
\omega^2 = \omega_p^2 \frac{c^2}{(c + v)^2 + \frac{1}{(c - v)^2}}
\]
the solution of which is of the form

\[ \xi_i = A_i \sin (\omega t + \theta_i) \quad (i = 1, 2) \]

\( A_i, \theta_i \) being constants. An approximate solution to the system of equations (3.6) for \( \mu \) small but different from zero are assumed to be of the form

\[ \xi_1 = A_1(\tau) \sin B_1(\tau) \]
\[ \xi_2 = A_2(\tau) \sin B_2(\tau) \]  

\[ \pi_1 = \frac{mN}{c} (c + v) A_1 \omega \cos B_1(\tau) \]  
\[ \pi_2 = \frac{mN}{c} (c - v) A_2 \omega \cos B_2(\tau) \]

with time varying \( A_i, \theta_i \) subject to the condition

\[ \dot{\xi}_i = A_i \omega \cos (\omega \tau + \theta_i) \]

(3.10)

where \( B_i = \omega_i \tau + \theta_i, (\omega_i = \omega + 0(\epsilon)) \).

Utilising the relations for the zeroth order solutions

\[ (c + v)\pi_1 = (c - v)\pi_2 \]
\[ (c + v)\xi_1 = (c - v)\xi_2 \]  

(3.11)

One can have from equation (3.6) with \( v \ll c \)

\[ \dot{A}_1 = \frac{\mu}{2c} \omega^2 A^2_1 A_2 [\sin B_2 + \sin (B_2 - 2B_1)] \]  
\[ \dot{A}_2 = -\frac{\mu}{2c} \omega^2 A_1^2 [\sin B_2 + \sin (B_2 - 2B_1)] \]

(3.12)

\[ \dot{B}_1 = \omega - \frac{\mu}{2c} A_2 \omega^2 [\cos B_2 + \cos (B_2 - 2B_1)] \]
\[ \dot{B}_2 = \omega - \frac{\mu}{2c} A_1 \omega^2 [\cos B_2 + \cos (B_2 - B_1)]. \]

(3.13)

Now we take the equations (3.12), (3.13) as the starting equations and analogous to equations (2.1) our next step will be to construct the Hamiltonian \( K \) from this system of equations (3.12), (3.13).
We introduce the adjoint vector $V$ with components $\lambda_{A_1}, \lambda_{A_2}, \lambda_{B_1}, \lambda_{B_2}$ and the Hamiltonian $K$ where

$$K = \lambda_{A_1} \dot{A}_1 + \lambda_{A_2} \dot{A}_2 + \lambda_{B_1} \dot{B}_1 + \lambda_{B_2} \dot{B}_2$$

(3.14)

so that we have

$$K^{(0)} = \omega (\lambda_{B_1} + \lambda_{B_2}) = H_0$$

(3.15)

$$K^{(1)} = \lambda_{A_1} \frac{\omega^2}{2c} \{ A_1 A_2 [\sin B_2 + \sin (B_2 - 2B_1)] \}$$

$$- \lambda_{A_2} \frac{\omega^2}{2c} \{ A_2 [\cos B_2 + \cos (B_2 - 2B_1)] \}$$

$$- \lambda_{B_1} \frac{\omega^2}{2c} \{ A_1^2 [\cos B_2 + \cos (B_2 - 2B_1)] \}$$

$$- \lambda_{B_2} \frac{\omega^2}{2c} \{ A_2^2 [\cos B_2 + \cos (B_2 - 2B_1)] \}$$

(3.16)

$$K^{(n)} = 0, \ n \geq 2.$$

We transform $Y$ and $K$ into $Y'$ and $H$ using the algorithm defined by Kamel (1971). The dashed quantities will be determined in terms of the old variables using the generating function (Kamel, 1971). Choosing $H$, as the secular terms of $K'$

$$H_1 = \frac{\omega^2}{2c} \{ [\lambda_{A_1} A_1 A_2 ^2 - \lambda_{A_2} A_1^2] \} \sin (B_1' - 2B_1^2)$$

$$- \left\{ \lambda_{B_1} A_1^2 + \lambda_{B_2} A_2^2 \right\} \sin (B_1^2 - 2B_1')$$

(3.17)

From the relation $(\partial W_i / \partial \tau) = H_1 - K'$ we have

$$W_1 = \frac{\omega}{2c} \left[ (\lambda_{A_1} A_1 A_2 ^2 - \lambda_{A_2} A_1^2) \cos B_1' + \left\{ \lambda_{B_1} A_1^2 + \lambda_{B_2} A_2^2 \right\} \sin B_1' \right].$$

(3.18)

Computing $L_1(H_1 + K')$ and choosing $H_2$ to eliminate its secular part we have from the relation

$$\frac{\partial W_2}{\partial \tau} = H_2 + L_1(H_1 + K')$$

(3.19)

$$H_2 = (\lambda_{B_1} - \lambda_{B_2}) \frac{\omega^3}{2c^2} A_1^2$$

(3.20)
where $A_n$, $B_n$, $\lambda_{A_n}$, $\lambda_{B_n}$ are transformed to $A'_n$, $B'_n$, $\lambda_{A'_n}$, $\lambda_{B'_n}$, so that the resulting Hamiltonian will contain only long periodic terms. The equations of motion

$$
\dot{A}'_1 = \frac{\partial H}{\partial \lambda_{A'_1}} = \frac{\omega^2}{2c} A'_1 A'_2 \sin B'
$$

$$
\dot{A}'_2 = \frac{\partial H}{\partial \lambda_{A'_2}} = -\frac{\omega^2}{2c} A'_1^2 \sin B'
$$

\((B' = B'_2 - 2B'_1\) is slowly varying) gives $A'_1^2 + A'_2^2 = P$ (P being arbitrary constant).

