CHAPTER III

LP – SASAKIAN MANIFOLDS
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LP-SASAKIAN MANIFOLDS

Introduction

In 1989 Matsumoto [60] introduced the notion of Lorentzian para-Sasakian manifolds. Then Mihai and Rosca [63] defined the same notion independently and they obtained several results in this manifold. LP-Sasakian manifolds have also been studied by Matsumoto and Mihai [61], Matsumoto, Mihai and Rosca [62], De and Shaikh [32], Ozgur [75] and many others.

Let $M$ be an $n$-dimensional Lorentzian para Sasakian \((LP)-\text{Sasakian}\) manifold with structure $\sum = (\phi, \xi, \eta, g)$, $\phi$ a \((1,1)\) tensor field, $\xi$ a contravariant vector field, $\eta$ a 1-form and $g$ a Lorentzian metric, then by definition, they satisfies

\begin{align*}
\eta(\xi) &= -1, \quad \phi^2 = I + \eta \otimes \xi \\
\phi \xi &= 0, \quad \eta \cdot \phi = 0, \quad \text{rank}(\phi) = n - 1 \\
\eta(X) &= g(\xi, X), \quad g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y) \\
(\nabla_X \eta)(Y) &= \Omega(Y, X), \quad \Omega(X, Y) = \Omega(Y, X) = g(\phi Y, X)
\end{align*}

(III.i.1)

(III.i.2)

(III.i.3)

(III.i.4)

(III.i.5)

for any vector fields $X$, $Y$ and $Z$ on $M$, where $I$ denotes the identity map on $T_pM$ (the tangent vector space at $p$ of $M$) and the symbol $\otimes$ is the tensor product. An $n$-dimensional Lorentzian manifold $(M, g)$ is said to be Lorentzian special para Sasakian \((LSP)-\text{Sasakian}\) if $M$ admits a timelike unit vector field $\xi$ with its associated 1-form $\eta$ satisfies

\begin{align*}
\Omega(X, Y) &= (\nabla_X \eta)(Y) = \epsilon\{g(X, Y) + \eta(X)\eta(Y)\}, \quad \epsilon^2 = 1.
\end{align*}

(III.i.6)

Of course, an \(LSP\)-Sasakian manifold is \(LP\)-Sasakian.

On the other hand, the eigenvalues of $\phi$ are -1, 0 and 1. And the multiplicity of 0 is 1 by (III.i.2). Let $K$ and $l$ be the multiplicities of -1 and 1 respectively. Then $tr\phi = l - K$. So, if $(tr\phi)^2 = (n - 1)^2$, then $l = 0$ or $K = 0$. In this case, we call our structure is a trivial \(LP\)-Sasakian structure.

In the first section we study some Lemmas in an \(LP\)-Sasakian manifolds. In the next section we study an \(LP\)-Sasakian manifolds under a D-homothetic deformation.
Section 3 and 4 are devoted to study of Significance of the associated scalars in an \(LP\)-Sasakian \(\eta\)-Einstein manifold and \(\eta\)-Einstein manifolds. In the next section we study quasi-conformally flat \(LP\)-Sasakian manifold. In section 6 we study \(LP\)-Sasakian manifolds satisfying \(R(\xi, Y) \cdot C^* = 0\). Section 7 and 8 are devoted to study of globally \(\phi\)-quasiconformally symmetric \(LP\)-Sasakian manifolds and 3-dimensional locally \(\phi\)-quasiconformally symmetric \(LP\)-Sasakian manifolds. In section 9 we give some examples of such manifolds. In the remaining sections of this chapter we study pseudosymmetric, weyl-pseudosymmetric and Ricci-pseudosymmetric \(LP\)-Sasakian manifolds.

**Preliminaries**

In an \(LP\)-Sasakian manifold \(M^n (\phi, \xi, \eta, g)\), the following relations hold [60]:

\[
\eta(R(X, Y)Z) = [g(Y, Z)\eta(X) - g(X, Z)\eta(Y)], \quad (\text{III.p.1})
\]

\[
S(X, \xi) = 2n\eta(X), \quad (\text{III.p.2})
\]

\[
R(X, Y)\xi = [\eta(Y)X - \eta(X)Y], \quad (\text{III.p.3})
\]

\[
R(\xi, X)Y = g(X, Y)\xi - \eta(Y)X, \quad (\text{III.p.4})
\]

\[
(\nabla_X\phi)(Y) = [g(X, Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X], \quad (\text{III.p.5})
\]

for all vector fields \(X, Y, Z\), where \(R, S\) denote respectively the curvature tensor and the Ricci tensor of the manifold. Also since the vector field \(\eta\) is closed in an \(LP\)-Sasakian manifold, we have ([60],[61])

\[
(\nabla_X\eta)Y = \Omega(X, Y), \quad (\text{III.p.6})
\]

\[
\Omega(X, \xi) = 0, \quad (\text{III.p.7})
\]

\[
\nabla_X\xi = \phi X, \quad (\text{III.p.8})
\]

for any vector field \(X\) and \(Y\).
In this section we shall state and prove some Lemmas which will be needed to prove the main results.

**Lemma III.1.1.** \( [3] \) In an LP-Sasakian manifold, the following relation holds

\[
g(R(\phi X, \phi Y)\phi Z, \phi W) = g(R(X, Y)Z, W) + g(X, W)\eta(Y)\eta(Z) - g(X, Z)\eta(W)\eta(Y) + g(Y, Z)\eta(X)\eta(W) - g(Y, W)\eta(X)\eta(Z). \tag{III.1.1}\]

**Lemma III.1.2.** Let \((M^{2n+1}, g)\) be an LP-Sasakian manifold. Then the Ricci operator \(Q\) commutes with \(\phi\).

Proof: From (III.1.1), it follows that

\[
\phi R(\phi X, \phi Y)\phi Z = R(X, Y)Z - [\eta(Z)Y - g(Y, Z)\xi]\eta(X) + [X\eta(Z) - g(X, Z)\xi]\eta(Y). \tag{III.1.2}\]

Let \(\{e_i, \phi e_i, \xi\}, i = 1, 2, \ldots, n\) be an orthonormal frame at any point of the manifold. Then putting \(Y = Z = e_i\) in (III.1.2) and taking summation over \(i\) and using \(\eta(e_i) = 0\), we get

\[
\sum_{i=1}^{n} \epsilon_i R(\phi X, e_i)e_i = \sum_{i=1}^{n} \epsilon_i R(X, e_i)e_i - n\eta(X)\xi, \tag{III.1.3}\]

where \(\epsilon_i = g(e_i, e_i)\).

Again setting \(Y = Z = \phi e_i\) in (III.1.2) and taking summation over \(i\) and using \(\eta, \phi = 0\), we get

\[
\sum_{i=1}^{n} \epsilon_i R(\phi X, e_i)e_i = \sum_{i=1}^{n} \epsilon_i R(X, \phi e_i)e_i - n\eta(X)\xi. \tag{III.1.4}\]

Adding (III.1.3) and (III.1.4) and using the definition of the Ricci tensor, we obtain

\[
\phi(Q\phi X - R(\phi X, \xi)\xi) = QX - R(X, \xi)\xi - 2n\eta(X)\xi.
\]

Using (III.p.3) and \(\phi\xi = 0\) in the above relation, we have

\[
\phi(Q\phi X) = QX - 2n\eta(X)\xi.
\]
Operating both sides by $\phi$ and using (III.i.1), symmetry of $Q$ and $\phi \xi = 0$ we get $\phi Q = Q \phi$. This proves the lemma.

**Proposition III.1.1.** In an $2n + 1$-dimensional $\eta$-Einstein LP-Sasakian manifold, the Ricci tensor $S$ is expressed as

$$S(X, Y) = \left[ \frac{r}{2n} - 1 \right] g(X, Y) - \left[ \frac{r}{2n} - 2n - 1 \right] \eta(X) \eta(Y). \quad (III.1.5)$$

## SECTION 2

**D-Homothetic deformations**

In this section we study $\eta$-Einstein LP-Sasakian manifolds, $\phi$- sectional curvature, locally $\phi$- Ricci symmetry and $\eta$- parallelity of the Ricci tensor of an odd dimensional LP-Sasakian manifold under a D- homothetic deformation.

