CHAPTER I

KENMOTSU MANIFOLDS
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Introduction.

This chapter is divided into two parts A and B of which part A deals with Kenmotsu manifolds and part B is concerned with $\beta$-Kenmotsu manifolds.

PART A

The product of an almost contact manifold $M$ and the real line $\mathbb{R}$ carries a natural almost complex structure. However if one takes $M$ to be an almost contact metric manifold and suppose that the product metric $G$ on $M \times \mathbb{R}$ is Kaeehlerian, then the structure on $M$ is cosymplectic [46] and not Sasakian. On the other hand, Oubina [72] pointed out that if the conformally related metric $e^{2t}G$, $t$ being the coordinates on $\mathbb{R}$, is Kaeehlerian, then $M$ is Sasakian and conversely.

In [98] Tanno classified connected almost contact metric manifolds whose automorphism groups have the maximum dimension. For such a manifold $M$, the sectional curvature of plane section containing $\xi$ is a constant, say $c$. If $c > 0$, $M$ is a homogeneous Sasakian manifold of constant $\phi$-sectional curvature. If $c = 0$, $M$ is the product of a line or circle with a Kaeehler manifold of constant holomorphic curvature. If $c < 0$, $M$ is a warped product space $\mathbb{R} \times fC^n$. In [53] Kenmotsu abstracted the differential geometric properties of the third case. In particular the almost contact metric structure in this case satisfies

$$(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X \quad (I.A.i.1)$$

and an almost contact metric manifold satisfying this condition is called a Kenmotsu manifold ([50],[53]).

\footnote{The equations of the thesis are in the form (C.S.E), where C stands for chapter, S for section number and E for equation number. Thus (I.A.1.1) means the equation (1) of section 1 of part A of chapter I, (II.3.7) means the equation (3) of section 7 of chapter II etc.}
In the first section of part A we introduce globally φ−concircularly symmetric Kenmotsu manifolds. In the next section we study 3-dimensional locally φ−concircularly symmetric Kenmotsu manifold. Section 3 and 4 are devoted to study of conharmonically flat and φ-conharmonically flat Kenmotsu manifold. In the next section we study 3-dimensional Kenmotsu manifold admitting a non-null concircular vector field. In section 6 we study locally φ- conharmonically symmetric 3-dimensional Kenmotsu manifold and in section 7 we give an example of such a manifold. In the remaining sections of this chapter we study different curvature properties of Kenmotsu manifolds with respect to the quarter-symmetric metric connection.

**PRELIMINARIES**

Let $M$ be a $(2n + 1)$- dimensional connected almost contact metric manifold with an almost contact metric structure $(\phi, \xi, \eta, g)$, that is, $\phi$ is an $(1,1)$ tensor field, $\xi$ is a vector field, $\eta$ is a 1 - form and $g$ is a compatible Riemannian metric such that

\[
\phi^2(X) = -X + \eta(X)\xi, \eta(\xi) = 1, \phi\xi = 0, \eta\phi = 0 \quad (I.A.p.1)
\]

\[
g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \quad (I.A.p.2)
\]

\[
g(X, \xi) = \eta(X) \quad (I.A.p.3)
\]

for all $X, Y \in T(M)$([11],[13], [80]).

If an almost contact metric manifold satisfies

\[
(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X, \quad (I.A.p.4)
\]

then $M$ is called a Kenmotsu manifold [53], where $\nabla$ is the Levi-Civita connection of $g$. From the above equation it follows that

\[
\nabla_X \xi = X - \eta(X)\xi, \quad (I.A.p.5)
\]

and

\[
(\nabla_X \eta)Y = g(X, Y) - \eta(X)\eta(Y). \quad (I.A.p.6)
\]

Moreover the curvature tensor $R$ and the Ricci tensor $S$ satisfy

\[
R(X, Y)\xi = \eta(X)Y - \eta(Y)X \quad (I.A.p.7)
\]
and
\[ S(X, \xi) = -2n\eta(X). \] (I.A.p.8)

From [31] we know that for a 3-dimensional Kenmotsu manifold
\[ R(X, Y)Z = \left( r + \frac{4}{2} \right) [g(Y, Z)X - g(X, Z)Y] \]
\[ - (\frac{r + 8}{2}) [g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi] \]
\[ + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y \] (I.A.p.9)

and
\[ S(X, Y) = \frac{1}{2} [(r + 2)g(X, Y) - (r + 6)\eta(X)\eta(Y)] \] (I.A.p.10)

where \( S \) is the Ricci tensor of type (0,2), \( R \) is the curvature tensor of type (1,3) and \( r \) is the scalar curvature of the manifold \( M \).

In a \((2n + 1)\)-dimensional almost contact metric manifold, if \( \{e_1, ..., e_{2n}, \xi\} \) is a local orthonormal basis of vector fields, then \( \{\phi e_1, ..., \phi e_{2n}, \xi\} \) is also a local orthonormal basis. It is easy to verify that
\[ \sum_{i=1}^{2n} g(e_i, e_i) = \sum_{i=1}^{2n} g(\phi e_i, \phi e_i) = 2n. \] (I.A.p.11)

\[ \sum_{i=1}^{2n} g(e_i, Z)S(Y, e_i) = \sum_{i=1}^{2n} g(\phi e_i, Z)S(Y, \phi e_i) = S(Y, Z) - S(Y, \xi)\eta(Z), \] (I.A.p.12)

for \( Y, Z \in T(M) \). In particular in view of \( \eta \circ \phi = 0 \), we get
\[ \sum_{i=1}^{2n} g(e_i, \phi Z)S(Y, e_i) = \sum_{i=1}^{2n} g(\phi e_i, \phi Z)S(Y, \phi e_i) = S(Y, \phi Z), \] (I.A.p.13)

for \( Y, Z \in T(M) \). If \( M \) is a Kenmotsu manifold then it is known that
\[ R(X, \xi)\xi = \eta(X)\xi - X, \quad X \in T(M) \] (I.A.p.14)

and
\[ S(\xi, \xi) = -2n. \] (I.A.p.15)

From (I.A.p.15) we get
\[ \sum_{i=1}^{2n} S(e_i, e_i) = \sum_{i=1}^{2n} S(\phi e_i, \phi e_i) = r + 2n, \] (I.A.p.16)

where \( r \) is the scalar curvature. In a Kenmotsu manifold we also have
\[ \overline{R}(\xi, Y, Z, \xi) = -g(\phi Y, \phi Z), \quad Y, Z \in T(M). \] (I.A.p.17)
Consequently
\[ \sum_{i=1}^{2n} R(e_i, Y, Z, e_i) = \sum_{i=1}^{2n} \tilde{R}(\phi e_i, Y, Z, \phi e_i) = S(Y, Z) + g(\phi Y, \phi Z). \] (I.A.p.18)

Now we state the following Lemmas:

**Lemma (I.A.p.1.)**\[31\] A 3-dimensional Kenmotsu manifold is a manifold of constant negative curvature if and only if the scalar curvature \( r = -6 \).

**Lemma (I.A.p.2.)**\[31\] A 3-dimensional Kenmotsu manifold is locally \( \phi \)-symmetric if and only if the scalar curvature \( r \) is constant.

**Lemma (I.A.p.3.)**\[51\] Any \( \eta \)-Einstein Kenmotsu manifold of dimension \( \geq 5 \) with \( b = \text{constant} \) is Einstein.
SECTION 1

Globally $\phi$-concircularly symmetric Kenmotsu manifolds

Definition I.A.1.1. A Kenmotsu manifold $M$ is said to be globally $\phi$-concircularly symmetric if the concircular curvature tensor $\tilde{C}$ satisfies

$$\phi^2 \left( \left( \nabla_X \tilde{C} \right) (Y, Z, W) \right) = 0,$$

(I.A.1.1)

for all vector fields $X, Y, Z \in \chi(M)$.

Let us suppose that $M$ is a globally $\phi$-concircularly symmetric Kenmotsu manifold. Then by definition

$$\phi^2 \left( \left( \nabla_W \tilde{C} \right) (X, Y, Z) \right) = 0.$$

Using (I.A.p.1) we have

$$- \left( \nabla_W \tilde{C} \right) (X, Y) Z + \eta \left( \left( \nabla_W \tilde{C} \right) (X, Y) Z \right) \xi = 0.$$

From (1.4) it follows that

$$- g((\nabla_WR) (X, Y) Z, U) + \frac{dr(W)}{n} \left[ g(Y, Z)g(X, U) - g(X, Z)g(Y, U) \right]
+ \eta((\nabla_WR) (X, Y) Z) \eta(U) - \frac{dr(W)}{n(n-1)} [g(Y, Z) \eta(X) - g(X, Z) \eta(Y)] \eta(U) = 0.$$

Putting $X = U = e_i$, where $\{e_i\}$, $(i = 1, 2, \ldots, n)$ is an orthonormal basis of the tangent space at each point of the manifold, and taking summation over $i$, we get

$$- (\nabla_WS) (Y, Z) + \frac{dr(W)}{n} g(Y, Z)
+ \eta \left( \left( \nabla_WR \right) (e_i, Y) Z \right) \eta(e_i) - \frac{dr(W)}{n(n-1)} [g(Y, Z) - \eta(Y) \eta(Z)] = 0.$$

Putting $Z = \xi$, we obtain

$$- (\nabla_WS) (Y, \xi) + \frac{dr(W)}{n} \eta(Y) + \eta \left( \left( \nabla_WR \right) (e_i, Y) \xi \right) \eta(e_i) = 0.$$