The equation (3.21) may be interpreted as the Manley Rowe relation of the averaged momenta $A'_1$, $A'_2$ conjugate to the new coordinates $\lambda_{A'_1}$, $\lambda_{A'_2}$. In the unperturbed state $A_1$, $A_2$, $B_1$, $B_2$ are all constants. Under the influence of perturbation these variables undergo large but slow deviations from their unperturbed state.

The second order averaged Hamiltonian is a second order constant of motion which yields

$$
M A'_1^2 = \text{constant} \left(\text{where } M = \frac{\omega^3}{2c^2} (\lambda_{B'_1} - \lambda_{B'_2})\right)
$$

is angle independent and may be interpreted in terms of the nonlinear frequency shift of the waves. The above result (3.22) is the same as obtained by Sedlacek (1976).

4. DISCUSSION AND CONCLUSION

In the field of nonlinear wave problem the phenomenon of explosive instability is of great interest. The resonant interaction of the waves in a plasma under certain conditions lead to nonlinear instabilities in which all waves grow faster than exponentially and reach infinite amplitudes in a finite time. To discuss explosive instabilities we consider averaged Hamiltonians of different orders

$$
H = H_0 + \mu H_1 + \mu^2 H_2.
$$

Considering the dominating terms of $H$ the first order averaged Hamiltonian (using 3.21) yield the first order constant of motion of the form

$$
(\lambda_{A'_1} A'_1 \sqrt{(P - A'_1^2) - \lambda_{A'_2} A'_2^2}) \sin B' = \text{constant}.
$$

As $B'$ tends to zero, $A'_1$ grows indefinitely so also $A'_2$ by virtue of the equation (3.21). Thus the first order averaged Hamiltonian yields the explosive behaviour of the amplitudes independent of the initial conditions.

Next, we consider the second order averaged Hamiltonian. Though, there is no quartic nonlinear interaction term in the original Hamiltonian the second order averaged Hamiltonian contains terms which are independent of the angle like coordinates and may be interpreted in terms of the nonlinear frequency shift.
of the waves. The second order constant of motion is of the form

\[ MA_1^2 = \text{constant} \left( M = \lambda B_1 - \lambda B_2 \right) \frac{\omega^3}{2c^2}. \]

If \( \lambda_1 \neq \lambda_2 \), the equation (3.22) can be interpreted as the equation of concentric circles around the origin. These circles bound the motion for all values of the initial wave amplitudes. Thus the second order motion stabilizes the infinite growth.

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Nonlinear coupling of two three-wave systems in plasma

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The nonlinear interaction of two three-wave systems, including the possibility of negative energy waves in the presence of linear damping or growth and frequency mismatch, is investigated in a plasma, where one system of two transverse and one longitudinal wave interacts with a system of three longitudinal waves, and one of the longitudinal waves introduces coupling between the two subsystems. The solutions are analysed under various initial conditions and it is shown that, if one triplet be explosively unstable by itself, the presence of the second triplet can stabilize the solutions, depending on the relative strength of the coupling factor.

1. Introduction

Two basic characteristics of plasma, considering it as a medium through which waves can propagate, are its dispersion and nonlinearity. Recently, the nonlinear effects in plasma have been treated extensively both in random phase approach and coherent phase description. The nonlinear dynamics of plasma can involve both wave–particle and wave–wave interactions. In the field of fusion plasma research, nonlinear effects are essential, for example, in the explanation of anomalous resistance, anomalous diffusion, plasma heating and confinement, laser–plasma interaction and for enhanced radiation losses from a plasma. Nonlinear effects in plasma are of great interest in astrophysics in the study of radio emission from stars, the origin of cosmic rays and cosmic masers as well as in ionospheric physics (e.g. investigation of the ionosphere by powerful radio waves).

The three-wave interaction is the lowest-order nonlinear effect for a system approximately described by a linear superposition of discrete waves. In some cases the three-wave resonance condition is not satisfied and it requires a higher-order process to transfer energy (Davidson 1972). Also, with more intense power sources now being available, studies of higher order processes are feasible. The interaction between longitudinal–longitudinal or longitudinal–transverse waves in a three-wave system have been studied by several authors (Wilhelmsson 1969; Stenflo 1970; Bonnedal & Wilhelmsson 1974). The basic study of a three-wave system can be extended to the interaction of two three-wave systems of longitudinal and transverse waves which have one wave in common, i.e. to a...
five-wave system. Weiland & Wilhelmsson (1977) studied the coherent interaction in a nonlinearily unstable resonant five-wave system of longitudinal and transverse waves in a plasma and obtained exact analytical solutions in terms of the Jacobian elliptic function without considering the effect of dissipation and frequency mismatch. Wilhelmsson & Pavlenko (1973) included the effect of dissipation by means of a convenient but approximate time transform whose validity may be questionable when there are large differences in the damping rates (Fuchs 1975). They studied the problem of explosive instability in the five-wave system with special emphasis on the influence of dissipation and frequency indeterminacy on the dynamical phase locking but did not consider their effects on the threshold value and growth rate of the amplitudes of the excited waves. Recently, Menyuk, Chen & Lee (1983) have studied multiple three-wave interactions in which a set of wave triads interact through one shared wave. Painlevé analysis is used to investigate the integrability of the system. They give numerical results in particular in the case of two triads and the results are consistent with the Painlevé analysis.