In virtue of (III.p.6), the relation (21) reduces to

$$W(X, Y) = (1 - a) \left[ \eta(Y) \phi X + \eta(X) \phi Y \right] + (1 - \frac{1}{a}) g(\phi X, Y) \xi. \quad (III.2.1)$$

In view of (III.p.5), (III.p.6) and (III.p.8), the relation (III.2.1) yields

$$\nabla_z W(X, Y) = (1 - a) \left[ \{g(\phi Z, Y) \phi X \right.
\left. + g(X, Z) \eta(Y) \xi + 2 \eta(X) \eta(Y) Z + 4 \eta(X) \eta(Y) \eta(Z) \xi \right. 
\left. + g(\phi Z, X) \phi Y \right] (X, Y) Z) \xi 
\left. + \frac{a-1}{a} g(\phi X, Y) \phi Z. \right) \quad (III.2.2)$$

Using (III.2.1) and (III.2.2) into (22), we obtain by virtue of (III.p.3) and (III.p.6) that

$$\tilde{R}(X, Y) Z = R(X, Y) Z + (1 - a) [g(X, Z) \eta(Y) \xi - g(Y, Z) \eta(X) \xi ] + 2 \eta(Y) \eta(Z) X - 2 \eta(X) \eta(Z) Y + g(\phi Z, Y) \phi X - g(\phi Z, X) \phi Y + (1 - a)^2 [g(X, Z) Y - \eta(Y) \eta(Z) X] - \frac{(1 - a)^2}{a} [g(X, Z) \phi Y - g(\phi Z, Y) \phi X]. \quad (III.2.3)$$

Putting $Y = Z = \xi$ in (III.2.3) and using (III.i.1) we obtain

$$\tilde{R}(X, \xi) \xi = R(X, \xi) \xi + 2(1 - a) [-X + \eta(X) \xi] - (1 - a)^2 \phi^2 X. \quad (III.2.4)$$
Let \( \{e_i, \phi e_i, \xi\}, i = 1, 2, \ldots, n \) be an orthonormal frame at any point of the manifold. Then putting \( Y = Z = e_i \) in (III.2.3) and taking summation over \( i \) and using \( \eta(e_i) = 0 \), we get
\[
\sum_{i=1}^{n} e_i \bar{R}(X, e_i) e_i = \sum_{i=1}^{n} e_i R(X, e_i) e_i - (1 - a)n \eta(X) \xi, \tag{III.2.5}
\]
where \( e_i = g(e_i, e_i) \).

Again setting \( Y = Z = \phi e_i \) in (III.2.3) and taking summation over \( i \) and using \( \eta, \phi = 0 \), we get
\[
\sum_{i=1}^{n} e_i \bar{R}(X, \phi e_i) \phi e_i = \sum_{i=1}^{n} e_i R(X, \phi e_i) \phi e_i - (1 - a)n \eta(X) \xi. \tag{III.2.6}
\]

Adding (III.2.5) and (III.2.6) and using the definition of Ricci operator we have
\[
\bar{Q}X - \bar{R}(X, \xi) \xi = QX - R(X, \xi) \xi - 2(1 - a)n \eta(X) \xi. \tag{III.2.7}
\]

In view of (III.2.4) we get from (III.2.7)
\[
\bar{S}(X, Y) = S(X, Y) - [2(1 - a) + (1 - a)^2] g(X, Y) - [2(1 - a)(n - 1) + (1 - a)^2] \eta(X) \eta(Y), \tag{III.2.8}
\]
which implies that
\[
\bar{Q}X = QX - [2(1 - a) + (1 - a)^2] X - [2(1 - a)(n - 1) + (1 - a)^2] \eta(X) \xi. \tag{III.2.9}
\]

Operating \( \tilde{\phi} \) = \( \phi \) on both sides of (III.2.9) from the left we have
\[
\tilde{\phi} \bar{Q}X = \phi QX - [2(1 - a) + (1 - a)^2] \phi X. \tag{III.2.10}
\]

Again, putting \( \tilde{\phi}X = \phi X \) in (III.2.9) from the right we have
\[
\bar{Q} \tilde{\phi} X = Q \phi X - [2(1 - a) + (1 - a)^2] \phi X. \tag{III.2.11}
\]

Subtracting (III.2.10) and (III.2.11) we get
\[
(\tilde{\phi} \bar{Q} - \bar{Q} \tilde{\phi}) X = (\phi Q - Q \phi) X. \tag{III.2.12}
\]

Therefore using Lemma III.1.2 we can state the following:

**Theorem III.2.1.** Under a D-homothetic deformation, the expression \( \bar{Q} \tilde{\phi} = \tilde{\phi} \bar{Q} \) holds in an \((2n + 1)\)-dimensional LP-Sasakian manifold.
III.2.1 η-Einstein LP-Sasakian manifolds

Let $M(\phi, \xi, \eta, g)$ be a $(2n + 1)$-dimensional η-Einstein LP-Sasakian manifold which reduces to $M(\bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$ under a D-homothetic deformation. Then from (III.2.8) it follows by virtue of (III.1.5) that

$$\bar{S}(X, Y) = \bar{\lambda}\bar{g}(X, Y) + \bar{\mu}\bar{\eta}(X)\bar{\eta}(Y),$$  \hspace{1cm} (III.2.13)

where $\bar{\lambda}, \bar{\mu}$ are smooth functions given by

$$\bar{\lambda} = \left[ \frac{r}{2n} - (a - 2)^2 \right] \hspace{1cm} (III.2.14)$$

and

$$\bar{\mu} = \left[ \frac{r}{2n} - 4n + 2an - a^2 \right]. \hspace{1cm} (III.2.15)$$

In view of the relation (III.2.13) we state the following:

**Theorem III.2.1.1** Under a D-homothetic deformation, a $(2n + 1)$-dimensional η-Einstein LP-Sasakian manifold is invariant.

III.2.2 ϕ-sectional curvature of LP-Sasakian manifolds

In this section we consider the ϕ-sectional curvature on a $(2n + 1)$-dimensional LP-Sasakian manifold. From (III.2.3) it can be easily seen that

$$\bar{K}(\phi X) - K(\phi X) = -2(a - 1) \hspace{1cm} (III.2.16)$$

and hence we state the following:

**Theorem III.2.2.1.** Under a D-homothetic deformation, the ϕ-sectional curvature of a $(2n + 1)$-dimensional LP-Sasakian manifold is not an invariant.

If a $(2n + 1)$-dimensional LP-Sasakian manifold $M(\bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$ satisfies $R(X, Y)\xi = 0$ for all $X, Y$, then it can be easily seen that $\bar{K}(X, \phi X) = 0$ and hence from (III.2.16) it follows that

$$\bar{K}(X, \phi X) = -2(a - 1) \neq 0$$

where $X$ is a unit vector field orthogonal to $\xi$ and $K(X, \phi X)$ is the ϕ-sectional curvature. This implies that the ϕ-sectional curvature $\bar{K}(X, \phi X)$ is non-vanishing. Therefore we state the following:

**Theorem III.2.2.2.** There exists $(2n + 1)$-dimensional LP-Sasakian manifold with non-zero ϕ-sectional curvature.

III.2.3 Locally ϕ-Ricci symmetric LP-Sasakian manifolds

In this section we study locally ϕ-Ricci symmetry on an LP-Sasakian manifold.
Differentiating (III.2.9) covariantly with respect to $W$ we obtain
\[
(\nabla_W \bar{Q})(X) = (\nabla_W Q)(X) \\
- [2(1 - a)(n - 1) + (1 - a)^2] \eta(X) \nabla_W \xi.
\] (III.2.17)

Operating $\phi^2$ on both sides of (III.2.17) and taking $X$ as an orthonormal vector to $\xi$ we obtain
\[
\bar{\phi}^2(\nabla_W \bar{Q})(X) = \phi^2(\nabla_W Q)(X).
\] (III.2.18)

In view of the relation (III.2.18) we state the following:

**Theorem III.2.3.1.** Under a D-homothetic deformation a locally $\phi$-Ricci symmetry on an LP-Sasakian manifold is invariant.