(I.A.1.2)

Now

$$\eta \left( \left( \nabla_WR \right) (e_i, Y) \xi \right) \eta(e_i) = g \left( \left( \nabla_WR \right) (e_i, Y) \xi, \xi \right) g(e_i, \xi).$$

(I.A.1.3)

$$g \left( \left( \nabla_WR \right) (e_i, Y) \xi, \xi \right) = g \left( \nabla_WR(e_i, Y) \xi, \xi \right) - g \left( R(\nabla_WE_i, Y) \xi, \xi \right)
- g \left( R(e_i, \nabla_W Y) \xi, \xi \right) - g \left( R(e_i, Y) \nabla_W \xi, \xi \right).$$
Since \( \{e_i\} \) is an orthonormal basis \( \nabla_X e_i = 0 \) and using (I.A.p.7) we find
\[
g(R(e_i, \nabla W Y) \xi, \xi) = g(\eta(e_i) \nabla W Y - \eta(\nabla W Y) e_i, \xi) \\
= \eta(e_i) \eta(\nabla W Y) - \eta(\nabla W Y) \eta(e_i) \\
= 0.
\]
As
\[
g(R(e_i, Y) \xi, \xi) + g(R(\xi, Y) e_i, \xi) = 0
\]
we have
\[
g(\nabla W R(e_i, Y) \xi, \xi) + g(R(e_i, Y) \xi, \nabla W \xi) = 0.
\]
Using this we get
\[
g( (\nabla W R) (e_i, Y) \xi, \xi) = 0. \quad (I.A.1.4)
\]
By the use of (I.A.1.3) and (I.A.1.4), from (I.A.1.2) we obtain
\[
(\nabla W S) (Y, \xi) = \frac{1}{n} dr(W)\eta(Y), \quad (I.A.1.5)
\]
Putting \( Y = \xi \) in (I.A.1.5), we get \( dr(W) = 0 \). This implies \( r \) is constant. So from (I.A.1.5), we have \( \nabla W S(Y, \xi) = 0 \). This implies that
\[
S(Y, W) = (1 - n) g(Y, W). \quad (I.A.1.6)
\]
Hence we can state the following:

**Theorem I.A.1.1.** If a Kenmotsu manifold is globally \( \phi \)-concircularly symmetric, then the manifold is an Einstein manifold. Next suppose \( S(X, Y) = \lambda g(X, Y) \), that is, the manifold is an Einstein manifold. Then from we have
\[
\left(\nabla W \tilde{C}\right) (X, Y) Z = (\nabla W R) (X, Y) Z.
\]
Applying \( \phi^2 \) on both sides of the above equation we have
\[
\phi^2 \left(\nabla W \tilde{C}\right) (X, Y) Z = \phi^2 (\nabla W R) (X, Y) Z.
\]
Hence we can state:

**Theorem I.A.1.2.** A globally \( \phi \)-concircularly symmetric Kenmotsu manifold is globally \( \phi \)-symmetric.

**Remark I.A.1.1.** Since a globally \( \phi \)-symmetric Kenmotsu manifold is always a globally \( \phi \)-concircularly symmetric manifold, from Theorem I.A.1.2, we conclude that on a Kenmotsu manifold, globally \( \phi \)-symmetry and globally \( \phi \)-concircularly symmetry are equivalent.
SECTION 2

3-dimensional locally $\phi$-concircularly symmetric Kenmotsu manifolds

Using (I.A.p.9) in (9), in a 3-dimensional Kenmotsu manifold the concircular curvature tensor is given by

$$\tilde{\mathcal{C}}(X,Y)Z = (r+4)\left[g(Y,Z)X - g(X,Z)Y\right]$$

$$-(r+6)\left[g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi \right]$$

$$+\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y$$

$$-(\frac{r}{6})\left[g(Y,Z)X - g(X,Z)Y\right]$$


Taking the covariant differentiation to the both sides of the equation (I.A.2.1), we have

$$\left(\nabla_{W}\tilde{\mathcal{C}}\right)(X,Y)Z = \frac{dr(W)}{2}\left[g(Y,Z)X - g(X,Z)Y\right]$$

$$-\frac{dr(W)}{2}\left[g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi \right]$$

$$+\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y$$

$$-(r\frac{r}{6})\left[g(Y,Z)(\nabla_{W}\eta)(X)\xi - g(X,Z)(\nabla_{W}\eta)(Y)\xi \right]$$

$$+g(Y,Z)\eta(X)(\nabla_{W}\xi) - g(X,Z)\eta(Y)(\nabla_{W}\xi)$$

$$+(\nabla_{W}\eta)(Y)\eta(Z)X + \eta(Y)(\nabla_{W}\eta)(Z)X$$

$$-(\nabla_{W}\eta)(X)\eta(Z)Y - \eta(X)(\nabla_{W}\eta)(Z)Y$$

$$-(\frac{dr(W)}{6})\left[g(Y,Z)X - g(X,Z)Y\right]$$

Now assume that $X,Y$ and $Z$ are horizontal vector fields. So the equation (I.A.2.2) becomes

$$\left(\nabla_{W}\tilde{\mathcal{C}}\right)(X,Y)Z = \frac{dr(W)}{3}\left[g(Y,Z)X - g(X,Z)Y\right]$$

$$-\left(r\frac{r}{6}\right)\left[g(Y,Z)(\nabla_{W}\eta)(X)\xi - g(X,Z)(\nabla_{W}\eta)(Y)\xi \right]$$

From (I.A.2.3) it follows that

$$\phi^{2}(\left(\nabla_{W}\tilde{\mathcal{C}}\right)(X,Y)Z) = -\frac{dr(W)}{3}\left[g(Y,Z)X - g(X,Z)Y\right]$$

Hence we can state the following:

**Theorem I.A.2.1.** A 3-dimensional Kenmotsu manifold is locally $\phi$-concircularly symmetric if and only if the scalar curvature $r$ is constant.

In [31], De and Pathak prove that
Corollary I.A.2.1. A 3-dimensional Kenmotsu manifold is locally $\phi$-symmetric if and only if the scalar curvature $r$ is constant.

Using Corollary I.A.2.1., we can state the following theorem:

Theorem I.A.2.2. A 3-dimensional Kenmotsu manifold is locally $\phi$-concircularly symmetric if and only if it is locally $\phi$-symmetric.

SECTION 3

Conharmonically flat Kenmotsu manifold

In this section we study conharmonically flat Kenmotsu manifold.

Definition I.A.3.1. A Kenmotsu manifold is said to be conharmonically flat if

$$g(H(X,Y)Z,W) = 0. \quad (I.A.3.1)$$

Let a $(2n + 1)$-dimensional Kenmotsu manifold $M$ be conharmonically flat. Then using (I.A.3.1) in (16) we have

$$R(X,Y)Z = \frac{1}{2n-1}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY]. \quad (I.A.3.2)$$

Taking $Z = \xi$ and using (I.A.p.7) and (I.A.p.8) we have

$$\eta(X)Y - \eta(Y)X = \frac{1}{2n-1}[2n\{\eta(X)Y - \eta(Y)X\} - \eta(X)QY + \eta(Y)QX]. \quad (I.A.3.3)$$

Again putting $Y = \xi$ in (I.A.3.3) we get

$$\eta(X)\xi - X = \frac{1}{2n-1}[2n\{\eta(X)\xi - X\} - \eta(X)Q\xi + QX] \quad (I.A.3.4)$$

and after simplification the above equation reduces to

$$S(X,Y) = g(X,Y) - (2n + 1)\eta(X)\eta(Y). \quad (I.A.3.5)$$

So in view of (I.A.3.5) and Lemma I.A.p.3 we state the following:

Theorem I.A.3.1. A conharmonically flat Kenmotsu manifold is an Einstein manifold.

Using Theorem I.A.3.1 in equation (I.A.3.2) we obtain the following:

Corollary I.A.3.1. A conharmonically flat Kenmotsu manifold is a manifold of constant curvature.
SECTION 4

φ-Conharmonically flat Kenmotsu manifold

In this section we study φ-conharmonically flat Kenmotsu manifolds.

Definition I.A.4.1. A Kenmotsu manifold is said to be φ-conharmonically flat if
\[ g(H(\phi X, \phi Y) \phi Z, \phi W) = 0, \]  
(I.A.4.1)

where \( X, Y, Z, W \in T(M) \).