Following Wilhelmsson & Pavlenko (1973) we consider the coherent nonlinear interaction of two three-wave systems in a plasma in the presence of linear damping or growth and frequency mismatch, where one system of two transverse and one longitudinal wave interacts with a system of three longitudinal waves and one of the longitudinal waves introduces coupling between the two subsystems thus forming a five-wave system. It is assumed that there is free energy in the medium (Tsytovich 1967) so that longitudinal as well as transverse waves may carry negative wave energy and therefore there is a possibility of explosively unstable solutions to the system of waves. One of the plasma waves is assumed to be at resonance with the beat frequency of two transverse electromagnetic waves. Furthermore, the same plasma wave is at resonance with two other plasma waves and forms with them an explosively unstable triplet of waves.

Five-wave systems of this type allow for enhancement of optical or microwave radiation by nonlinear effects. This kind of nonlinear effect might play a significant role in high-power laser devices and also in astrophysical plasma.

In the case of weak nonlinearity, when the equations of the dynamical quantities are formulated some procedure is needed to eliminate the secular terms. The perturbation method used here is due to Coffey & Ford (1969). This method was successfully applied to the study of nonlinear three-wave interactions and their stabilization (Khan, De, Roychowdhury & Roy 1980; De, Khan & Roychowdhury 1981). Analytical solutions obtained here describe the temporal behaviour of the amplitudes. Under certain restrictions on the initial amplitudes and by making use of the constants of motion, the solutions obtained may be classified into the following types: (i) periodic solutions, (ii) shock like solutions, (iii) soliton solutions. The periodic solutions, obtained in terms of Jacobian elliptic functions, show that the waves either exchange energy in a periodic way or are singular at some finite time corresponding to explosive behaviour. The threshold value, explosion time and growth rate which characterize the instability have been calculated and the effects of dissipation and frequency mismatch on these two parameters have been demonstrated. Also it is shown
that, if one triplet be explosively unstable by itself, the presence of the second triplet can stabilize the solutions, depending on the relative strength of the coupling factor.

2. Basic coupled mode equations

For the nonlinear interaction of two three-wave systems of electromagnetic and plasma waves having one wave in common, i.e. a five-wave system, assuming the resonance conditions to be of the form

\[
\begin{align*}
  k_{0L} &= k_{1L} + k_{2L}, \\
  k_{0T} &= k_{1T} + k_{2L}, \\
  \omega_{0L} &= \omega_{1L} + \omega_{2L}, \\
  \omega_{0T} &= \omega_{1T} + \omega_{2L}.
\end{align*}
\]

the coupled mode equations are

\[
\begin{align*}
  \frac{\partial}{\partial t} - i\omega_{0L} a_{0L} &= c_{0L} a_{1L} a_{2L}, \\
  \frac{\partial}{\partial t} - i\omega_{1L} a_{1L} &= c_{0L} a_{0L} a_{2L}^\ast, \\
  \frac{\partial}{\partial t} - i\omega_{2L} a_{2L} &= c_{0L} a_{0L} a_{1L}^\ast + c_{0T} a_{0T} a_{1T}^\ast, \\
  \frac{\partial}{\partial t} - i\omega_{0T} a_{0T} &= c_{1T} a_{1T} a_{2L}, \\
  \frac{\partial}{\partial t} - i\omega_{1T} a_{1T} &= c_{0T} a_{0T} a_{2L}^\ast.
\end{align*}
\]

Here \(iL\) and \(jT\) refer to different longitudinal and transverse modes respectively \((i = 0, 1, 2, j = 0, 1)\).

Following Wilhelmsson & Stenflo (1970) one can take

\[
\begin{align*}
  a_k &= \epsilon A_k \exp (i \Re \omega_k t),
  & A_k = \tilde{a}_k \exp (i \phi_k),
  & \tilde{a}_k = |A_k|,

  \nu_k &= \Im \omega_k,
  & c_{ij} = \alpha_{ij} \exp (i \theta_{ij}),
\end{align*}
\]

(where \(\epsilon\) is an ordering parameter which, as will be shown later, may be absorbed in redefined amplitudes). In the presence of dissipation the coupling coefficients become complex. We introduce the renormalization

\[
\begin{align*}
  u_{0L} &= (\alpha_{1L} a_{2L}) \tilde{a}_{0L}, \\
  u_{1L} &= (\alpha_{1L} a_{2L}) \tilde{a}_{1L}, \\
  u_{2L} &= (\alpha_{1L} a_{2L}) \tilde{a}_{2L}, \\
  u_{0T} &= (\alpha_{2T} a_{1T} a_{0T}) \tilde{a}_{0T}, \\
  u_{1T} &= (\alpha_{2T} a_{1T} a_{0T}) \tilde{a}_{1T}.
\end{align*}
\]
Then from (2.2) using (2.3), (2.4) and separating real and imaginary parts, we obtain the following system of equations:

\[
\begin{align*}
\frac{\partial u_{1L}}{\partial t} + v_{1L} u_{1L} &= e \omega_{kL} u_{1L} \cos (\phi_1 + \theta_{k1L}) \quad (k = 0, 1, j + k), \\
\frac{\partial u_{2L}}{\partial t} + v_{2L} u_{2L} &= e \left( u_{0L} u_{1L} \cos (\phi_1 + \theta_{01L}) + \Lambda u_{0T} u_{1T} \cos (\phi_2 + \theta_{0T1T}) \right), \\
\frac{\partial u_{1T}}{\partial t} + v_{1T} u_{1T} &= e \Lambda u_{kT} u_{2L} \cos (\phi_2 + \theta_{k2L}).
\end{align*}
\]