### III.2.4 $\eta$-parallel Ricci tensor of an LP-Sasakian manifolds

Let us consider the $\eta$-parallelity of the Ricci tensor on an LP-Sasakian manifold. Differentiating (III.2.8) covariantly with respect to $W$ and using (III.p.6) we obtain
\[
(\nabla_W S)(X, Y) = (\nabla_W S)(X, Y) \\
- [2(1 - a)(n - 1) + (1 - a)^2] \\
\quad [g(\phi W, X) \eta(Y) + g(\phi W, Y) \eta(X)].
\] (III.2.19)

In (III.2.19) replacing $X$ by $\phi X$, $Y$ by $\phi Y$ and using (III.i.3) we get
\[
(\nabla_W \bar{S})(\phi X, \phi Y) = (\nabla_W S)(\phi X, \phi Y).
\] (III.2.20)

Hence we can state the following:

**Theorem III.2.4.1.** Under a D-homothetic deformation $\eta$-parallelity of the Ricci tensor on an LP-Sasakian manifold is invariant.

### SECTION 3

**Significance of the associated scalars in an LP-Sasakian $\eta$-Einstein manifold**

We can express (13) as follows:
\[
S(X, \xi) = (a - b)g(X, \xi).
\] (III.3.1)
From (III.3.1), we conclude that \((a - b)\) is an eigen value of the Ricci operator \(Q\) defined by \(S(X, Y) = g(QX, Y)\) and \(\xi\) is an eigen vector corresponding to this eigen value.

Let \(V\) be any other vector orthogonal to \(\xi\) so that

\[
\eta(V) = 0. \tag{III.3.2}
\]

From (13), we obtain

\[
S(X, V) = ag(X, V) + b\eta(X)\eta(V) \tag{III.3.3}
\]

Hence in virtue of (III.3.2), we get

\[
S(X, V) = ag(X, V). \tag{III.3.4}
\]

From (III.3.4), we see that \(a\) is an eigen value of the Ricci operator \(Q\) and \(V\) is an eigen vector corresponding to this eigen value. If the manifold under consideration is \(n\)-dimensional and \(V\) is any vector orthogonal to \(\xi\), it follows from a known result in linear algebra [89] that the eigen value \(a\) is of multiplicity \((n - 1)\). Hence the multiplicity of the eigen value \((a - b)\) must be 1. Therefore we can state the following:

**Theorem III.3.1.** In an \(LP\)-Sasakian \(\eta\)-Einstein manifold of dimension \(n\), the Ricci operator \(Q\) has only two distinct eigen values \((a - b)\) and \(a\) of which the former is simple and the later is of multiplicity \((n - 1)\).

**SECTION 4**

\(\eta\)-Einstein manifolds

This section deals with \(\eta\)-Einstein \(LP\)-Sasakian manifolds.

From (13) we have

\[
S(\phi X, Y) = ag(\phi X, Y), \tag{III.4.1}
\]

\[
S(\xi, \xi) = -a + b. \tag{III.4.2}
\]

**Theorem III.4.1.** The Ricci curvature of an \(\eta\)-Einstein \(LP\)-Sasakian manifold in the direction of \(\xi\) is equal to \(-(n - 1)\).

Proof: Substituting \(\xi\) for \(X\) in (III.p.2) we have the theorem.
Theorem III.4.2. The functions $a$ and $b$ of the defining equation (13) are constants, provided $\text{tr} \phi = 0$.

Proof: Equation (III.4.2) and (III.p.2) imply

$$-a + b = 1 - n.$$  \hspace{1cm} (III.4.3)

So we need only to show that $a$ is constant. Taking a frame field we get from (13),

$$\sum_{i=1}^{n} \epsilon_i S(e_i, e_i) = a \sum_{i=1}^{n} \epsilon_i g(e_i, e_i) + b \sum_{i=1}^{n} \epsilon_i \eta(e_i) \eta(e_i),$$

which gives

$$r = na - b,$$

where $r$ is the scalar curvature of the manifold. Now differentiating the above equation we have

$$dr(X) = n da(X) - db(X) = (n + 1) da(X).$$  \hspace{1cm} (III.4.4)

Again from (13) we have

$$QX = aX + b \eta(X) \xi.$$  \hspace{1cm} (III.4.5)

Differentiating (III.4.5) along $Y$, we get

$$(\nabla_Y Q)X = (Ya)X + (Yb) \eta(X) \xi + bg(\phi X, Y) \xi + b \eta(X) \phi Y.$$  \hspace{1cm} (III.4.6)

Contracting the above equation with respect to $Y$, we get

$$(\text{div} Q)X = Xa + (\xi b) \eta(X) + b \eta(X) \text{tr} \phi.$$  \hspace{1cm} (III.4.7)

Using the identity [78] $$(\text{div} Q)X = \frac{dr(X)}{2},$$ (III.4.4) and $\text{tr} \phi = 0$, we get

$$(n - 1) da(X) = 2db(\xi) \eta(X).$$  \hspace{1cm} (III.4.8)

Putting $X = \xi$ in it, we get

$$n - 1) da(\xi) = -2db(\xi) = 2da(\xi),$$

which gives $da(\xi) = 0$ and hence $db(\xi) = 0$. Consequently (III.4.8) yields $da(X) = 0$. We now obtain a necessary and sufficient condition for an $LP$-Sasakian manifold to
be an $\eta$-Einstein manifold. In an $LP$-Sasakian manifold, the following relation holds

\[ R(X,Y)\phi Z = \phi R(X,Y)Z + g(Y,Z)\phi X \]
\[ -g(X,Z)\phi Y + g(X,\phi Z)Y - g(Y,\phi Z)X \]
\[ +2[g(X,\phi Z)\eta(Y) - g(Y,\phi Z)\eta(X)]\xi \]
\[ +2[\eta(Y)\phi X - \eta(X)\phi Y]\eta(Z). \]  

(III.4.9)

Taking a frame field and contracting (III.4.9) with respect to $X$, we get

\[ S(Y,\phi Z) = (C^1_1\overline{R})(Y,Z) \]
\[ +[g(Y,Z) + 2\eta(Y)\eta(Z)]\text{tr}\phi - (n+1)g(Y,\phi Z), \]

(III.4.10)

where $C^1_1$ denotes contraction at the first slot and $\overline{R} = \phi R$.

Since $(C^1_1\overline{R})(Y,Z) = (C^1_1\overline{R})(Z,Y)$, from the above it is obvious that

\[ S(Y,\phi Z) = S(Z,\phi Y). \]

(III.4.11)

**Theorem III.4.3.** In order that an $LP$-Sasakian manifold to be an $\eta$-Einstein manifold it is necessary and sufficient that the symmetric tensor $(C^1_1\overline{R})$ and $\Omega$ should be linearly dependent, provided tr$\phi = 0$.

Proof: At first we assume that $(C^1_1\overline{R})$ and $\Omega$ are linearly dependent. Then from (III.4.10) we have

\[ S(Y,\phi Z) = \lambda g(Y,\phi Z), \]

where $\lambda$ is a scalar. Now using Theorem III.4.2 we can easily seen that the manifold is a $\eta$-Einstein manifold.