Let a \((2n + 1)\)-dimensional Kenmotsu manifold \( M \) be φ-conharmonically flat. Then using (I.A.4.1) in (16) we have
\[
\begin{align*}
R(\phi X, \phi Y, \phi Z, \phi W) &= \frac{1}{2n-1} [S(\phi Y, \phi Z)g(\phi X, \phi W) \\
&- S(\phi X, \phi Z)g(\phi Y, \phi W) \\
&+ S(\phi X, \phi W)g(\phi Y, \phi Z) \\
&- S(\phi Y, \phi W)g(\phi X, \phi Z)].
\end{align*}
\]  
(I.A.4.2)

Let \( \{e_1, ..., e_{2n}, \xi\} \) be a local orthonormal basis of vector fields in \( M \). Putting \( X = W = e_i \) in (I.A.4.2) and summing up from 1 to \( 2n \) we have
\[
\sum_{i=1}^{2n} R(\phi e_i, \phi Y, \phi Z, \phi e_i) = \frac{1}{2n-1} \sum_{i=1}^{2n} [S(\phi Y, \phi Z)g(\phi e_i, \phi e_i) \\
&- S(\phi e_i, \phi Z)g(\phi Y, \phi e_i) \\
&+ S(\phi e_i, \phi e_i)g(\phi Y, \phi Z) \\
&- S(\phi Y, \phi e_i)g(\phi e_i, \phi Z)].
\]  
(I.A.4.3)

Using (I.A.p.11), (I.A.p.12), (I.A.p.16) and (I.A.p.18) in (I.A.4.3) we get
\[
S(\phi Y, \phi Z) + g(\phi^2 Y, \phi^2 Z) = \frac{2n-2}{2n-1} S(\phi Y, \phi Z) \\
+ \frac{r+2n}{2n-1} g(\phi Y, \phi Z).
\]

that is,
\[ S(\phi Y, \phi Z) = (r + 1)g(\phi Y, \phi Z). \]  
(I.A.4.4)

Substituting \( Y \) by \( \phi Y \) and \( Z \) by \( \phi Z \) in (I.A.4.4) we have
\[ S(\phi^2 Y, \phi^2 Z) = (r + 1)g(\phi Y, \phi Z). \]  
(I.A.4.5)

Using (I.A.p.1), (I.A.p.2) and (I.A.p.8) in (I.A.4.5) we get
\[ S(Y, Z) = (r + 1)g(Y, Z) - (2n + 1 + r)\eta(Y)\eta(Z). \]  
(I.A.4.6)
Contacting (I.A.4.6) we have
\[ r = 0. \quad \text{(I.A.4.7)} \]

In view of (I.A.4.6) and (I.A.4.7) we have the following:

**Theorem I.A.4.1.** A \( \phi \)-conharmonically flat Kenmotsu manifold is an \( \eta \)-Einstein manifold with vanishing scalar curvature.

### SECTION 5

3-dimensional Kenmotsu manifold admitting a non-null concircular vector field

**Definition I.A.5.1.** A vector field \( V \) on a Riemannian manifold is said to be a concircular vector field [105] if it satisfies an equation of the form
\[ \nabla_X V = \rho X \quad \text{(I.A.5.1)} \]
for all \( X \), where \( \rho \) is a scalar function. In particular if \( \rho = 0 \), then \( V \) is parallel.

We suppose that a 3-dimensional Kenmotsu manifold admits a non-null concircular vector field. Then differentiating (I.A.5.1) covariantly we get
\[ \nabla_Y \nabla_X V = \rho \nabla_Y X + d\rho(X)X. \quad \text{(I.A.5.2)} \]

From (I.A.5.2) it follows that (since the torsion tensor \( T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] = 0 \))
\[ \nabla_Y \nabla_X V - \nabla_X \nabla_Y V - \nabla_{[X,Y]} V = d\rho XY - d\rho(Y)X. \quad \text{(I.A.5.3)} \]

Hence by Ricci identity we obtain from (I.A.5.3)
\[ R(X,Y)V = d\rho(X)Y - d\rho(Y)X, \quad \text{(I.A.5.4)} \]
which implies that
\[ \overline{R}(X,Y,V,Z) = d\rho(X)g(Y,Z) - d\rho(Y)g(X,Z), \quad \text{(I.A.5.5)} \]
where \( \overline{R}(X,Y,V,Z) = g(R(X,Y)V, Z) \).

Replacing \( Z \) by \( \xi \) in (I.A.5.5) we get
\[ \eta(R(X,Y)V) = d\rho(X)\eta(Y) - d\rho(Y)\eta(X). \quad \text{(I.A.5.6)} \]
Again
\[ \eta(R(X,Y)V) = \eta(Y)g(X,V) - \eta(X)g(Y,V). \]  \hspace{1cm} \text{(I.A.5.7)}

From (I.A.5.6) and (I.A.5.7) we have
\[ d\rho(X)\eta(Y) - d\rho(Y)\eta(X) = \eta(Y)g(X,V) - \eta(X)g(Y,V). \]  \hspace{1cm} \text{(I.A.5.8)}

Putting \( X = \phi X \) and \( Y = \xi \) in (I.A.5.8), we get
\[ d\rho(\phi X) = g(\phi X, V). \]  \hspace{1cm} \text{(I.A.5.9)}

Substituting \( X \) by \( \phi X \) in (I.A.5.9), we obtain
\[ d\rho(X) - d\rho(\xi)\eta(X) = g(X, V) - \eta(X)\eta(V). \]  \hspace{1cm} \text{(I.A.5.10)}

Here \( g(X, V) \neq 0 \) for all \( X \). For, if \( g(X, V) = 0 \) for all \( X \), then \( g(V, V) = 0 \) which means that \( V \) is a null vector field. This is contradicting our assumption. Hence multiplying both sides of (I.A.5.10) by \( g(X, V) \) we get
\[ d\rho(X)g(X, V) - d\rho(\xi)g(X, V)\eta(X) = g(X, V)[g(X, V) - \eta(X)\eta(V)]. \]  \hspace{1cm} \text{(I.A.5.11)}

Also putting \( Z = V \) in (I.A.5.5), we get
\[ d\rho(X)g(Y, V) = d\rho(Y)g(X, V). \]  \hspace{1cm} \text{(I.A.5.12)}

For \( Y = \xi \), we obtain from (I.A.5.12) that
\[ d\rho(X)\eta(V) = d\rho(\xi)g(X, V). \]  \hspace{1cm} \text{(I.A.5.13)}

Since \( \eta(X) \neq 0 \) for all \( X \), multiplying both sides of (I.A.5.13) by \( \eta(X) \), we have
\[ d\rho(X)\eta(V)\eta(X) = d\rho(\xi)\eta(X)g(X, V). \]  \hspace{1cm} \text{(I.A.5.14)}

By virtue of (I.A.5.11) and (I.A.5.14) we get
\[ [d\rho(X) - g(X, V)][g(X, V) - \eta(X)\eta(V)] = 0. \]  \hspace{1cm} \text{(I.A.5.15)}

Hence it follows from (I.A.5.15) that
\[ \text{either} \quad d\rho(X) = g(X, V) \quad \text{for all} \ X \]  \hspace{1cm} \text{(I.A.5.16)}
\[ \text{or} \quad g(X, V) - \eta(X)\eta(V) = 0 \quad \text{for all} \ X. \]  \hspace{1cm} \text{(I.A.5.17)}
First we consider the case of (I.A.5.16). Then we obtain from (I.A.5.5)

\[ \widehat{R}(X,Y,V,Z) = g(X,V)g(Y,Z) - g(Y,V)g(X,Z). \]  
(I.A.5.18)

Then putting \( X = Z = e_i, \ i = 1, 2, 3 \) in (I.A.5.18) and taking summation over \( 1 \leq i \leq 3 \), we get

\[ S(Y,V) = -2g(Y,V). \]  
(I.A.5.19)

By virtue of (I.A.p.10) and (I.A.5.19) we obtain

\[ (r + 6)[g(Y,V) - \eta(Y)\eta(V)] = 0. \]  
(I.A.5.20)

Since in this case \( g(Y,V) - \eta(Y)\eta(V) \neq 0 \), it follows from (I.A.5.20) that

\[ r = -6. \]  
(I.A.5.21)

Next, we consider case (I.A.5.17). Differentiating (I.A.5.17) covariantly along \( Z \), we get

\[ (\nabla_Z\eta)(X)\eta(V) + (\nabla_Z\eta)(V)\eta(X) = 0. \]  
(I.A.5.22)

Using (I.A.p.6) in (I.A.5.22), we obtain

\[ g(X,Z)\eta(V) + g(V,Z)\eta(X) - 2\eta(X)\eta(Z)\eta(V) = 0. \]  
(I.A.5.23)

Then putting \( X = Z = e_i, \ i = 1, 2, 3 \) in (I.A.5.23) and taking summation over \( 1 \leq i \leq 3 \), we get \( \eta(V) = 0 \), which contradicts our assumption.

Therefore, by virtue of (I.A.5.21) and Lemma I.A.p.1, we can state the following:

**Theorem I.A.5.1.** If a 3-dimensional Kenmotsu manifold admits a non-null concircular vector field, then the manifold is a manifold of constant negative curvature.

---

**SECTION 6**

**Locally \( \phi \)- conharmonically symmetric three dimensional Kenmotsu manifolds**

The notion of locally \( \phi \)-symmetry was first introduced by Takahashi [95] on a Sasakian manifold. In a recent paper [29] De and Sarkar introduced the notion of locally \( \phi \)-Ricci symmetric Sasakian manifolds again. In this paper we consider a locally \( \phi \)- conharmonically symmetric 3- dimensional Kenmotsu manifolds.
Definition I.A.6.1. A three-dimensional Kenmotsu manifold is said to be locally \( \phi \)-conharmonically symmetric if the conharmonic curvature tensor \( H \) satisfies
\[
\phi^2(\nabla_w H)(X,Y)Z = 0, \quad \text{(I.A.6.1)}
\]
where \( X, Y \) and \( Z \) are horizontal vector fields.