(2.5)

\[
\begin{align*}
\frac{\partial \phi_1}{\partial t} &= \Delta \omega_1 - e \left[ \frac{u_{0L} u_{2L}}{u_{1L}} \sin (\phi_1 + \theta_{11L}) + \frac{u_{0L} u_{2L}}{u_{1L}} \sin (\phi_1 + \theta_{01L}) + \frac{u_{0T} u_{1T}}{u_{2L}} \sin (\phi_2 + \theta_{12L}) \\
&\quad + \frac{u_{0T} u_{1T}}{u_{2L}} \sin (\phi_2 + \theta_{12L}) + \frac{u_{0T} u_{1T}}{u_{2L}} \sin (\phi_2 + \theta_{0T1T}) \right], \\
\frac{\partial \phi_2}{\partial t} &= \Delta \omega_2 - e \left[ \frac{u_{0L} u_{1L}}{u_{2L}} \sin (\phi_1 + \theta_{01L}) + \Lambda u_{0T} u_{1T} \sin (\phi_2 + \theta_{01T}) + \frac{u_{0T} u_{1T}}{u_{2L}} \sin (\phi_2 + \theta_{0T1T}) \right].
\end{align*}
\]

(2.6)

where,

\[
\begin{align*}
\phi_1 &= \phi_{0L} - \phi_{1L} - \phi_{2L} + \Delta \omega_1 t, \\
\phi_2 &= \phi_{0T} - \phi_{1T} - \phi_{2T} + \Delta \omega_2 t, \\
\Delta \omega_1 &= \text{Re} (\omega_{0L}) - \text{Re} (\omega_{1L}) - \text{Re} (\omega_{2L}), \\
\Delta \omega_2 &= \text{Re} (\omega_{0T}) - \text{Re} (\omega_{1T}) - \text{Re} (\omega_{2T}).
\end{align*}
\]

(2.7)

For two three-wave systems it is not possible to normalize the amplitudes in such a way that all the coupling factors become equal to unity and

\[
\Lambda = (e_{012L} e_{01T2L} / e_{0L2L} e_{1L2L})^2,
\]

(2.8)

is the coupling factor as a measure of coupling strength between the sets of transverse and longitudinal waves.

3. Solutions of the secular motion in second order

In order to solve the system of equations (2.5)-(2.6) we apply the method of perturbation due to Coffey & Ford (1969). To separate the secular motion from the rapidly fluctuating motion the perturbation scheme used is

\[
\begin{align*}
u_{1L} &= y_{1L} + e F_{11L}(y, \phi_1, \phi_2) + e^2 F_{112L}(y, \phi_1, \phi_2), \\
u_{2L} &= y_{2L} + e F_{21L}(y, \phi_1, \phi_2) + e^2 F_{212L}(y, \phi_1, \phi_2), \\
\phi_m &= y_m + e G_{1m}(y, \phi_1, \phi_2) + e^2 G_{2m}(y, \phi_1, \phi_2), \\
\dot{y}_{1L} &= \sum_{n=0}^{\infty} e^n a_{1n}(y), \\
\dot{y}_m &= \Delta \omega_m + \sum_{n=1}^{\infty} e^n b_{mn}(y).
\end{align*}
\]

(3.1)

(3.2)

(3.3)

(3.4)
Nonlinear coupling of two three-wave systems

Inserting (3.1), (3.2) in (2.5), (2.6) and using (3.3)-(3.4), one can obtain successive systems of equations for each power of $\epsilon$ (for details see Khan et al. 1980, De et al. 1981).

The zeroth-order terms give

$$a^{(0)}_{n,j,k,T} + \nu_{n,j,k,T} = 0. \quad (3.5)$$

The first-order terms give

$$a^{(1)}_{n,j,k,T} = 0, \quad \delta^{(1)}_{m} = 0, \quad (3.6)$$

$$F^{(1)}_{n} = \frac{y_{nL}y_{nT}}{\Delta \omega_{nL}} \sin (\psi + \theta_{KLnL} + \eta_{L}) \quad (k = 0, 1, j + k), \quad (3.7)$$

$$F^{(1)}_{n} = \frac{y_{nL}y_{nT}}{\Delta \omega_{nL}^{2} + \nu_{nL}^{2}} \left\{ \nu_{nL} \cos (\psi + \theta_{O2L}) + \Delta \omega_{n} \sin (\psi + \theta_{O2L}) \right\}$$

$$+ \frac{\Delta \omega_{nL}^{2} + \nu_{nL}^{2}}{\Delta \omega_{nL}^{2} + \nu_{nL}^{2}} \left\{ \nu_{nL} \cos (\psi + \theta_{O2L}) + \Delta \omega_{n} \sin (\psi + \theta_{O2L}) \right\}, \quad (3.7)$$

$$F^{(1)}_{n} = \frac{\Delta \omega_{nL}y_{nT}}{\Delta \omega_{nL}} \sin (\pi + \psi + \eta_{LT} + \theta_{KTnL}), \quad (3.8)$$