Conversely, let the manifold is an $\eta$-Einstein manifold. Then we have

\[ S(Y,Z) = ag(Y,Z) + b\eta(Y)\eta(Z). \]

Replacing $Y$ by $\phi Y$ in the above equation we obtain

\[ S(Y,\phi Z) = ag(Y,\phi Z). \]

(III.4.12)

Using (III.4.12) in (III.4.10) we see that $(C^1_1\overline{R})$ and $\Omega$ are linearly dependent.
SECTION 5
Quasi-conformally flat $LP$-Sasakian manifolds

When the quasi-conformal curvature tensor vanishes identically on the Lorentzian manifold, then we find from (10)

$$a \tilde{R}(X, Y, Z, W) = b \{S(X, Z)g(Y, W) - S(Y, Z)g(X, W) + S(Y, W)g(X, Z) - S(X, W)g(Y, Z)\} + \frac{r}{n} \left(\frac{a}{n - 1} + 2b\right) \{g(Y, Z)g(X, W) - g(X, Z)g(Y, W)\},$$

which implies that

$$\{a + (n - 2)b\} \{S(Y, Z) - \frac{r}{n} g(Y, Z)\} = 0. \quad \text{(III.5.2)}$$

Thus we obtain $a + (n - 2)b = 0$ or $S(Y, Z) = \frac{r}{n} g(Y, Z)$. If $a + (n - 2)b = 0$, then the conformal curvature tensor vanishes identically. It is known that a conformally flat $LP$-Sasakian manifold is of constant curvature [25]. When $M$ is an Einstein $LP$-Sasakian manifold, we get $r = n(n - 1)$. It is easy to see from (III.5.1) that $M$ is of constant curvature 1. Conversely, if $M$ is of constant curvature, then the quasi-conformal curvature tensor vanishes. Hence we have

**Theorem III.5.1.** Let $M^n$ ($n > 3$) be an $LP$-Sasakian manifold. Then $M$ is quasi-conformally flat if and only if it is of constant curvature.

From [71], we have

**Theorem III.5.2.** Let $M^n$ ($n > 3$) be a complete simply connected $LP$-Sasakian manifold. If $M$ is quasi-conformally flat, then $M$ is isometric to the Lorentz sphere $S^n_1(1)$.

SECTION 6

$LP$-Sasakian manifolds satisfying $R(\xi, Y) \cdot C^* = 0$

In this section we consider an $LP$-Sasakian manifold $M^n$ ($n > 3$) satisfying the condition

$$(R(\xi, Y) \cdot C^*)(U, V)W = 0, \quad \text{(III.6.1)}$$
which yields from (III.6.4) that
\[ g(C^*(U,V)W,Y)\xi - \eta(C^*(U,V)W)Y - g(Y,U)C^*(\xi,V)W + \eta(U)C^*(Y,V)W - g(Y,V)C^*(U,\xi)W \]
\[ + 2S(U,W) - g(U,W) + (n-1)\eta(U)\eta(W) \] 
\[ \{ g(V,Y) - \eta(V)\eta(W) \} S(U,Y) + \{ g(U,W) - \eta(U)\eta(W) \} S(V,Y) = 0. \]

Operating \( \eta \) to the above equation and using of (10), (III.6.3) to (III.6.4) we obtain
\[ g(C^*(U,V)W,Y) + bg(U,Y)\{ S(V,W) + (n-1)\eta(V)\eta(W) \} \]
\[ - bg(Y,V)\{ S(U,W) + (n-1)\eta(U)\eta(W) \} \]
\[ + b\{ S(V,Y)\eta(U) - S(Y,U)\eta(V) \} \eta(W) \]
\[ - \{ a + (n-1)b - \frac{a}{n-1} + 2b \} \]
\[ \{ g(V,W)g(Y,U) - g(U,W)g(V,Y) \} = 0. \]

Putting \( Y = U = e_i \) in the above equation and taking summation over \( i \), we get
\[ (a-b)S(V,W) - \{ (n-1)a + (n-1)^2b - br \} g(V,W) \]
\[ + b\{ r - n(n-1) \} \eta(V)\eta(W) = 0, \] (III.6.3)

moreover, we find \( \{ a + (n-2)b \} \{ r - n(n-1) \} = 0. \)

We can consider the two cases. At first, in the case of \( r = n(n-1) \), we have form (III.6.3)
\[ (a-b)\{ S(V,W) - (n-1)g(V,W) \} = 0. \]

If \( a \neq b \), then \( S(V,W) = (n-1)g(V,W) \). Therefore it is clear from (III.6.2) that the quasi-conformal curvature tensor vanishes, namely, \( M \) is of constant curvature 1 from Theorem III.5.1. Also, if \( a = b(\neq 0) \), then we get form (10) and (III.6.2)
\[ g(R(U,V)W,Y) + 2S(U,W) - ng(V,W) + (n-1)\eta(V)\eta(W) \] 
\[ - 2\{ S(U,W) - ng(U,W) + (n-1)\eta(U)\eta(W) \} \eta(V) \eta(W) \]
\[ + \{ g(V,W) - \eta(V)\eta(W) \} S(U,Y) - \{ g(U,W) - \eta(U)\eta(W) \} S(V,Y) = 0. \]

If we put \( W = \xi \), then we have \( \eta(V)S(U,Y) - \eta(U)S(V,Y) = 0. \) Furthermore, putting \( U = \xi \), we get \( S(V,Y) = -(n-1)\eta(V)\eta(Y), \) that is \( r = n - 1. \) This is the contradiction. Thus \( a \neq b \) holds.

Secondly, in the case of \( a + (n-2)b = 0, \) equation (III.6.2) is rewritten as follows:
\[ (n-2)g(R(U,V)W,Y) \]
\[ - 2\{ S(U,W) - ng(U,W) + (n-1)\eta(U)\eta(W) \} \eta(V) \]
\[ + \{ g(U,W) - \eta(U)\eta(W) \} S(U,Y) \]
\[ + \{ g(V,W) - \eta(V)\eta(W) \} S(V,Y) = 0. \] (III.6.4)
We put \( U = W = \xi \). Then we find \( S(V,Y) = -(n-1)\eta(V)\eta(Y) \), which yields form (III.6.4) that

\[
R(U,V)W = \frac{1}{4}(c+3)\{g(V,W)U-g(U,W)V\} + \frac{1}{4}(c-1)\{\eta(V)\eta(W)U-\eta(U)\eta(W)V\} + g(V,W)\eta(U)\xi - g(U,W)\eta(V)\xi,
\]

where \( c = -\frac{3n-2}{n-2} \). Hence we have

**Theorem III.6.1.** Let \( M^n(n > 3) \) be an LP-Sasakian manifold satisfying \( R(\xi,Y)\cdot C^* = 0 \) for any \( Y \).

1. If \( a + (n-2)b \neq 0 \), then \( M \) is of constant curvature 1.
2. If \( a + (n-2)b = 0 \), then \( M \) is a space satisfying (III.6.5).

**Corollary III.6.1.** Let \( M^n(n > 3) \) be an LP-Sasakian manifold. If \( M \) is a quasi-conformally semi-symmetric, then

1. when \( a + (n-2)b \neq 0 \), then \( M \) is of constant curvature 1.
2. when \( a + (n-2)b = 0 \), then \( M \) is a space satisfying (III.6.5).

From [71], we have

**Theorem III.6.2.** Let \( M^n(n > 3) \) be a complete simply connected LP-Sasakian manifold satisfying \( R(\xi,Y)\cdot C^* = 0 \) for any \( Y \). If \( a + (n-2)b \neq 0 \), then \( M \) is isometric to the Lorentz sphere \( S^n_1(1) \).

**Corollary III.6.2.** Let \( M^n(n > 3) \) be a complete simply connected LP-Sasakian manifold. If \( M \) is a quasi-conformally semi-symmetric and \( a + (n-2)b \neq 0 \), then \( M \) is isometric to the Lorentz sphere \( S^n_1(1) \).