Using (I.A.p.10) in (16), in a 3-dimensional Kenmotsu manifold the conharmonic curvature tensor is given by
\[
H(X,Y)Z = \left( \frac{\xi}{r} \right) [g(X,Z)Y - g(Y,Z)X] \\
- \frac{\xi}{r} [g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi] \\
+ \eta(Y)\xi - \eta(X)\xi. \quad \text{(I.A.6.2)}
\]

Taking the covariant differentiation to both sides of equation (I.A.6.2), we have
\[
(\nabla_w H)(X,Y)Z = \frac{dr(W)}{2} [g(X,Z)Y - g(Y,Z)X] \\
- \frac{dr(W)}{2} [g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi] \\
+ \eta(Y)\xi - \eta(X)\xi \\
- \frac{\xi}{r^2} [g(Y,Z)(\nabla_w \eta)(X)\xi - g(X,Z)(\nabla_w \eta)(Y)\xi] \\
+ g(Y,Z)\eta(X)\nabla_w \xi - g(X,Z)\eta(Y)\nabla_w \xi \\
+ (\nabla_w \eta)(Y)\xi + \eta(Y)(\nabla_w \xi) \\
- (\nabla_w \eta)(X)\xi - \eta(X)(\nabla_w \xi). \quad \text{(I.A.6.3)}
\]

Now assume that \( X, Y \) and \( Z \) are horizontal vector fields. So equation (I.A.6.3) becomes
\[
(\nabla_w H)(X,Y)Z = \frac{dr(W)}{2} [g(X,Z)Y - g(Y,Z)X] \\
- \frac{\xi}{r^2} [g(Y,Z)(\nabla_w \eta)(X)\xi - g(X,Z)(\nabla_w \eta)(Y)\xi] \\
- (\nabla_w \eta)(X)\xi - \eta(X)(\nabla_w \xi). \quad \text{(I.A.6.4)}
\]

From (I.A.6.4) it follows that
\[
\phi^2(\nabla_w H)(X,Y)Z = \frac{dr(W)}{2} [g(Y,Z)X - g(X,Z)Y]. \quad \text{(I.A.6.5)}
\]

Hence we can state the following:

**Theorem I.A.6.1** A 3-dimensional Kenmotsu manifold is locally \( \phi \)-conharmonically symmetric if and only if the scalar curvature \( r \) is constant.

Using Lemma I.A.p.2., we can state the following theorem:

**Theorem I.A.6.2** A 3-dimensional Kenmotsu manifold is locally \( \phi \)-conharmonically symmetric if and only if it is locally \( \phi \)-symmetric.


SECTION 7
Examples of 3-dimensional Kenmotsu manifolds

Example I.A.7.1: In [45], the authors prove that if \( R(\xi, X)\tilde{C} = 0 \) for any \( X \in \chi(M) \), then \( M \) has constant sectional curvature \(-1\). Hence the manifold is an Einstein manifold. Therefore from the definition of concircular curvature tensor we find that globally \( \phi \)-symmetry and globally \( \phi \)-concircularly symmetry are equivalent. Hence in a concircular semi-symmetric \([R, \tilde{C}] = 0\) Kenmotsu manifold globally \( \phi \)-symmetry and globally \( \phi \)-concircularly symmetry are equivalent. Thus Theorem I.A.1.2 is verified.

Example I.A.7.2: In [53], Kenmotsu proved that a conformally flat Kenmotsu manifold of dimension \( \geq 5 \) has constant sectional curvature equal to \(-1\). Hence the manifold is an Einstein manifold. Therefore by the same argument as in Example I.A.7.1, in a conformally flat Kenmotsu manifold of dimension \( \geq 5 \) globally \( \phi \)-symmetry and globally \( \phi \)-concircularly symmetry are equivalent. Thus Theorem I.A.1.2 is verified.

Example I.A.7.3: In [51], Jun, De and Pathak prove that any \( \eta \)-Einstein \([S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y)]\) Kenmotsu manifold of dimension \( n \geq 5 \) with \( b = \) constant is Einstein. Hence by the similar argument as in Example I.A.7.1, in an \( \eta \)-Einstein Kenmotsu manifold of dimension \( \geq 5 \) globally \( \phi \)-symmetry and globally \( \phi \)-concircularly symmetry are equivalent. Thus Theorem I.A.1.2 is verified.

Example I.A.7.4: In [51], the authors prove that a Ricci recurrent \([\nabla S = \alpha \otimes S]\) manifold is an Einstein manifold. Hence by the similar argument as in Example I.A.7.1, in a Ricci-recurrent Kenmotsu manifold globally \( \phi \)-symmetry and globally \( \phi \)-concircularly symmetry are equivalent. Thus Theorem I.A.1.2 is verified.

Example I.A.7.5: We consider the 3-dimensional manifold \( M = \{(x, y, z) \in \mathbb{R}^3, z \neq 0\} \), where \((x, y, z)\) are standard coordinate of \( \mathbb{R}^3 \). The vector fields
\[
e_1 = z \frac{\partial}{\partial x}, \quad e_2 = z \frac{\partial}{\partial y}, \quad e_3 = -z \frac{\partial}{\partial z}
\]
are linearly independent at each point of \( M \).

Let \( g \) be the Riemannian metric defined by
\[
g(e_1, e_3) = g(e_1, e_2) = g(e_2, e_3) = 0, \\
g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1.
\]
Let $\eta$ be the 1-form defined by $\eta(Z) = g(Z, e_3)$ for any $Z \in \chi(M)$.

Let $\phi$ be the $(1,1)$ tensor field defined by

$$
\phi(e_1) = -e_2, \quad \phi(e_2) = e_1, \quad \phi(e_3) = 0.
$$

Then using the linearity of $\phi$ and $g$, we have

$$
\eta(e_3) = 1,
$$

$$
\phi^2 Z = -Z + \eta(Z)e_3,
$$

$$
g(\phi Z, \phi W) = g(Z, W) - \eta(Z)\eta(W),
$$

for any $Z, W \in \chi(M)$.

Then for $e_3 = \xi$, the structure $(\phi, \xi, \eta, g)$ defines an almost contact metric structure on $M$.

Let $\nabla$ be the Levi-Civita connection with respect to metric $g$. Then we have

$$
[e_1, e_3] = e_1 e_3 - e_3 e_1
= z \frac{\partial}{\partial x}(-z \frac{\partial}{\partial z}) - (-z \frac{\partial}{\partial z})(z \frac{\partial}{\partial x})
= -z^2 \frac{\partial^2}{\partial x \partial z} + z^2 \frac{\partial^2}{\partial z \partial x} + z \frac{\partial}{\partial x}
= e_1.
$$

Similarly

$$
[e_1, e_2] = 0 \quad \text{and} \quad [e_2, e_3] = e_2.
$$

The Riemannian connection $\nabla$ of the metric $g$ is given by

$$
2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y)
- g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]), \quad \text{(I.A.7.1)}
$$

which known as Koszul’s formula.

Using (I.A.7.1) we have

$$
2g(\nabla_{e_1} e_3, e_1) = -2g(e_1, -e_1) = 2g(e_1, e_1). \quad \text{(I.A.7.2)}
$$

Again by (I.A.7.1)

$$
2g(\nabla_{e_1} e_3, e_2) = 0 = 2g(e_1, e_2) \quad \text{(I.A.7.3)}
$$

and

$$
2g(\nabla_{e_1} e_3, e_3) = 0 = 2g(e_1, e_3). \quad \text{(I.A.7.4)}
$$
From (I.A.7.2), (I.A.7.3) and (I.A.7.4) we obtain

$$2g(\nabla e_1 e_3, X) = 2g(e_1, X),$$

for all $X \in \chi(M)$.

Thus

$$\nabla e_1 e_3 = e_1.$$

Therefore, (I.A.7.1) further yields

$$\nabla e_1 e_3 = e_1, \quad \nabla e_1 e_2 = 0, \quad \nabla e_2 e_1 = -e_3,$$

$$\nabla e_2 e_3 = e_2, \quad \nabla e_2 e_2 = e_3, \quad \nabla e_3 e_1 = 0,$$

$$\nabla e_3 e_3 = 0, \quad \nabla e_3 e_2 = 0, \quad \nabla e_2 e_1 = 0. \quad (I.A.7.5)$$

From the above it follows that the manifold satisfies $\nabla X \xi = X - \eta(X) \xi$, for $\xi = e_3$. Hence the manifold is a Kenmotsu manifold. It is known that

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \quad (I.A.7.6)$$

With the help of the above results and using (I.A.7.6), it can be easily verified that

$$R(e_1, e_2)e_3 = 0, \quad R(e_2, e_3)e_3 = -e_2, \quad R(e_1, e_3)e_3 = -e_1,$$

$$R(e_1, e_2)e_2 = -e_1, \quad R(e_2, e_3)e_2 = e_3, \quad R(e_1, e_3)e_2 = 0,$$

$$R(e_1, e_2)e_1 = e_2, \quad R(e_2, e_3)e_1 = 0, \quad R(e_1, e_3)e_1 = e_3.$$

From the above expressions of the curvature tensor $R$ we obtain

$$S(e_1, e_1) = g(R(e_1, e_2)e_2, e_1) + g(R(e_1, e_3)e_3, e_1)$$

$$= -2.$$ 

Similarly, we have

$$S(e_2, e_2) = S(e_3, e_3) = -2.$$ 

Therefore,

$$r = S(e_1, e_1) + S(e_2, e_2) + S(e_3, e_3) = -6.$$

We note that here $r$ is constant. Thus Theorem I.A.2.1. and Theorem I.A.6.1. are verified.