where

$$\tan \eta_{LT} = \nu_{LT}/\Delta \omega_{LT}, \quad \Delta \omega_{LT} = \Delta \omega_{LT}(1 + \nu_{LT}^{2}/\Delta \omega_{LT}^{2}), \quad (3.9a)$$

$$G^{(1)}_{L} = \frac{\Delta \omega_{nL}y_{nL}}{\Delta \omega_{nL}} \cos (\psi + \theta_{KLnL}) + \frac{\Delta \omega_{nL}y_{nL}}{\Delta \omega_{nL}} \cos (\psi + \theta_{KLnL})$$

$$+ \frac{\Delta \omega_{nL}^{2} + \nu_{nL}^{2}}{\Delta \omega_{nL}^{2} + \nu_{nL}^{2}} \left\{ \nu_{nL} \cos (\psi + \theta_{O2L}) + \Delta \omega_{n} \sin (\psi + \theta_{O2L}) \right\}, \quad (3.9b)$$

We assume all $\nu$'s to be equal (which is a reasonable approximation if damping is due to collisions in the long-wavelength regime). Then,

$$\Delta \omega_{nL,T} = \Delta \omega_{nL,T},$$

$$\tan \eta_{LT} = \tan \eta_{LT}.$$

Using the relation (3.3) one gets from the next powers of $\epsilon$ the following system of equations:

$$\frac{dx_{nL}}{dT} = x_{nL}x_{nL} \left\{ \frac{\sin (\theta_{LLnL} - \theta_{KLnL} + \eta_{L})}{\Delta \omega_{nL}} + \frac{\sin (\theta_{LLnL} - \theta_{KLnL})}{\Delta \omega_{nL}} \right\}$$

$$+ \frac{x_{nL}x_{nL}^{2}k}{\Delta \omega_{nL}} \left\{ \sin (\theta_{LLnL} - \theta_{KLnL} + \beta) \right\}$$

$$+ \Lambda x_{nL}^{3}x_{nL}x_{nL}x_{nL}^{2}k \sin (\psi_{nL} + \theta_{O2L} + \theta_{KLnL} + \delta), \quad (3.10a)$$

$$\frac{dx_{nT}}{dT} = \Lambda x_{nT}x_{nT} \left\{ \frac{\sin (\pi + \eta_{LT} + \theta_{T4nL} - \theta_{KTnL})}{\Delta \omega_{nT}} + \frac{\sin (\theta_{T4nL} - \theta_{KTnL})}{\Delta \omega_{nT}} \right\}$$

$$+ \Lambda x_{nT}x_{nT}^{2}k \sin (\theta_{T4nL} + \theta_{KTnL} + \beta)$$

$$+ \Lambda x_{nT}x_{nT}^{2}x_{nL}^{2}x_{nL}^{2}k \sin (\psi_{nL} - \psi_{nL} + \theta_{KLnL} - \theta_{LT} + \beta), \quad (3.10b)$$
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\[
\frac{dx_{3L}}{dt} = x_{1L} x_{2L} \left( \frac{\sin (\theta_{13L} - \theta_{03L} + \gamma_L)}{\Delta \omega_L} + \frac{\sin (\theta_{13L} - \theta_{03L})}{\Delta \omega_1} \right)
+ x_{03L} \left( \frac{\sin (\theta_{03L} - \theta_{03L} + \gamma_L)}{\Delta \omega_L} + \frac{\sin (\theta_{03L} - \theta_{03L})}{\Delta \omega_1} \right)
+ \Lambda^3 x_{0T} x_{2L} \left( \frac{\sin (\pi + \gamma_T + \theta_{0T1L} - \theta_{0T1})}{\Delta \omega_T} + \frac{\sin (\theta_{0T1L} - \theta_{0T1})}{\Delta \omega_2} \right)
+ \Lambda^3 x_{1T} x_{2L} \left( \frac{\sin (\pi + \gamma_T + \theta_{1T1L} - \theta_{0T1})}{\Delta \omega_T} + \frac{\sin (\theta_{1T1L} - \theta_{0T1})}{\Delta \omega_2} \right)
+ \Lambda x_{0T}^2 x_{1T}^2 x_{0L} x_{1L}^2 \sin (\psi_2 - \psi_1 + \theta_{0T1L} - \theta_{03L}) \left( \frac{1}{\Delta \omega_2} - \frac{1}{\Delta \omega_1} \right),
\]
(3.10c)

where
\[
x = \frac{y^2}{e^\phi}, \quad \tau = \frac{1}{2\nu} \left( 1 - e^{-2\nu t} \right),
\]
\[
\tan \beta, \delta = \frac{\nu \Delta \omega_{1,2}}{\Delta \omega_{1,2}^4 + \Delta \omega_{1,T}^2}, \quad \nu^2, \nu^* = \left( \frac{\Delta \omega_{1,2}}{\Delta \omega_{1,T}^2 + \Delta \omega_{1,2}^2} \right)^2 + \nu^2
\]
(3.11)

and \((\psi_1 - \psi_3)\) is assumed to be a slowly varying function of time.

4. Constants of motion and wave solution

According to a variety of initial states one can classify the solutions of the five-wave interactions into the following types (Inoue 1975).