**SECTION 7**

**Globally \( \phi \)-quasiconformally symmetric LP-Sasakian manifolds**

Let us suppose that \( M \) is a globally \( \phi \)-quasiconformally symmetric LP-Sasakian manifold. Then by definition

\[
\phi^2 (\nabla_W C^*) (X,Y) Z = 0.
\]

Using (III.i.1) we have

\[
(\nabla_W C^*) (X,Y) Z + \eta ((\nabla_W C^*) (X,Y) Z) \xi = 0.
\]
From (11) it follows from (III.p.3) and (III.p.5) that
\[
\begin{align*}
a \{g((\nabla WR)(X,Y)Z,U) + \eta(U)\eta((\nabla WR)(X,Y)Z)} \\
b \{g(X,U) + \eta(X)\eta(U)\}\end{align*}
\]
we get
\[
(\nabla WS)(X,U) = \eta(X)\{S(U,\phi W) - (n-1)g(U,\phi W)} \\
+ \eta(U)\{S(X,\phi W) - (n-1)g(X,\phi W)} \\
+ \frac{dr(W)}{n-1} \{g(X,U) + \eta(X)\eta(U)}.
\]

Moreover, putting \(Z = \xi\), in (III.7.1) and using of (III.p.3), (III.p.4) and (III.p.5), we obtain
\[
\begin{align*}
a \{g(Y,\phi W)g(X,U) - g(X,\phi W)g(Y,U) - g(R(X,Y)\phi W, U)} \\
b \{\eta(Y)(\nabla WS)(X,U) - \eta(X)(\nabla WS)(Y,U)} \\
- g(X,U)\{S(Y,\phi W) - (n-1)g(Y,\phi W)} \\
+ g(Y,U)\{S(X,\phi W) - (n-1)g(X,\phi W)} \\
- \frac{dr(W)}{n-1} \{\eta(Y)g(X,U) \\
- \eta(X)g(Y,U)} \} = 0.
\]

Putting \(Z = \xi\), in (III.7.1) and using of (III.p.3), (III.p.4) and (III.p.5), we obtain
\[
\begin{align*}
a \{g(Y,\phi W)g(X,U) - g(X,\phi W)g(Y,U) - g(R(X,Y)\phi W, U)} \\
b \{\eta(Y)(\nabla WS)(X,U) - \eta(X)(\nabla WS)(Y,U)} \\
- g(X,U)\{S(Y,\phi W) - (n-1)g(Y,\phi W)} \\
+ g(Y,U)\{S(X,\phi W) - (n-1)g(X,\phi W)} \\
- \frac{dr(W)}{n-1} \{\eta(Y)g(X,U) \\
- \eta(X)g(Y,U)} \} = 0.
\]

Moreover, putting \(X = U = \varepsilon_i\) in (III.7.2) and taking summation over \(i\), we obtain
\[
\{a + (n-2)b\} \{S(Y,\phi W) - (n-1)g(Y,\phi W) + \frac{dr(W)}{n} \eta(Y)} = 0.
\]

Thus if \(a + (n-2)b \neq 0\), then we find
\[
S(Y,\phi W) = (n-1)g(Y,\phi W) - \frac{dr(W)}{n} \eta(Y). \quad \text{Setting } Y = \xi, \text{ we have } dr(W) = 0, \text{ that is, the scalar curvature is constant. It is easy to see form (III.p.2) that } S(Y,W) = (n-1)g(Y,W), \text{ that is, } M \text{ is an Einstein. Since (III.7.2), we find that } M \text{ is of constant curvature 1. Next, when } a + (n-2)b = 0, \text{ it is clear from (III.7.2) that }\]
\[
\begin{align*}(n-2)\{g(R(X,Y)\phi W, U) - g(Y,\phi W)g(X,U) + g(X,\phi W)g(Y,U)} \\
+ \eta(Y)(\nabla WS)(X,U) - g(X,U)\{S(Y,\phi W) - (n-1)g(Y,\phi W)} \\
- \eta(X)(\nabla WS)(Y,U) + g(Y,U)\{S(X,\phi W) - (n-1)g(X,\phi W)} \\
- \frac{dr(W)}{n-1} \{\eta(Y)g(X,U) - \eta(X)g(Y,U)} \} = 0.
\]

Setting \(Y = \xi\) in the above equation, we get
\[
(\nabla WS)(X,U) = \eta(X)\{S(U,\phi W) - (n-1)g(U,\phi W)} \\
+ \eta(U)\{S(X,\phi W) - (n-1)g(X,\phi W)} \\
+ \frac{dr(W)}{n-1} \{g(X,U) + \eta(X)\eta(U)}.
\]
Therefore, we obtain
\[ R(X,Y)Z = g(Y,Z)X - g(X,Z)Y \]
\[ + \frac{1}{n-2} \{ S(Y,Z) - (n-1)g(Y,Z) \} \phi^2 X \]
\[ - \frac{1}{n-2} \{ S(X,Z) - (n-1)g(X,Z) \} \phi^2 Y. \]  

Hence we have

**Theorem III.7.1.** Let \( M^n \) \( (n>3) \) be a globally \( \phi \)-quasiconformally symmetric \( LP \)-Sasakian manifold.

1. If \( a + (n-2)b \neq 0 \), then \( M \) is of constant curvature \( 1 \).
2. If \( a + (n-2)b = 0 \), then \( M \) is a space satisfying (III.7.3).

From [71], we have

**Theorem III.7.2.** Let \( M^n(n>3) \) be a complete simply connected \( LP \)-Sasakian manifold. If \( M \) is globally \( \phi \)-quasiconformally symmetric and \( a + (n-2)b \neq 0 \), then \( M \) is isometric to the Lorentz sphere \( S^n_1(1) \).

Moreover, by virtue of (III.7.1) and Theorem III.7.1., we find
\[ a[(\nabla W)(X,Y)Z + \eta((\nabla W)(X,Y)Z)\xi] = 0, \]
which implies that \( \phi^2((\nabla W)(X,Y)Z) = 0 \) if \( a \neq 0 \). Hence we can state:

**Theorem III.7.3.** A globally \( \phi \)-quasiconformally symmetric \( LP \)-Sasakian manifold is globally \( \phi \)-symmetric if \( a \neq 0 \).

**SECTION 8**

3-dimensional locally \( \phi \)-quasiconformally symmetric \( LP \)-Sasakian manifolds

Let us consider a 3-dimensional \( LP \)-Sasakian manifold. It is known that the conformal curvature tensor vanishes idetically in the 3-dimensional Riemannian manifold. Thus we find
\[ R(X,Y)Z = g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y \]
\[ - \frac{r}{2} [g(Y,Z)X - g(X,Z)Y], \]  
where \( Q \) is the Ricci operator, that is, \( g(QX,Y) = S(X,Y) \) and \( r \) is the scalar curvature of the manifold.

Putting \( Z = \xi \) in (III.8.1) and using (III.p.3) we have
\[ \eta(Y)QX - \eta(X)QY = (\frac{r}{2} - 1)[\eta(Y)X - \eta(X)Y]. \]  

(III.8.2)
Putting \( Y = \xi \) in (III.8.2) and using (III.i.1) and (III.p.2), we get
\[
QX = \frac{1}{2}[(r-2)X + (r-6)\eta(X)\xi],
\]
that is,
\[
S(X, Y) = \frac{1}{2}[(r-2)g(X, Y) + (r-6)\eta(X)\eta(Y)].
\]
Using (III.8.3) in (III.8.1), we get

**Lemma III.8.1.** Let
\[
\text{Thus we have}
\]
Putting (III.8.3), (III.8.4) and (III.8.5) into (10) we have
\[
C^*(X, Y)Z = (a + b)(r - 6)[\frac{1}{2}g(Y, Z)X - g(X, Z)Y]
+ \frac{1}{2}g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi
+ \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y].
\]
Thus we have

**Lemma III.8.1.** Let \( M \) be a 3-dimensional \( LP \)-Sasakian manifold.

*If \( a + b = 0 \) or \( r = 6 \), then the quasi-conformal curvature tensor vanishes identically.*