SECTION 8

η- parallel Ricci tensor with respect to the quarter-symmetric metric connection

Definition I.A.8.1. The Ricci tensor $S$ of a Kenmotsu manifold is said to be $\eta$-parallel if it satisfies

$$(\nabla_X S)(\phi Y, \phi Z) = 0,$$

(I.A.8.1)

for all vector fields $X, Y$ and $Z$.

Let $M$ be a 3-dimensional Kenmotsu manifold. From [92] we know that for a quarter-symmetric metric connection in a Kenmotu manifold

$$\tilde{\nabla}XY = \nabla XY - \eta(X)\phi Y$$

(I.A.8.2)

and

$$\tilde{S}(Y, Z) = S(Y, Z) + g(\phi Y, Z)$$

(I.A.8.3)

where $\tilde{\nabla}$ be a quater-symmetric metric connection in $M$ and $\tilde{S}$ is the Ricci tensor of the connection $\tilde{\nabla}$.

We know that

$$(\tilde{\nabla}_X \tilde{S})(Y, Z) = \tilde{\nabla}_X \tilde{S}(Y, Z)$$

(I.A.8.4)

Using (I.A.8.2) and (I.A.8.3) from (I.A.8.4), we have

$$(\tilde{\nabla}_X \tilde{S})(Y, Z) = \nabla XS(Y, Z) + \nabla X g(\phi Y, Z) - S(\nabla XY, Z)$$

+ $\eta(X)S(\phi Y, Z) - g(\phi \nabla_X Y, Z)$

+ $\eta(X)g(\phi^2 Y, Z) - S(Y, \nabla_X Z)$

+ $\eta(X)S(Y, \phi Z) - g(\phi Y, \nabla_X Z)$

+ $\eta(X)g(\phi Y, \phi Z).$  

(I.A.8.5)

Now using (I.A.p.1),(I.A.8.5) yields

$$(\tilde{\nabla}_X \tilde{S})(\phi Y, \phi Z) = (\nabla X S)(\phi Y, \phi Z)$$

+ $\eta(X)[S(\phi Y, Z) + S(\phi Z, Y)]$

+ $\eta(Z)g(\phi X, \phi Y) - \eta(Y)g(\phi X, \phi Z).$  

(I.A.8.6)

In (I.A.8.6) replacing $Y$ by $\phi Y$, $Z$ by $\phi Z$ and using (I.A.p.1) we get

$$(\tilde{\nabla}_X \tilde{S})(\phi Y, \phi Z) = (\nabla X S)(\phi Y, \phi Z)$$

+ $\eta(X)[-S(\phi Y, \phi Z) + \eta(Y)S(\xi, \phi Z)]$

+ $\eta(Z)S(\phi Y, Z) + \eta(Z)S(\phi Y, \xi)].$  

(I.A.8.7)
Now using (I.A.p.10),(I.A.8.7) yields
\[
(\tilde{\nabla}_X \tilde{S})(\phi Y, \phi Z) = (\nabla_X S)(\phi Y, \phi Z).
\]  
(I.A.8.8)

Hence we can state the following:

**Theorem I.A.8.1.** In a 3-dimensional Kenmotsu manifold, $\eta$-parallelity of the Ricci tensor with respect to the quater-symmetric metric connection and the Levi-Civita connection are equivalent.

## SECTION 9

**Cyclic Parallel Ricci tensor with respect to the quarter-symmetric metric connection**

A.Gray [41] introduced two classes of Riemannian manifolds determined by the covariant differentiation of the Ricci tensor. The class $A$ consisting of all Riemannian manifolds whose Ricci tensor $S$ is a Codazzi tensor, that is, $(\nabla_X S)(Y, Z) = (\nabla_Y S)(X, Z)$.

The class $B$ consisting of all Riemannian manifolds whose Ricci tensor is cyclic parallel, that is, $(\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y) = 0$.

Let $M$ be a 3-dimensional Kenmotsu manifold. Then its Ricci tensor $\tilde{S}$ is given by (I.A.8.3). Now using (I.A.8.6) we have

\[
(\tilde{\nabla}_X \tilde{S})(Y, Z) + (\tilde{\nabla}_Y \tilde{S})(Z, X) + (\tilde{\nabla}_Z \tilde{S})(X, Y)
= (\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y) + \eta(X)[S(\phi Y, Z) + S(Y, \phi Z)]
+ \eta(Y)[S(\phi Z, X) + S(Z, \phi X)] + \eta(Z)[S(\phi X, Y) + S(X, \phi Y)].
\]  
(I.A.9.1)

Now using (I.A.p.10),(I.A.9.1) yields

\[
(\tilde{\nabla}_X \tilde{S})(Y, Z) + (\tilde{\nabla}_Y \tilde{S})(Z, X) + (\tilde{\nabla}_Z \tilde{S})(X, Y)
\]  
(I.A.9.2)

Hence we can state the following:

**Theorem I.A.9.1** Cyclic parallel Ricci tensor of a 3-dimensional Kenmotsu manifold with respect to the quater-symmetric metric connection and the Levi-Civita connection are equivalent.
SECTION 10

Locally $\phi$-Symmetric Kenmotsu manifolds with respect to 
the quarter-symmetric metric connection

**Definition I.A.10.1.** A Sasakian manifold is said to be locally $\phi$-symmetric if

$$\phi^2(\nabla_W R)(X,Y)Z = 0$$  \hspace{1cm} (I.A.10.1)

for all vector fields $W, X, Y, Z$ orthogonal to $\xi$. This notion was introduced for Sasakian manifolds by Takahashi [95].

Analogous to the definition of $\phi$-symmetric Sasakian manifold with respect to the Riemannian connection, we define locally $\phi$-symmetric Kenmotsu manifold with respect to the quarter-symmetric metric connection by

$$\phi^2(\tilde{\nabla}_W \tilde{R})(X,Y)Z = 0,$$  \hspace{1cm} (I.A.10.2)

for all vector fields $W, X, Y, Z$ orthogonal to $\xi$.

Using (I.A.10.2) we can write

$$(\tilde{\nabla}_W \tilde{R})(X,Y)Z = (\nabla_W \tilde{R})(X,Y)Z - \eta(W)\phi \tilde{R}(X,Y)Z.$$  \hspace{1cm} (I.A.10.3)

From [13] we know that for a Kenmotsu manifold

$$\tilde{R}(X,Y)Z = R(X,Y)Z + \eta(X)g(\phi Y, Z)\xi$$

$$-\eta(Y)g(\phi X, Z)\xi - \eta(X)\eta(Z)\phi Y$$

$$+\eta(Y)\eta(Z)\phi X.$$  \hspace{1cm} (I.A.10.4)

Using (I.A.p.9),(I.A.10.4) yields

$$\tilde{R}(X,Y)Z = \frac{\theta+\delta}{2}[g(Y, Z)X - g(X, Z)Y]$$

$$-\frac{\theta-\delta}{2}[g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi]$$

$$+\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y$$

$$+\eta(X)g(\phi Y, Z)\xi$$

$$-\eta(Y)g(\phi X, Z)\xi - \eta(X)\eta(Z)\phi Y$$

$$+\eta(Y)\eta(Z)\phi X.$$  \hspace{1cm} (I.A.10.5)

Now differentiating (I.A.10.5) with respect to $W$ and using (I.A.p.4), we get from
\( (\tilde{\nabla}_W \tilde{R})(X, Y)Z = \frac{dr(W)}{2} [g(Y, Z)X - g(X, Z)Y] \)

\(- \frac{dr(W)}{2} [g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi] + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y \)

\(- \frac{(r+6)}{2} [g(Y, Z)(\nabla W\eta)(X)\xi - g(X, Z)(\nabla W\eta)(Y)\xi] + g(Y, Z)\eta(X)\nabla W\xi - g(X, Z)\eta(Y)\nabla W\xi \)

\((- (\nabla W\eta)(Y)\eta(Z)X + \eta(Y)(\nabla W\eta)(Z)X \)

\(- (\nabla W\eta)(X)\eta(Z)Y - \eta(X)(\nabla W\eta)(Z)Y \)

\(+ (\nabla W\eta)(X)g(\phi Y, Z)\xi + \eta(X)g(\phi Y, Z)W \)

\(- \eta(X)g(\phi Y, Z)\eta(W)\xi - (\nabla W\eta)(Y)g(\phi X, Z)\xi \)

\(- \eta(Y)g(\phi X, Z)W + \eta(Y)g(\phi X, Z)\eta(W)\xi \)

\(- g(W, X)\eta(Z)\phi Y + 2\eta(W)\eta(X)\eta(Z)\phi Y \)

\(- \eta(X)g(W, Z)\phi Y - \eta(X)\eta(Z)g(\phi W, Y)\xi \)

\(+ g(W, Y)\eta(Z)\phi X - 2\eta(W)\eta(Y)\eta(Z)\phi X \)

\(+ \eta(Y)g(W, Z)\phi X + \eta(Y)\eta(Z)g(\phi W, X)\xi \)

\(- \eta(W)\phi \tilde{R}(X, Y)Z. \)

\((\text{I.A.10.3})\)

Now taking \( W, X, Y, Z \) are horizontal vector fields, that is, \( W, X, Y, Z \) are orthogonal to \( \xi \), then we get from the above

\( \phi^2(\tilde{\nabla}_W \tilde{R})(X, Y)Z = - \frac{dr(W)}{2} [g(Y, Z)X - g(X, Z)Y]. \)

\((\text{I.A.10.7})\)

Hence we can state the following:

**Theorem I.A.10.1** A 3-dimensional Kenmotsu manifold is locally \( \phi \)-symmetric with respect to the.quarter-symmetric connection if and only if the scalar curvature \( r \) is constant.
PART B

β-Kenmotsu manifolds

An almost contact metric manifold satisfying the condition (I.A.i.1) is called a Kenmotsu manifold ([50],[53]). Again one has the more general notion of a β-Kenmotsu structure [50] which may be defined by

\[(\nabla_X\phi)Y = \beta(g(\phi X,Y)\xi - \eta(Y)\phi X)\] (I.B.i.1)

where β is a non-zero constant. From the condition one may readily deduce that

\[\nabla_X\xi = \beta(X - \eta(X)\xi).\] (I.B.i.2)

Kenmotsu manifolds appear as examples of β-Kenmotsu manifolds, with β = 1. β-Kenmotsu manifolds have been studied by several authors such as Matamba [100], Janssens, and Vanhecke [50] and many others.