4.1. Periodic solutions

We assume,
\[
\begin{align*}
\theta_{03L} &= \theta_{13L}, \\
\theta_{0T1L} &= \theta_{1T1L}
\end{align*}
\]
which gives
\[
\begin{align*}
x_{0L} &= x_{1L}, \\
x_{0T} &= x_{1T}.
\end{align*}
\]
(4.1)

The other constants of motion are:
\[
\begin{align*}
x_{2L} x_{0T} x_{0L} &= Q^2, \\
x_{0L} - x_{0T} &= (1 - \Lambda) x_{2L} + R
\end{align*}
\]
(4.2)
(4.3)

\((Q, R\) are arbitrary constants determined by initial conditions), and the \(\theta\)'s, \(\gamma\)'s and \(\Delta \omega\)'s satisfy the relations
\[
\begin{align*}
\psi_1 - \psi_3 + \theta_{03L} - \theta_{0T1T} = mn, \\
\theta_{03L} - \theta_{0L3L} + \beta = rm, \\
\theta_{0T1T} - \theta_{0T2L} + \delta = sn,
\end{align*}
\]
\[
\Delta \omega_L^2 \left( 1 + (m^2 + \Delta \omega_L^2) \right) = \Delta \omega_T^2 \left( 1 + (r^2 + \Delta \omega_T^2) (2 - \Lambda) \Delta \omega_L^2 \right)
\]
\[
\Delta \omega_L^2 \left( 1 + (m^2 + \Delta \omega_L^2) \right) = \Delta \omega_T^2 \left( 1 + (r^2 + \Delta \omega_T^2) (2 - \Lambda) \Delta \omega_L^2 \right)
\]
((\(m, r, s\) are integers including zero).

One can use the constants of motions (4.1)–(4.3) to eliminate \(x_{0L}, x_{1L}, x_{0T}, x_{1T}\) in favour of \(x_{2L}\) which leads to
\[
\left( \frac{dx_{2L}}{dt} \right)^2 + \tau (x_{2L}) = 0,
\]
(4.4)
Nonlinear coupling of two three-wave systems

**Figure 1.** The potential \( n(x^2) \) for \( \Lambda > 1 \) with two real (+ve and -ve) roots and two complex roots.

\[
\pi(x_{2L})
\]

where

\[
\pi(x_{2L}) = -\frac{1}{\Delta \omega_L^2} \left( (1-\Lambda)x_{2L}^4 + 2Rx_{2L}^2 + \frac{R^2}{1-\Lambda}x_{2L}^2 - \frac{4Q^2}{1-\Lambda} \right).
\]

\( \Lambda \) is the coupling factor as a measure of coupling strength between the two triplets.

When \( \Lambda > 1 \),

\[
\frac{dx_{2L}}{d\tau} = \frac{(\Lambda-1)^4}{\Delta \omega_L} \left( -x_{2L}^4 + \frac{2R}{\Lambda-1} x_{2L}^2 - \frac{R^2}{(\Lambda-1)^3} x_{2L} + \frac{4Q^2}{(\Lambda-1)^3} \right)^{\frac{1}{2}} \tag{4.5}
\]

and for \( \Lambda < 1 \),

\[
\frac{dx_{2L}}{d\tau} = \frac{(1-\Lambda)^4}{\Delta \omega_L} \left( x_{2L}^4 + \frac{2R}{1-\Lambda} x_{2L}^2 + \frac{R^2}{(1-\Lambda)^3} x_{2L}^2 - \frac{4Q^2}{(1-\Lambda)^3} \right)^{\frac{1}{2}} \tag{4.6}
\]

Solutions to (4.5)-(4.6) can be found in terms of Jacobian elliptic functions. However, the character of the solutions is greatly dependent on the nature of the roots, the ordering of the roots in magnitude, and the relative strength of the coupling factor.

**Case I: \( \Lambda > 1 \).**

The potential \( \pi(x_{2L}) \) has two (+ve, -ve) real roots \( \alpha_1, \alpha_2 \) (\( \alpha_1 > \alpha_2 \)) and two complex roots

\[
\frac{dx_{2L}}{d\tau} = \frac{(\Lambda-1)^4}{\Delta \omega_L} \left( (x_{2L} + \alpha_1)(x_{2L} - \alpha_2)(x_{2L}^2 - 2\alpha_2 x_{2L} + \alpha_4) \right)^{\frac{1}{2}} \tag{4.5} \quad (\alpha_2 < \alpha_4).
\]

The bounded solutions of the coupled mode equations correspond to \( x_{2L} \) oscillating between \( A \) and \( B \) of the curve shown in figure 1. No explosive solution occurs.
Case II: $\Lambda < 1$.

The potential $\pi(x_{2L})$ has two real roots and two complex roots,

$$\frac{dx_{2L}}{d\tau} = \frac{(1 - \Lambda)^{1/2}}{\Delta \omega_L} \left((x_{2L} - \alpha_1)(x_{2L} + \alpha_2)(x_{2L}^2 - 2\alpha_3 x_{2L} + \alpha_4)^{1/2}\right).$$

The explosive instability corresponds to $x_{2L}$ lying on the $D\infty$ portion of the curve shown in figure 2 ($x_{2L}$ being non-negative for all $\tau$).