Next, we assume that \( a + b \neq 0 \) or \( r \neq 6 \). Taking the covariant differentiation of (III.8.6), we get
\[
(\nabla^W C^*)(X, Y)Z = \left. \frac{dr(W)}{3}(a + b)\right\{g(Y, Z)X - g(X, Z)Y\}
+ \frac{dr(W)}{2}(a + b)\left\{g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi\right\}
- g(X, Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y\}
+ \frac{1}{2}(r - 6)(a + b)[\{g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\}X
+ \{g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\}Y
\]
Operating \( \phi^2 \) to the above equation, then we find
\[
\phi^2((\nabla^W C^*)(X, Y)Z) = \left. \frac{dr(W)}{3}(a + b)\right\{g(Y, Z)\phi^2 X - g(X, Z)\phi^2 Y\}
+ \frac{dr(W)}{2}(a + b)\eta(Z)\left\{\eta(Y)\phi^2 X - \eta(X)\phi^2 Y\right\}
+ \frac{1}{2}(r - 6)(a + b)[\{g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\}\phi^2 W
+ \{g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\}Y
- \{g(Y, Z)\phi^2 W + \eta(X)g(Z, \phi W)\}X
- \{g(Y, Z)\phi^2 W + \eta(X)g(Z, \phi W)\}Y].
\]
If the vector fields $X$, $Y$ and $Z$ are horizontal, then the above equation is rewritten as follows:

$$
\phi^2((\nabla_W C^*) (X,Y)Z) = \frac{dr(W)}{3} (a+b) \{ g(Y,Z)X - g(X,Z)Y \}.
$$

Hence we conclude the following theorem:

**Theorem III.8.1.** A 3-dimensional $LP$-Sasakian manifold is locally $\phi$-quasiconformally symmetric if and only if the scalar curvature $r$ is constant if $a+b \neq 0$ and $r \neq 6$.

**SECTION 9**

**Examples**

**Example III.9.1:** [62] Let $\mathbb{R}^5$ be the 5-dimensional real number space with a coordinate system $(x,y,z,t,s)$. Denoting

$$
\eta = ds - ydx - tdz, \quad \xi = \frac{\partial}{\partial s}, \quad g = \eta \otimes \eta - (dx)^2 - (dy)^2 - (dz)^2 - (dt)^2
$$

and

$$
\phi(\frac{\partial}{\partial x}) = -\frac{\partial}{\partial x} - y\frac{\partial}{\partial s}, \quad \phi(\frac{\partial}{\partial y}) = -\frac{\partial}{\partial y},
$$

$$
\phi(\frac{\partial}{\partial z}) = -\frac{\partial}{\partial z} - t\frac{\partial}{\partial s}, \quad \phi(\frac{\partial}{\partial t}) = -\frac{\partial}{\partial t}, \quad \phi(\frac{\partial}{\partial s}) = 0,
$$

the structure $(\phi, \xi, \eta, g)$ becomes an $LP$-Sasakian structure on $\mathbb{R}^5$. The metric tensor $g$ can be expressed by the matrix

$$
g = \begin{pmatrix}
1 + y^2 & 0 & ty & 0 & -y \\
0 & -1 & 0 & 0 & 0 \\
ty & 0 & -1 + t^2 & 0 & -t \\
0 & 0 & 0 & -1 & 0 \\
-y & 0 & -t & 0 & 1
\end{pmatrix}.
$$

**Example III.9.2:** Let $\mathbb{R}^4$ be the 4-dimensional real number space with a coordinate system $(x,y,z,t)$. In $\mathbb{R}^4$ we define

$$
\eta = dt - ydz - dx, \quad \xi = \frac{\partial}{\partial t},
$$

$$
g = e^{2t}(dx)^2 + e^{2t}(dy)^2 + (e^{2t} + y^2)(dz)^2 + ydz \otimes dx + ydx \otimes dz - ydz \otimes dt - ydt \otimes dz - \eta \otimes \eta,
$$
and
\[ \phi\left( \frac{\partial}{\partial x} \right) = \frac{\partial}{\partial x} + \frac{\partial}{\partial t}, \quad \phi\left( \frac{\partial}{\partial y} \right) = \frac{\partial}{\partial y}, \quad \phi\left( \frac{\partial}{\partial z} \right) = \frac{\partial}{\partial z}, \quad \phi\left( \frac{\partial}{\partial t} \right) = 0. \]

Then it can be seen that the structure \((\phi, \xi, \eta, g)\) becomes an \(LP\)-Sasakian structure on \(\mathbb{R}^4\). The metric \(g\) can be expressed by
\[
g = \begin{pmatrix}
e^{2t} - 1 & 0 & 0 & 1 \\
0 & e^{2t} & 0 & 0 \\
0 & 0 & e^{2t} & 0 \\
1 & 0 & 0 & -1
\end{pmatrix}.
\]

**Example III.9.3:** [25] A conformally flat \(LP\)-Sasakian manifold is an \(\eta\)-Einstein manifold.

**Example III.9.4:** [75] A \(\phi\)-conformally flat \(LP\)-Sasakian manifold is an \(\eta\)-Einstein manifold.

**Example III.9.5:** Let \((M^{n-1}, \tilde{g})\) be a hypersurface of \((M^n, g)\). If \(A\) is the \((1,1)\) tensor corresponding to the normal valued second fundamental tensor \(H\), then we have ([22], p.41),
\[
\tilde{g}(A_\xi(X), Y) = g(H(X, Y), \xi) \quad \text{(III.9.1)}
\]
where \(\xi\) is the unit normal vector field and \(X, Y\) are tangent vector fields.

Let \(H_\xi\) be the symmetric \((0,2)\) tensor associated with \(A_\xi\) in the hypersurface defined by
\[
\tilde{g}(A_\xi(X), Y) = (H_\xi(X, Y)). \quad \text{(III.9.2)}
\]

A hypersurface of a Riemannian manifold \((M^n, g)\) is called quasi-umbilical ([22], p.147) if its second fundamental tensor has the form
\[
H_\xi(X, Y) = \alpha g(X, Y) + \beta \omega(X) \omega(Y) \quad \text{(III.9.3)}
\]
where \(\omega\) is a 1-form, the vector field corresponding to the 1-form \(\omega\) is a unit vector field, and \(\alpha, \beta\) are scalars. If \(\alpha = 0\) (respectively \(\beta = 0\) or \(\alpha = \beta = 0\)) holds, then it is called cylindrical (respectively umbilical or geodesic).