In the classification of Gray and Hervella [42] of almost Hermitian manifolds there appears a class, \(W_4\), of Hermitian manifolds which are closely related to locally conformally Kaehler manifolds. An almost contact metric structure \((\phi, \xi, \eta, g)\) on \(M\) is trans-Sasakian [72] if \((M \times \mathbb{R}, J, G)\) belongs to the class \(W_4\), where \(J\) is the almost complex structure on \(M \times \mathbb{R}\) defined by

\[J(X, f \frac{d}{dt}) = (\phi X - f \xi, \eta(X)\frac{d}{dt}),\] for all vector fields \(X\) on \(M\), \(f\) is a smooth function on \(M \times \mathbb{R}\) and \(G\) is the product metric on \(M \times \mathbb{R}\). This may be expressed by the condition [16]

\[(\nabla_X\phi)Y = \alpha(g(X,Y)\xi - \eta(Y)X) + \beta(g(\phi X,Y)\xi - \eta(Y)\phi X)\] (I.B.i.3)

for smooth functions \(\alpha\) and \(\beta\) on \(M\). Hence we say that the trans-Sasakian structure is of type \((\alpha, \beta)\). In particular, it is normal and it generalizes both \(\alpha\)-Sasakian and \(\beta\)-Kenmotsu structures. From the formula one easily obtains

\[\nabla_X\xi = -\alpha(\phi X) + \beta(X - \eta(X)\xi).\] (I.B.i.4)

Hence a trans-Sasakian structure of type \((\alpha, \beta)\) with \(\alpha, \beta \in \mathbb{R}\) and \(\alpha = 0\) is a \(\beta\)-Kenmotsu structure.

In the first section of this chapter we introduce globally \(\phi\)–quasiconformally symmetric \(\beta\)-Kenmotsu manifolds. In the next section we study 3-dimensional locally
$\phi$–quasiconformally symmetric $\beta$-Kenmotsu manifold and in section 3 we study second order parallel tensor. Section 4 deals with the study of Ricci solitons. In the last section we give some examples of such a manifold.

**PRELIMINARIES**

Let $M$ be a connected almost contact metric manifold with an almost contact metric structure $(\phi, \xi, \eta, g)$, that is, $\phi$ is an $(1,1)$ tensor field, $\xi$ is a vector field, $\eta$ is a 1-form and $g$ is a compatible Riemannian metric such that

$$\phi^2(X) = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta\phi = 0 \quad \text{(I.B.p.1)}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \quad \text{(I.B.p.2)}$$

$$g(X, \xi) = \eta(X) \quad \text{(I.B.p.3)}$$

for all $X, Y \in T(M)$. If an almost contact metric manifold satisfies

$$(\nabla_X \phi) Y = \beta(g(\phi X, Y)\xi - \eta(Y)\phi X), \quad \text{(I.B.p.4)}$$

then $M$ is called a $\beta$-Kenmotsu manifold, where $\nabla$ is the Levi-Civita connection of $g$. From the above equation it follows that

$$\nabla_X \xi = \beta(X - \eta(X)\xi), \quad \text{(I.B.p.5)}$$

and

$$(\nabla_X \eta) Y = \beta(g(X, Y) - \eta(X)\eta(Y)). \quad \text{(I.B.p.6)}$$

Moreover the curvature tensor $R$ and the Ricci tensor $S$ satisfy

$$R(X, Y)\xi = \beta(\eta(X)Y - \eta(Y)X) \quad \text{(I.B.p.7)}$$

and

$$S(X, \xi) = -\beta(n - 1)\eta(X). \quad \text{(I.B.p.8)}$$
SECTION 1
Globally $\phi$-quasiconformally symmetric $\beta$-Kenmotsu manifolds

Definition I.B.1.1. A $\beta$-Kenmotsu manifold $M$ is said to be globally $\phi$-quasiconformally symmetric if the quasi-conformal curvature tensor $C^*$ satisfies

$$\phi^2((\nabla_X C^*)(Y, Z) W) = 0,$$ (I.B.1.1)

for all vector fields $X, Y, Z \in \chi(M)$.

It is well-known that if the Ricci tensor $S$ for the manifold is of the form $S(X, Y) = \lambda g(X, Y)$, where $\lambda$ is a constant and $X, Y \in \chi(M)$, then the manifold is called an Einstein manifold.

Let us suppose that $M$ is a globally $\phi$-quasiconformally symmetric $\beta$-Kenmotsu manifold. Then by definition

$$\phi^2((\nabla_W C^*)(X, Y) Z) = 0,$$

Using (I.B.p.1) we have

$$-(\nabla_W C^*)(X, Y) Z + \eta ((\nabla_W C^*)(X, Y) Z) \xi = 0.$$

From (11) it follows that


$$-bg(Y, Z) g((\nabla_W Q) X, U) + bg(X, Z) g((\nabla_W Q) Y, U)$$

$$+\frac{1}{n} \hat{d}r(W) \left[ \frac{a}{n-1} + 2b \right] (g(Y, Z) g(X, U) - g(X, Z) g(Y, U))$$

$$+an ((\nabla_W R)(X, Y) Z) \eta(U)$$

$$+b(\nabla_W S)(Y, Z) \eta(U) \eta(X) - b(\nabla_W S)(X, Z) \eta(U) \eta(Y)$$

$$+bg(Y, Z) \eta ((\nabla_W Q) X) \eta(U) - bg(X, Z) \eta ((\nabla_W Q) Y) \eta(U)$$

$$-\frac{1}{n} \hat{d}r(W) \left[ \frac{a}{n-1} + 2b \right] (g(Y, Z) \eta(X) - g(X, Z) \eta(Y)) \eta(U) = 0$$

Putting $X = U = e_i$, where $\{e_i\}, (i = 1, 2, ..., n)$ is an orthonormal basis of the tangent space at each point of the manifold, and taking summation over $i$, we get

$$-(a + nb - 2b) (\nabla_W S)(Y, Z) - \{ bg ((\nabla_W Q) e_i, e_i) - \frac{n-1}{n} \hat{d}r(W) \left( \frac{a}{n-1} + 2b \right) \} g(Y, Z) + bg((\nabla_W Q) Y, Z)$$

$$-bn ((\nabla_W Q) e_i, e_i) \eta(e_i) + \frac{1}{n} \hat{d}r(W) \left( \frac{a}{n-1} + 2b \right) \} g(Y, Z) + bg((\nabla_W Q) Y, Z)$$

$$+an ((\nabla_W R)(e_i, Y) Z) \eta(e_i) - b(\nabla_W S)(\xi, Z) \eta(Y) - bn ((\nabla_W Q) Y) \eta(Z)$$

$$+\frac{1}{n} \hat{d}r(W) \left( \frac{a}{n-1} + 2b \right) \} \eta(Y) \eta(Z) = 0.$$
Putting $Z = \xi$, we obtain

$$-(a + nb - 2b) (\nabla_W S) (Y, \xi) - \eta(Y) \{ bdr(W) - \frac{n-1}{n} dr(W) \left( \frac{a}{n-1} + 2b \right) - b\eta((\nabla_W Q) e_i) \eta(e_i) + \ldots \} = 0.$$  

Putting $Y = \xi$ in (I.B.1.6) we get $dr(W) = 0$. This implies $r$ is constant. So from (I.B.1.6), we have

$$(\nabla_W S) (Y, \xi) = 0. \quad \text{(I.B.1.2)}$$

Now

$$\eta((\nabla_W Q) e_i) \eta(e_i) = g((\nabla_W Q) e_i, \xi) = \eta((\nabla_W Q) \xi) = g(Q\phi X, \xi) = S(\phi X, \xi) = 0. \quad \text{(I.B.1.3)}$$

$$\eta((\nabla_W R) (e_i, Y) \xi) \eta(e_i) = g((\nabla_W R) (e_i, Y) \xi, \xi) \eta(e_i) = 0. \quad \text{(I.B.1.4)}$$