Depending on the assignments of the roots to the values $a, b, c, \bar{c}$ ($a, b$ real, $a > x_{2L} > b; c, \bar{c}$ complex) the solution for $x_{2L}$ (Byrd & Friedman 1954, p. 135) is

$$x_{2L}(\tau) = \frac{(aB + bA) - (aB - bA) \text{cn}(\gamma(\tau - \tau_0), k)}{(B + A) + (A - B) \text{cn}(\gamma(\tau - \tau_0), k)}, \tag{4.7}$$

where

$$\gamma = \frac{(AB(\Lambda - 1))^{1/2}}{\Delta \omega_L}, \quad k^2 = \frac{(a-b)^2 - (A-B)^2}{4AB}$$

$$a^2 = -\frac{1}{3}(c - \bar{c})^2, \quad b_2 = \frac{1}{3}(c + \bar{c}), \quad A^2 = (a-b)^2 + a_1^2,$$

$$B^2 = (b-b_1)^2 + a_1^2, \quad g = (AB)^{-1}.$$

$\tau_0$ is a constant defined by

$$\tau_0 = \frac{\Delta \omega_L}{(AB(\Lambda - 1))^{3/2}} \text{cn}^{-1}\left(\frac{aB + bA - x_{2L}(0)(A + B)}{(a - B)x_{2L}(0) + aB - bA}, k\right). \tag{4.8}$$
The solutions for the other wave amplitudes can be obtained from (4.7) and the conservation laws. The solutions (4.7) are generally periodic with period of oscillation $4K/\gamma$ where $K$ is the complete elliptic integral of the first kind,

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{(1 - k^2 \sin^2 \theta)^{\frac{1}{2}}}.$$

All the amplitudes will become infinite when

$$(A + B) = (B - A) \text{cn} (\gamma (\tau - \tau_0), k),$$

and the explosion time $\tau_\infty$ is given by

$$\tau_\infty = \tau_0 + \left( \frac{\Delta \omega_1^2 + \nu^2}{AB(\Lambda - 1)} \right)^{\frac{1}{2}} \text{cn}^{-1} \left( \frac{A + B}{B - A}, k \right). \tag{4.9}$$

From figure 2, for the instability to occur we require $x_{2L}(0) > \alpha_1$ so that a threshold exists for the onset of the instability.

The growth rate is given by the reciprocal of the explosion time

$$\Gamma_{\text{growth}} = 1/\tau_\infty.$$

The effect of $\Delta \omega$ and $\nu$ is to introduce threshold values of the initial amplitudes for the explosion to occur and to increase the time of explosion.

### 4.2. Shock-like solution

Let the values of $\theta_k$ be so chosen that

$$\theta_{0LL} = \theta_{0LL} = \theta_{1LL} = \theta_1,$$

$$\theta_{0TT} = \theta_{0TT} = \theta_{1TT} = \theta_2.$$

Then

$$x_{0L} = x_{1L}, \quad x_{0T} = x_{1T}.$$

If, in particular, $x_{0L} = x_{1L} = \text{constant} = a$,

$$\psi_2 - \psi_1 + \theta_2 - \theta_1 = mn,$$

$$x_{2L} = A' x_{2L} - B' x_{2L}^2,$$

where

$$A' = \frac{2a \sin \eta_L - 2\Lambda \cosec \sin \eta_T \nu a}{\Delta \omega_L \Delta \omega_T \Delta \omega_L^2},$$

$$B' = \frac{2\Lambda \sin \eta_T \sin \eta_L \cosec \delta}{\Delta \omega_T \Delta \omega_L \Delta \omega_L^2}. \tag{4.10}$$

The solution for $x_{2L}$ is

$$x_{2L}(\tau) = \frac{A'/B'}{1 + C' \exp \left( -A'(\tau - \tau_0) \right)} \tag{4.11}$$

$C'$ being an arbitrary constant depending on initial conditions. The solution (4.11) represents a shock-like solution (see, for example, Bullough 1977). Similarly for the other wave amplitudes. The values of $x_{2L}$ vary between zero and $A'/B'$.