Now from (III.9.1), (III.9.2) and (III.9.3) we obtain
\[
g(H(X, Y), \xi) = \alpha g(X, Y)g(\xi, \xi) + \beta \omega(X) \omega(Y)g(\xi, \xi)
\]
which implies that
\[
H(X, Y) = \alpha g(X, Y)\xi + \beta \omega(X) \omega(Y)\xi, \quad \text{(III.9.4)}
\]
since $\xi$ is the only unit normal vector field.
We have the following equation of Gauss ([22], p.45) for any vector fields $X, Y, Z, W$
tangent to the hypersurface
\[
g(R(X, Y)Z, W) = g(\tilde{R}(X, Y)Z, W) - g(H(X, W), H(Y, Z)) + g(H(Y, W), H(X, Z)),
\]
(III.9.5)
where $\tilde{R}$ is the curvature tensor of the hypersurface.
Let us assume that the hypersurface is quasi-umbilical. Then from (III.9.4) and
(III.9.5) it follows that
\[
g(R(X, Y)Z, W) = \tilde{g}(\tilde{R}(X, Y)Z, W) + \alpha^2[g(Y, W)g(X, Z)
\]
\[\quad - g(X, W)g(Y, Z)] + \alpha \beta [g(Y, W)\omega(X)\omega(Z)
\]
\[\quad + g(X, Z)\omega(Y)\omega(W) - g(X, W)\omega(Y)\omega(Z)
\]
\[\quad - g(Y, Z)\omega(X)\omega(W)].
\]
(III.9.6)
We know that every $LP$-Sasakian space form is of constant curvature 1 [25]. Hence
we have
\[
R(X, Y)Z = g(Y, Z)X - g(X, Z)Y
\]
which implies that
\[
g(R(X, Y)Z, W) = g(Y, Z)g(X, W) - g(X, Z)g(Y, W).
\]
(III.9.7)
Using (III.9.7) in (III.9.6) we have
\[
\tilde{g}(\tilde{R}(X, Y)Z, W) = (\alpha^2 - 1)[g(X, W)g(Y, Z) - g(X, Z)g(Y, W)]
\]
\[\quad - \alpha \beta [g(Y, W)\omega(X)\omega(Z) + g(X, Z)\omega(Y)\omega(W)
\]
\[\quad - g(X, W)\omega(Y)\omega(Z) - g(Y, Z)\omega(X)\omega(W)].
\]
(III.9.8)
Let $\{e_i\}, i = 1, 2, \ldots, n$ be an orthonormal frame at any point of the manifold. Then
putting $X = W = \{e_i\}$ in (III.9.8) and taking summation over $i$ , we get
\[
\sum_{i=1}^{n} e_i \tilde{g}(\tilde{R}(e_i, Y)Z, e_i) = \alpha^2(n - 1)\sum_{i=1}^{n} e_i [g(e_i, e_i)g(Y, Z) - g(e_i, Z)g(Y, e_i)]
\]
\[\quad - \alpha \beta \sum_{i=1}^{n} e_i [g(Y, e_i)\omega(e_i)\omega(Z) + g(e_i, Z)\omega(Y)\omega(e_i)
\]
\[\quad - g(e_i, e_i)\omega(Y)\omega(Z) - g(Y, Z)\omega(e_i)\omega(e_i)],
\]
which implies that
\[
\tilde{S}(Y, Z) = [(\alpha^2 - 1)(n - 1) - \alpha \beta]g(Y, Z) + \alpha \beta(n - 2)\omega(Y)\omega(Z).
\]
(III.9.9)
Thus a quasi-umbilical hypersurface of an $LP$-Sasakian space form is $\eta$-Einstein.
Example III.9.6: We consider the 3-dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3\},$
where \((x, y, z)\) are standard coordinate of \(\mathbb{R}^3\).

The vector fields
\[
e_1 = e^z \frac{\partial}{\partial y}, \quad e_2 = e^z \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right), \quad e_3 = \frac{\partial}{\partial z}
\]
are linearly independent at each point of \(M\).

Let \(g\) be the Lorentzian metric defined by
\[
g(e_1, e_3) = g(e_1, e_2) = g(e_2, e_3) = 0,
\]
\[
g(e_1, e_1) = g(e_2, e_2) = 1,
\]
\[
g(e_3, e_3) = -1.
\]

Let \(\eta\) be the 1-form defined by \(\eta(Z) = g(Z, e_3)\) for any \(Z \in \chi(M)\).

Let \(\phi\) be the \((1, 1)\) tensor field defined by
\[
\phi(e_1) = -e_1, \quad \phi(e_2) = -e_2, \quad \phi(e_3) = 0.
\]

Then using the linearity of \(\phi\) and \(g\), we have
\[
\eta(e_3) = -1,
\]
\[
\phi^2 Z = Z + \eta(Z)e_3,
\]
\[
g(\phi Z, \phi W) = g(Z, W) + \eta(Z)\eta(W),
\]
for any \(Z, W \in \chi(M)\).

Then for \(e_3 = \xi\), the structure \((\phi, \xi, \eta, g)\) defines a Lorentzian paracontact structure on \(M\).

Let \(\nabla\) be the Levi-Civita connection with respect to the Lorentzian metric \(g\) and \(R\) be the curvature tensor of \(g\). Then we have
\[
[e_1, e_2] = 0, \quad [e_1, e_3] = -e_1 \quad \text{and} \quad [e_2, e_3] = -e_2.
\]

Taking \(e_3 = \xi\) and using Koszul’s formula for the Lorentzian metric \(g\), we can easily calculate
\[
\nabla_{e_1}e_3 = -e_1, \quad \nabla_{e_1}e_2 = 0, \quad \nabla_{e_1}e_1 = -e_3,
\]
\[
\nabla_{e_2}e_3 = -e_2, \quad \nabla_{e_2}e_2 = -e_3, \quad \nabla_{e_2}e_1 = 0,
\]
\[
\nabla_{e_3}e_3 = 0, \quad \nabla_{e_3}e_2 = 0, \quad \nabla_{e_3}e_1 = 0.
\]

(III.9.10)
From the above it can be easily seen that $M^3(\phi, \xi, \eta, g)$ is an $LP$-Sasakian manifold. With the help of the above results it can be easily verified that

$$
R(e_1, e_2)e_3 = 0, \quad R(e_2, e_3)e_3 = -e_2, \quad R(e_1, e_3)e_3 = -e_1,
$$

$$
R(e_1, e_2)e_2 = e_1, \quad R(e_2, e_3)e_2 = -e_3, \quad R(e_1, e_3)e_2 = 0,
$$

$$
R(e_1, e_2)e_1 = -e_2, \quad R(e_2, e_3)e_1 = 0, \quad R(e_1, e_3)e_1 = -e_3.
$$

From the above expressions of the curvature tensor we obtain

$$
S(e_1, e_1) = g(R(e_1, e_2)e_2, e_1) - g(R(e_1, e_3)e_3, e_1) = 2.
$$

Similarly we have

$$
S(e_2, e_2) = 2
$$

and

$$
S(e_3, e_3) = -2.
$$

Therefore,

$$
r = S(e_1, e_1) + S(e_2, e_2) - S(e_3, e_3) = 6.
$$

Therefore Lemma III.8.1. is verified.

From [32] we know that in a 3- dimensional $LP$-Sasakian manifold

$$
R(X, Y)Z = (\frac{r-4}{2})[g(Y, Z)X - g(X, Z)Y] + (\frac{r-6}{2})[g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi + \eta(Y)\eta(X)\eta(Z)X - \eta(X)\eta(Z)Y].
$$

(III.9.11)

Now using (III.9.11) we get

$$
g(R(X, Y)Z, W) = (\frac{r-4}{2})[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)]
+ (\frac{r-6}{2})[g(Y, Z)\eta(X)\eta(W) - g(X, Z)\eta(Y)\eta(W)]
+ \eta(Y)\eta(Z)g(X, W) - \eta(X)\eta(Z)g(Y, W)].
$$

(III.9.12)

From (III.9.12), it follows that the $\phi$- sectional curvature of the manifold is given by

$$
K(X, \phi X) = \frac{r - 4}{2}
$$

for any vector field $X$ orthogonal to $\xi$. In view of the above relation we get

$$
K(e_1, \phi e_1) = K(e_2, \phi e_2) = \frac{r - 4}{2}
$$
Again it can be easily shown from (III.2.3) that
\[ \bar{K}(e_1, \phi e_1) - K(e_1, \phi e_1) = -2(a - 1) \]
and
\[ \bar{K}(e_2, \phi e_2) - K(e_2, \phi e_2) = \cdots \]
Therefore Theorem III.2.2.1. is verified.