Since $\{e_i\}$ is an orthonormal basis $\nabla_X e_i = 0$ and using (I.B.p.7) we find

$$g(R(e_i, \nabla_W Y) \xi, \xi) = \beta(\eta(e_i) \nabla_W Y - \eta(\nabla_W Y) e_i, \xi)$$

$$\quad = \beta(\eta(e_i) \eta(\nabla_W Y) - \eta(\nabla_W Y) \eta(e_i))$$

$$\quad = 0.$$

As

$$g(R(e_i, Y) \xi, \xi) + g(R(\xi, \xi) Y, e_i) = 0$$

we have

$$g((\nabla_W R) (e_i, Y) \xi) + g(R(e_i, Y) \xi, \nabla_W Y) = 0.$$

Using this we get

$$g((\nabla_W R) (e_i, Y) \xi, \xi) = 0. \quad \text{(I.B.1.5)}$$

By the use of (I.B.1.3), (I.B.1.4) and (I.B.1.5), from (I.B.1.2) we obtain

$$(\nabla_W S) (Y, \xi) = \frac{1}{n} dr(W) \eta(Y), \quad \text{(I.B.1.6)}$$

since $a + (n - 2)b \neq 0$. Because if $a + (n - 2)b = 0$ then from (10), it follows that $C^* = aC$. So we can not take $a + (n - 2)b = 0$. Putting $Y = \xi$ in (I.B.1.6) we get $dr(W) = 0$. This implies $r$ is constant. So from (I.B.1.6), we have

$$(\nabla_W S) (Y, \xi) = 0.$$
Using (I.B.p.8), this implies

\[ S(Y, W) = \lambda g(Y, W), \quad (I.B.1.7) \]

where \( \lambda = -\beta(n - 1) \). Hence we can state the following:

**Theorem I.B.1.1.** If a \( \beta \)-Kenmotsu manifold is globally \( \phi \)-quasiconformally symmetric, then the manifold is an Einstein manifold.

Next suppose \( S(X, Y) = \lambda g(X, Y) \), i.e. \( QX = \lambda X \). Then from (10) we have

\[
C^*(X, Y)Z = aR(X, Y)Z \\
+ \left[ 2\delta \lambda - \frac{\ell}{n} \left( \frac{a}{n-1} + 2\delta \right) \right] [g(Y, Z)X - g(X, Z)Y],
\]

which gives us

\[
(\nabla_W C^*) (X, Y)Z = a (\nabla_W R) (X, Y)Z.
\]

Applying \( \phi^2 \) on both sides of the above equation we have

\[
\phi^2 (\nabla_W C^*) (X, Y)Z = a \phi^2 (\nabla_W R) (X, Y)Z.
\]

Hence we can state:

**Theorem I.B.1.2.** A globally \( \phi \)-quasiconformally symmetric \( \beta \)-Kenmotsu manifold is globally \( \phi \)-symmetric.

**Remark I.B.1.1.** Since a globally \( \phi \)-symmetric \( \beta \)-Kenmotsu manifold is always a globally \( \phi \)-quasiconformally symmetric manifold, from Theorem I.B.1.2 we conclude that on a \( \beta \)-Kenmotsu manifold, globally \( \phi \)-symmetry and globally \( \phi \)-quasiconformally symmetry are equivalent.

### SECTION 2

3-dimensional locally \( \phi \)-quasiconformally symmetric \( \beta \)-Kenmotsu manifolds

Let us consider a 3-dimensional \( \beta \)-Kenmotsu manifold. It is known that the conformal curvature tensor vanishes identically in the 3-dimensional Riemannian manifold. Thus we find

\[
R(X, Y)Z = g(Y, Z)QX - g(X, Z)QY + S(Y, Z)X - S(X, Z)Y \\
- \frac{\ell}{2} [g(Y, Z)X - g(X, Z)Y], \quad (I.B.2.1)
\]

where $Q$ is the Ricci operator, that is, $g(QX,Y) = S(X,Y)$ and $r$ is the scalar curvature of the manifold.

Putting $Z = \xi$ in (I.B.2.1) and using (I.B.p.8) we have

$$\eta(Y)QX - \eta(X)QY = \left(\frac{r}{2} + \beta\right)[\eta(Y)X - \eta(X)Y]. \quad \text{(I.B.2.2)}$$

Putting $Y = \xi$ in (I.B.2.2) and using (I.B.2.1) and (I.B.p.8), we get

$$QX = \frac{1}{2}(r + 2\beta)X - (r + 6\beta)\eta(X)\xi, \quad \text{(I.B.2.3)}$$

that is,

$$S(X,Y) = \frac{1}{2}(r + 2\beta)g(X,Y) - (r + 6\beta)\eta(X)\eta(Y). \quad \text{(I.B.2.4)}$$

Using (I.B.2.3) in (I.B.2.1), we get

$$R(X,Y)Z = (r + 4\beta)[g(Y,Z)X - g(X,Z)Y] - (r + 6\beta)[g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi - g(X,Z)\eta(X)\eta(Y)Z]. \quad \text{(I.B.2.5)}$$

Putting (I.B.2.3), (I.B.2.4) and (I.B.2.5) into (10) we have

$$C^*(X,Y)Z = (a + b)(r + 6\beta)[\frac{1}{6}g(Y,Z)X - g(X,Z)Y] \quad \text{(I.B.2.6)}$$

Thus we have

**Lemma I.B.2.1.** Let $M$ be a 3-dimensional $\beta$-Kenmotsu manifold.

*If* $a + b = 0$ *or* $r = -6\beta$, *then the quasi-conformal curvature tensor vanishes identically.*

Next, we assume that $a + b \neq 0$ or $r \neq -6\beta$. Taking the covariant differentiation of (I.B.2.6), we get

$$(\nabla_W C^*)(X,Y)Z = \frac{dr(W)}{3}(a + b)\{g(Y,Z)X - g(X,Z)Y\} \quad \text{(I.B.2.7)}$$
If the vector fields $X$, $Y$, and $Z$ are horizontal, then the above equation is rewritten as follows:

$$(\nabla_{C^*}W)(X,Y)Z = \frac{dr(W)}{3}(a+b)\{g(Y,Z)X - g(X,Z)Y\}
- \frac{1}{2}(r + 6\beta)(a + b)[g(Y,Z)(\nabla W\eta)(X) - g(X,Z)(\nabla W\eta)(Y)]\xi.$$ \text{(I.B.2.8)}

Operating $\phi^2$ to the above equation, then we find

$$\phi^2((\nabla_{C^*}W)(X,Y)Z) = -\frac{dr(W)}{3}(a + b)\{g(Y,Z)X - g(X,Z)Y\}. \text{ (I.B.2.9)}$$

Hence we conclude the following theorem:

**Theorem I.B.2.1.** A 3-dimensional $\beta$-Kenmotsu manifold is locally $\phi$-quasiconformally symmetric if and only if the scalar curvature $r$ is constant if $a + b \neq 0$ and $r \neq -6\beta$.

If $\beta = 1$, then the manifold reduces to a Kenmotsu manifold. Thus from the above theorem we get the following:

**Corollary I.B.2.1.** A 3-dimensional Kenmotsu manifold is locally $\phi$-quasiconformally symmetric if and only if the scalar curvature $r$ is constant if $a + b \neq 0$ and $r \neq -6$.

**SECTION 3**

Second order parallel tensor

Let us consider a parallel symmetric $(0,2)$-tensor $\delta$ on a 3-dimensional $\beta$-Kenmotsu manifold $M$.

Then, by $\nabla \delta = 0$, we have

$$\delta(R(U,V)X,Y) + \delta(X,R(U)V)Y = 0, \text{ (I.B.3.1)}$$

where $U, V, X$ and $Y$ are arbitrary vector fields on $M$.

As $\delta$ is symmetric, putting $U = X = Y = \xi$ in (I.B.3.1), we obtain

$$\delta(\xi, R(\xi, X))\xi = 0. \text{ (I.B.3.2)}$$

Now applying (I.B.p.7) in (I.B.3.2) we have

$$\beta\delta(Y,\xi) - \beta\eta(Y)\delta(\xi, \xi) = 0. \text{ (I.B.3.3)}$$

Differentiating (I.B.3.3) covariantly along $X$ we find

$$\beta\{\delta(\nabla_XY,\xi) + \delta(Y,\nabla_X\xi)\} - \beta\{g(\nabla_XY,\xi) + g(Y,\nabla_X\xi)\}\delta(\xi, \xi) - 2\beta g(Y,\xi)\delta(\nabla_X\xi, \xi) = 0. \text{ (I.B.3.4)}$$
Putting \( Y = \nabla_X Y \) in (I.B.3.2) we get
\[
\beta\{\delta(\nabla_X Y, \xi) - \beta\eta(\nabla_X Y)\delta(\xi, \xi)\} = 0. \tag{I.B.3.5}
\]
From (I.B.3.4) and (I.B.3.5) we have
\[
\beta\delta(Y, \nabla_X \xi) - \beta g(Y, \nabla_X \xi)\delta(\xi, \xi) - 2\beta g(Y, \xi)\delta(\nabla_X \xi, \xi) = 0,
\]
which implies that
\[
\beta^2\{\delta(Y, X) - g(Y, X)\delta(\xi, \xi)\} = 0.
\]
This implies either
\[
\delta(Y, X) = \delta(\xi, \xi)g(Y, X), \quad \text{or}, \quad \beta = 0. \tag{I.B.3.6}
\]
Since \( \delta \) and \( g \) are parallel tensor fields, \( \lambda = \delta(\xi, \xi) \) is constant on \( U \). By the parallelity of \( \delta \) and \( g \) it must be \( \lambda = \lambda g \) on whole of \( M \). Thus we have the following:

**Theorem I.B.3.1.** A parallel symmetric \((0,2)\) tensor in a 3-dimensional non-cosymplectic \( \beta \)-Kenmotsu manifold is a constant multiple of the associated metric tensor.