### 4.3. Soliton solution

Let

$$\theta_{0LL} = \theta_{1LL}, \quad \theta_{0TT} = \theta_{1TT},$$

which gives

$$x_{0L} = x_{1L}, \quad x_{0T} = x_{1T}.$$
The two other constants of motion are
\[ \dot{x}_{2L} = P'(x_{0L} - x_{0T}), \]
\[ x_{0L} x_{0T} = Q'^2 x_{2L}, \]
where \( \theta \) and \( \Delta \omega \) are chosen to satisfy the conditions
\[ \theta_{0LLL} - \theta_{0LLL} + \beta = n\pi, \]
\[ \theta_{0TTT} - \theta_{0TTT} + \delta = m\pi, \]
where \( \psi_{1} - \psi_{2} + \theta_{0LLL} - \theta_{0TTT} = \pi, \)
\[ \Delta \omega_{1} \left( 1 + \frac{\Delta \omega_{2} (\nu_{1} + \Delta \omega_{2})}{(\Delta \omega_{1}^{2} + \Delta \omega_{2}^{2})^{3}} \right) = \Delta \omega_{2} \left( 1 + \frac{\Delta \omega_{1} (\nu_{2} + \Delta \omega_{1})}{(\Delta \omega_{1}^{2} + \Delta \omega_{2}^{2})^{3}} \right), \] (4.13)
Using the constants of motion, one gets
\[ \dot{x}_{2L} = \frac{\nu}{\Delta \omega_{1}} x_{2L}(x_{2L}^{2} + 4P_{1}^{2}Q_{1}^{2}x_{2L})^{\frac{3}{2}}. \] (4.14)
The solution of (4.14) has the form
\[ x_{2L} = \frac{4P_{1}^{2}Q_{1}^{2}}{A_{1}(t - B_{1}/2A_{1})^{2} + D_{1}^{2}}, \] (4.15)
where
\[ A_{1} = \nu^{2} \left( k_{1} - \frac{2P_{1}^{2}Q_{1}^{2}}{\Delta \omega_{1}} \right)^{2} - 1, \]
\[ B_{1} = \nu^{2} \left( 1 + \frac{2k_{1}^{2}P_{1}^{2}Q_{1}^{2}}{\Delta \omega_{1}^{2}} - k_{1}^{2} \right), \]
\[ C_{1} = k_{1}^{2} - 1, \quad k_{1} = 1 + 4P_{1}^{2}Q_{1}^{2}/x_{2L}^{(0)}, \]
\[ D_{1} = C_{1} - \frac{B_{1}^{2}}{4A_{1}}. \] (4.16)
The initial amplitudes are so chosen that
\[ P_{1}^{2}Q_{1}^{2} \left( \frac{1}{\Delta \omega_{1}^{2}/x_{2L}^{(0)}} - \frac{4}{x_{2L}^{(0)}} \right) \left( 1 - \frac{1}{\nu^{2}} \right) < \frac{2}{x_{2L}^{(0)}} \left( 1 - \frac{1}{\nu^{2}} \right). \] (4.17)
The solution (4.17) represents a soliton, \( x_{2L} \) is limited to a maximum value \( 4P_{1}^{2}Q_{1}^{2}/D_{1}^{2} \) when \( t = B_{1}/2A_{1} \) and tends to zero for large times.

In fact \( x_{2L} \) also tends to zero if we let \( t \) assume large negative values. Similarly for the other wave amplitudes. The decrease of the amplitudes after the maximum corresponds to a collapse of the waves. This type of soliton solution was obtained in the case of three-wave interactions (Weiland & Wilhelmsson 1977; De et al. 1981).

5. Solutions for two coupled three-wave interactions with one triplet unstable

Consider the case where the longitudinal wave triplet (first) is unstable when viewed as an isolated three-wave interaction but the other triplet of two transverse and one longitudinal (second) is not.
From (4.5), (4.6) if $\Lambda < 1$ the solution is divergent. But if $\Lambda > 1$, which can be interpreted as the second triplet being stronger than the first, no divergent solution occurs. A significant conclusion is that even though one triplet is explosively unstable by itself, the presence of the second triplet (stronger) can stabilize the solutions whereas the converse does not happen. In the weakly turbulent case it can be shown that the wave system will be unstable if any explosive triplet exists (Coppi, Rosenbluth & Sudan 1969). This is not true for coherent waves. Karplyuk, Oraevskii & Pavlenko (1973) studied the coupling of two three-wave systems with two waves in common. They have shown that the presence of a second triplet can either increase, decrease or fully stabilize the usual decay instability. Also it is possible to generate a wave of higher frequency than the initial large-amplitude wave if the second triplet can occur. Walters & Lewak (1977) have shown that, in the case of coupling of two three-wave systems with two waves in common, the weaker triplet can be stabilized by the stronger one against explosive instabilities. Also under certain circumstances one of the common waves may act as a catalyst remaining fixed in amplitude, while the other waves oscillate or even grow exponentially.

6. Discussion

To illustrate the existence of five waves coupled as described in the theory one may consider the following example. In a warm isotropic plasma the synchronism condition (2.1) permits the interaction of two Langmuir and one ion-acoustic
wave with a system of two transverse electromagnetic and one Langmuir or ion-acoustic wave; and one of the Langmuir or ion-acoustic waves introduces coupling between the two subsystems (Boyd & Turner 1978). In this case one transverse wave may be taken as the incoming laser wave and the other as a reflected wave.

The coupling constants contain all the information necessary for studying how the efficiency of the wave coupling depends on the plasma parameters. The growth rate, explosion time and threshold value of the amplitude of the excited waves for such a system can be obtained by direct application of our theory; also the effects of dissipation and frequency mismatch on these parameters can be calculated.

In the absence of dissipation and frequency mismatch, Weiland & Wilhelmsson (1977) have shown stable five-wave interaction (bounded) resulting from the continuous interchange of energy between the interacting waves. In the presence of dissipation the amplitudes are damped out and the periodic oscillations ultimately lead to constant asymptotic values. The temporal behaviour of the amplitudes corresponding to the periodic solutions in the presence of dissipation and frequency mismatch are shown in figure 3. In this connexion it should be mentioned that the transformation used by Wilhelmsson & Pavlenko (1973) to eliminate different $\omega$'s from the coupled mode equations are not applicable in the presence of frequency mismatch ($\Delta \omega$'s) explicitly in the coupled mode equations. However, the Coffey method is suitable for $\Delta \omega \neq 0$ and has the distinct advantage of separating a given motion into a secular motion and a rapidly fluctuating motion of small amplitude. Though the three-wave and five-wave interaction processes are both described by coupled mode equations with quadratic non-linear terms, one of the significant differences between the behaviour of linked double three-wave interaction and the corresponding independent three-wave interaction is that in the coherent case, if one triplet be explosively unstable by itself, the presence of the second triplet can stabilize the solutions depending on the relative strength of the coupling factor; while in the incoherent case this is not true.

REFERENCES

Nonlinear coupling of two three-wave systems