**SECTION 10**

Pseudosymmetric $LP$-Sasakian manifolds

Let us consider a pseudosymmetric $LP$-Sasakian manifold. Then we have
\[ (R(X, Y), R)(U, V)W = f_R Q(g, R)(U, V, W; X, Y) \]  \hspace{1cm} (III.10.1)
for all $X, Y, U, V, W \in \chi(M)$. From the above relation it follows that
\[ = f_R [(X \wedge_g Y)R(U, V)W - R((X \wedge_g Y)U, V)W \]
\[ - R(U, (X \wedge_g Y)V)W - R(U, V)(X \wedge_g Y)W]. \]  \hspace{1cm} (III.10.2)

Therefore replacing $X$ by $\xi$ in (III.10.2) we have
\[ = f_R [(\xi \wedge_g Y)R(U, V)W - R((\xi \wedge_g Y)U, V)W \]
\[ - R(U, (\xi \wedge_g Y)V)W - R(U, V)(\xi \wedge_g Y)W]. \]  \hspace{1cm} (III.10.3)

By virtue of (III,p.4) we obtain from (III.10.3) that
\[ (1 - f_R)\bar{R}(U, V, W; Y)\xi - \eta(R(U, V)W)Y - g(Y, U)\bar{R}(\xi, V)W \]
\[ + \eta(U)R(Y, V)W - g(Y, V)R(U, \xi)W + \eta(V)R(U, Y)W \]
\[ - g(Y, W)R(U, V)\xi + \eta(W)R(U, V)Y] = 0, \]  \hspace{1cm} (III.10.4)
where $\bar{R}(U, V, W, X) = g(R(U, V)W, X)$. Taking the inner product of (III.10.4) with $\xi$ we get
\[ (1 - f_R)[-\bar{R}(U, V, W, Y) - \eta(Y)\eta(R(U, V)W) - g(Y, U)\eta(R(\xi, V)W) \]
\[ + \eta(U)\eta(R(Y, V)W) - g(Y, V)\eta(R(U, \xi)W) + \eta(V)\eta(R(U, Y)W) \]
\[ - g(Y, W)\eta(R(U, V)\xi) + \eta(W)\eta(R(U, V)Y)] = 0. \]  \hspace{1cm} (III.10.5)
Using (III.p.1) in (III.10.5) yields

\[(1-f_R)[\tilde{R}(U,V,W,Y) + g(Y,U)g(V,W) - g(Y,V)g(U,W)] = 0, \quad (\text{III.10.6})\]

from which it follows that

either \(f_R = 1\) or

\[\tilde{R}(U,V,W,Y) = g(Y,U)g(V,W) - g(Y,V)g(U,W)\]

and hence

\[R(U,V)W = g(V,W)U - g(U,W)V, \quad (\text{III.10.7})\]

which means that the manifold is of constant curvature 1 and hence the manifold is semisymmetric. Thus we can state the following:

**Theorem III.10.1.** A pseudosymmetric \(LP\)-Sasakian manifold is either a pseudosymmetric manifold of constant type or semisymmetric.

**Remark III.10.1.** From Theorem III.10.1. it follows that a proper pseudosymmetric \(LP\)-Sasakian manifold does not exit provided the associated function is not a constant.

**SECTION 11**

**Weyl-Pseudosymmetric \(LP\)-Sasakian manifolds**

In this section we assume that an \(LP\)-Sasakian manifold is Weyl-pseudosymmetric. Then we have

\[(R(X,Y)C)(U,V)W = f_C(Q(g,C)(U,V,W;X,Y)) \quad (\text{III.11.1})\]

for all \(X,Y,U,V,W \in \chi(M)\). From the above relation it follows that


Therefore replacing \(X\) by \(\xi\) in (III.11.2) we have

\[R(\xi,Y)C(U,V)W - C(R(\xi,Y)U,V)W - C(U,R(\xi,Y)V)W - C(U,V)R(\xi,Y)W = f_C(\xi \wedge g Y)(C(U,V)W - C((\xi \wedge g Y)U,V)W - C(U,(\xi \wedge g Y)V)W - C(U,V)((\xi \wedge g Y)W). \quad (\text{III.11.3})\]
By virtue of (III.p.4) we obtain from (III.11.3) that

\[(1 - f_C)[\tilde{C}(U, V, W, Y)\xi - \eta(C(U, V)W)Y - g(Y, U)C(\xi, V)W
+ \eta(U)C(Y, V)W - g(Y, V)C(U, \xi)W + \eta(V)C(U, Y)W
- g(Y, W)C(U, V)\xi + \eta(W)C(U, V)Y] = 0, \]

where \( \tilde{C}(U, V, W, X) = g(R(U, V)W, X) \).

Taking the inner product of (III.11.4) with \( \xi \) we get

\[(1 - f_C)[-\tilde{C}(U, V, W, Y) - \eta(Y)\eta(C(U, V)W) - g(Y, U)\eta(C(\xi, V)W)
+ \eta(U)\eta(C(Y, V)W) - g(Y, V)\eta(C(U, \xi)W)
+ \eta(V)\eta(C(U, Y)W) - g(Y, W)\eta(C(U, V)\xi)
+ \eta(W)\eta(C(U, V)Y)] = 0. \]

(III.11.5)

Let \( \{e_i\}, i = 1, 2, \ldots, n \) be an orthonormal frame at any point of the manifold. Then putting \( Y = U = e_i \) in (III.11.5) and taking summation over \( i \), we get

\[(1 - f_C)\sum_{i=1}^{n} e_i[-\tilde{C}(e_i, V, W, e_i) - \eta(e_i)\eta(C(e_i, V)W) - g(e_i, e_i)\eta(C(\xi, V)W)
+ \eta(e_i)\eta(C(e_i, V)W) - g(e_i, V)\eta(C(e_i, \xi)W)
+ \eta(V)\eta(C(e_i, e_i)W) - g(e_i, W)\eta(C(e_i, V)\xi)
+ \eta(W)\eta(C(e_i, V)e_i)] = 0, \]

(III.11.6)

where \( e_i = g(e_i, e_i) \).

Using \( \eta(C(V, W)\xi) = 0 \) in (III.11.6), we obtain

\[(1 - f_C)\eta(C(\xi, V)W) = 0, \]

from which it follows that

either \( f_C = 1 \) or \( \eta(C(\xi, V)W) = 0 \).

Suppose \( f_C \neq 1 \), then from \( \eta(C(\xi, V)W) = 0 \), it follows that \( C = 0 \) which contradicts the definition of Weyl pseudosymmetric manifolds. Hence \( f_C = 1 \). Therefore we can state the following:

**Theorem III.11.1.** A Weyl-pseudosymmetric LP-Sasakian manifold is a Weyl-pseudosymmetric manifold of constant type.
SECTION 12

Ricci-Pseudosymmetric $LP$-Sasakian manifolds

Let us consider a Ricci-pseudosymmetric $LP$-Sasakian manifold. Then we have

$$(R(X,Y).S)(U,V) = f_S Q(g, S)(U, V; X, Y)$$  \hspace{1cm} (III.12.1)$$

for all $X, Y, U, V, \in \chi(M)$. From the above relation it follows that

$$S(R(X,Y)U,V) + S(U, R(X,Y)V) = f_S [S((X \wedge Y)U,V)$$
$$- S(U, (X \wedge Y)V)].$$  \hspace{1cm} (III.12.2)$$

Therefore replacing $X$ by $\xi$ in (III.12.2) we have

$$S(R(\xi,Y)U,V) + S(U, R(\xi,Y)V) = f_S [S((\xi \wedge Y)U,V)$$
$$- S(U, (\xi \wedge Y)V)].$$  \hspace{1cm} (III.12.3)$$

By virtue of (III.p.4) we obtain from (III.12.3) that

$$(1 + f_S)[g(Y,U)S(V,\xi) - \eta(U)S(Y,V) + g(Y,V)S(Y,U)] = 0.$$  \hspace{1cm} (III.12.4)$$

Now setting $U = \xi$ in (III.12.4) we have

$$(1 + f_S)[S(Y,V) + (n-1)g(Y,V)] = 0,$$

from which it follows that

either $f_S = -1$ or

$$S(Y,V) = -(n-1)g(Y,V).$$

Theorem III.12.1. A Ricci-pseudosymmetric $LP$-Sasakian manifold is either a Ricci-pseudosymmetric manifold of constant type or Einstein.

Conversely, if the manifold is an Einstein manifold, then obviously (III.12.1) holds.

Hence we can state the following:

Theorem III.12.2. An $LP$-Sasakian manifold is Ricci-pseudosymmetric if and only if the manifold is an Einstein manifold, provided $f_R \neq -1.$