### SECTION 4

**Ricci solitons**

Suppose a 3-dimensional \( \beta \)-Kenmotsu manifold admits a Ricci soliton defined by (19). It is well known that \( \nabla g = 0 \). Since \( \lambda \) in the Ricci soliton equation (19) is a constant, so \( \nabla \lambda g = 0 \). Thus \( \mathcal{L}_V g + 2S \) is parallel. Hence using the previous theorem we have \( \mathcal{L}_V g + 2S \) is a constant multiple of metric tensors \( g \), that is, \( \mathcal{L}_V g + 2S = ag \), where \( a \) is constant. Hence \( \mathcal{L}_V g + 2S + 2\lambda g \) reduces to \( (a + 2\lambda)g \), that implies \( \lambda = -a/2 \). So we have the following:

**Theorem I.B.4.1.** In a 3-dimensional non-cosymplectic \( \beta \)-Kenmotsu manifold, the Ricci soliton \((g, V, \lambda)\) is shrinking or expanding according as \( a \) is positive or negative.

Now in particular we investigate the case \( V = \xi \). Then (19) reduces to
\[
\mathcal{L}_\xi g + 2S + 2\lambda g = 0. \tag{I.B.4.1}
\]
Using (I.B.p.5) in a 3-dimensional $\beta$-Kenmotsu manifold we have
\[
\mathcal{L}_\xi g(Y, Z) = 2\beta(g(Y, Z) - \eta(Y)\eta(Z)).
\] (I.B.4.2)

Then using (I.B.4.2) in (I.B.4.1) we get $\lambda = -S(\xi, \xi) = \beta(n - 1)$. Also from (I.B.4.1) it follows that the manifold is an $\eta$-Einstein manifold. Thus we have

**Corollary I.B.4.1.** In a 3-dimensional non-cosymplectic $\beta$-Kenmotsu manifold, the Ricci soliton $(g, \xi, \lambda)$ is shrinking and the manifold is an $\eta$-Einstein manifold.

## SECTION 5

**Examples of 3-dimensional $\beta$-Kenmotsu manifolds**

**Example I.B.5.1:** We consider the 3-dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3, z \neq 0\}$, where $(x, y, z)$ are standard co-ordinate of $\mathbb{R}^3$.

The vector fields
\[
e_1 = e^z \frac{\partial}{\partial x}, \quad e_2 = e_z(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}), \quad e_3 = \alpha \frac{\partial}{\partial z}
\]

are linearly independent at each point of $M$, where $\alpha$ is constant.

Let $g$ be the Riemannian metric defined by
\[
g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1
\]
\[
g(e_1, e_3) = g(e_1, e_2) = g(e_2, e_3) = 0,
\]

Let $\eta$ be the 1-form defined by $\eta(Z) = g(Z, e_3)$ for any $Z \in \chi(M)$.

Let $\phi$ be the $(1,1)$ tensor field defined by
\[
\phi(e_1) = -e_2, \quad \phi(e_2) = e_1, \quad \phi(e_3) = 0.
\]

Then using the linearity of $\phi$ and $g$, we have
\[
\eta(e_3) = 1,
\]
\[
\phi^2 Z = -Z + \eta(Z)e_3,
\]
\[
g(\phi Z, \phi W) = g(Z, W) - \eta(Z)\eta(W),
\]
for any $Z, W \in \chi(M)$.

Then for $e_3 = \xi$, the structure $(\phi, \xi, \eta, g)$ defines an almost contact metric structure.
on $M$.
Let $\nabla$ be the Levi-Civita connection with respect to metric $g$. Then we have $[e_1, e_2] = 0$, $[e_1, e_3] = -\alpha e_1$ and $[e_2, e_3] = -\alpha e_2$.
Taking $e_3 = \xi$ and using Koszul formula for the Riemannian metric $g$, we can easily calculate
\[
\nabla_{e_1}e_1 = \alpha e_3, \quad \nabla_{e_1}e_2 = 0, \quad \nabla_{e_1}e_3 = -\alpha e_1,
\n\nabla_{e_2}e_1 = 0, \quad \nabla_{e_2}e_2 = -\alpha e_3, \quad \nabla_{e_2}e_3 = -\alpha e_2,
\n\nabla_{e_3}e_1 = 0, \quad \nabla_{e_3}e_2 = 0, \quad \nabla_{e_3}e_3 = 0.
\]

We see that the structure $(\phi, \xi, \eta, g)$ satisfies the formula (I.B.p.5) for $\beta = -\alpha$.
Hence the manifold is a $\beta$-Kenmotsu manifold with $\beta =$ constant.

**Example I.B.5.2:** We consider the 3-dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3, z \neq 0\}$, where $(x, y, z)$ are standard co-ordinate of $\mathbb{R}^3$.
The vector fields
\[
e_1 = z \frac{\partial}{\partial x}, \quad e_2 = z \frac{\partial}{\partial y}, \quad e_3 = z \frac{\partial}{\partial z}
\]
are linearly independent at each point of $M$.
Let $g$ be the Riemannian metric defined by
\[
g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1
\]
\[
g(e_1, e_3) = g(e_1, e_2) = g(e_2, e_3) = 0,
\]
that is, the form of the metric becomes
\[
g = \frac{dx^2 + dy^2 + dz^2}{z^2}.
\]
Let $\eta$ be the 1-form defined by $\eta(Z) = g(Z, e_3)$ for any $Z \in \chi(M)$.
Let $\phi$ be the $(1, 1)$ tensor field defined by
\[
\phi(e_1) = -e_2, \quad \phi(e_2) = e_1, \quad \phi(e_3) = 0.
\]
Then using the linearity of $\phi$ and $g$, we have
\[
\eta(e_3) = 1,
\]
\[
\phi^2 Z = -Z + \eta(Z)e_3,
\]
\[
g(\phi Z, \phi W) = g(Z, W) - \eta(Z)\eta(W),
\]
for any \( Z, W \in \chi(M) \).
Then for \( e_3 = \xi \), the structure \((\phi, \xi, \eta, g)\) defines an almost contact metric structure on \( M \).

Let \( \nabla \) be the Levi-Civita connection with respect to metric \( g \). Then we have

\[
[e_1, e_3] = e_1 e_3 - e_3 e_1
\]
\[
= z \frac{\partial}{\partial x} (z \frac{\partial}{\partial z} \xi) - z \frac{\partial}{\partial z} (z \frac{\partial}{\partial x} \xi)
\]
\[
= z^2 \frac{\partial^2}{\partial x \partial z} - z^2 \frac{\partial^2}{\partial z \partial x} - z \frac{\partial}{\partial x}
\]
\[
= -e_1.
\]

Similarly, \([e_1, e_2] = 0 \) and \([e_2, e_3] = -e_2 \).

The Riemannian connection \( \nabla \) of the metric \( g \) is given by

\[
2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]),
\]
which known as Koszul’s formula.

Using (I.B.5.1) we have

\[
2g(\nabla_{e_1} e_3, e_1) = -2g(e_1, e_1) = 2g(-e_1, e_1).
\]

Again by (I.B.5.1)

\[
2g(\nabla_{e_1} e_3, e_2) = 0 = 2g(-e_1, e_2)
\]
and

\[
2g(\nabla_{e_1} e_3, e_3) = 0 = 2g(-e_1, e_3).
\]

From (I.B.5.2), (I.B.5.3) and (I.B.5.4) we obtain

\[
2g(\nabla_{e_1} e_3, X) = 2g(-e_1, X),
\]
for all \( X \in \chi(M) \).
Thus

\[
\nabla_{e_1} e_3 = -e_1.
\]

Therefore, (I.B.5.1) further yields

\[
\nabla_{e_1} e_1 = e_3, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = -e_1,
\]
\[
\nabla_{e_2} e_1 = 0, \quad \nabla_{e_2} e_2 = e_3, \quad \nabla_{e_2} e_3 = -e_2.
\]
\[ \nabla_{e_3} e_1 = 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_3 = 0. \tag{I.B.5.5} \]

(I.B.5.5) tells us that the manifold satisfies (I.B.p.5) for \( \beta = -1 \) and \( \xi = e_3 \). Hence the manifold is a \( \beta \)-Kenmotsu manifold with \( \beta = \text{constant} \).

It is known that

\[ R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{I.B.5.6} \]

With the help of the above results and using (I.B.5.6) it can be easily verified that

\[ R(e_1, e_2)e_3 = 0, \quad R(e_2, e_3)e_3 = -e_2, \quad R(e_1, e_3)e_3 = -e_1, \]
\[ R(e_1, e_2)e_2 = -e_1, \quad R(e_2, e_3)e_2 = e_3, \quad R(e_1, e_3)e_2 = 0, \]
\[ R(e_1, e_2)e_1 = e_2, \quad R(e_2, e_3)e_1 = 0, \quad R(e_1, e_3)e_1 = e_3. \]

From the above expressions of the curvature tensor we obtain

\[ S(e_1, e_1) = g(R(e_1, e_2)e_2, e_1) + g(R(e_1, e_3)e_3, e_1) = -2. \]

Similarly we have

\[ S(e_2, e_2) = S(e_3, e_3) = -2. \]

Therefore,

\[ r = S(e_1, e_1) + S(e_2, e_2) + S(e_3, e_3) = -6. \]

Thus the scalar curvature \( r \) is constant. Hence Theorem I.B.2.1. is verified.