Chapter 6

Ricci Tensor for \textit{GCR}-Lightlike Submanifolds of Indefinite Kaehler Manifolds

6.1 Introduction

In this chapter, we obtain the expression of Ricci tensor for a \textit{GCR}-lightlike submanifold of indefinite complex space form and discuss the properties of Ricci tensor on totally geodesic \textit{GCR}-lightlike submanifold of an indefinite complex space form. We also prove that every proper totally umbilical \textit{GCR}-lightlike submanifold of an indefinite Kaehler manifold is a totally geodesic \textit{GCR}-lightlike submanifold.

6.2 Ricci Tensor of \textit{GCR}-lightlike Submanifold

Let \( \{E_1, E_2, ..., E_m\} \) be a local orthonormal frame field on \( M \) such that

\[
\{E_1, E_2, ..., E_p, E_{p+1} = JE_1, E_{p+2} = JE_2, ..., E_{2p} = JE_p\}, \ \{\xi_1, \xi_2, ..., \xi_s, \xi_{s+1} = J\xi_1, \xi_{s+2} = J\xi_2, ..., \xi_{2s} = J\xi_s\}, \ \{\xi_{2s+1}, \xi_{2s+2}, ..., \xi_r\} \text{ and } \{J\xi_{2s+1}, J\xi_{2s+2}, ..., J\xi_r\}
\]

be local frame fields on \( D_0, D_1, D_2 \) and \( JD_2 \), respectively and \( \{F_1, F_2, ..., F_q\} \) be a
local frame field on \( D' \), then by direct computation, we have
\[
\sum_{i=1}^{m} g(U, E_i)g(E_i, V) = g(U, V),
\]
\[
\sum_{i=r+1}^{m} g(U, E_i)g(E_i, V) = g(\bar{P}U, \bar{P}V),
\]
\[
\sum_{i=1}^{m-q} g(U, E_i)g(E_i, V) = g(QU, QV)
\]
and
\[
\sum_{i=1}^{q} g(U, E_i)g(E_i, V) = g(\bar{P}U, \bar{P}V),
\]
for any \( U, V \in \Gamma(TM) \) and the Ricci tensor is given by
\[
\mathrm{Ric}(U, V) = \sum_{a=1}^{r} g(R(U, \xi_a)V, N_a) + \sum_{b=r+1}^{m} g(R(U, U_b)V, U_b).
\]
Using (5.2.1), we obtain
\[
\sum_{a=1}^{r} g(R(U, \xi_a)V, N_a) = -\frac{c}{4} \sum_{a=1}^{r} g(JV, \xi_a)g(JU, N_a) - \frac{cr}{4} g(U, V)
\]
\[
-\frac{c}{4} \sum_{a=1}^{r} g(JU, V)g(J\xi_a, N_a)
\]
\[
-\frac{c}{2} \sum_{a=1}^{r} g(JU, \xi_a)g(JV, N_a)
\]
\[
+ \sum_{a=1}^{r} g(A_{h^*}(\xi_a, V)U, N_a) - \sum_{a=1}^{r} g(A_{h^*}(U, V)\xi_a, N_a)
\]
\[
+ \sum_{a=1}^{r} g(A_{h^*}(\xi_a, V)U, N_a)
\]
\[
- \sum_{a=1}^{r} g(A_{h^*}(U, V)\xi_a, N_a).
\]
Now, using equation (2.30) of [15] at page 158, for any \( U \in \Gamma(T(M)) \), define a differential 1-form as
\[
\eta_a(U) = \bar{g}(U, N_a), \forall a \in \{1, 2, ..., r\},
\]
then any vector field $U$ on $M$ can be expressed as

$$\mathbf{\bar{P}U + \sum_{a=1}^{r} \eta_a(U)\xi_a},$$

(6.2.7)

where $\mathbf{\bar{P}}$ is the projection morphism of $TM$ on $S(TM)$. Therefore, we have

$$g(U, JV) = g(\mathbf{\bar{P}}U, JV) + \sum_{a=1}^{r} \bar{g}(U, N_a)g(\xi_a, JV).$$

(6.2.8)

Also, using (1.3.13), (1.3.14) and (6.2.8) in (6.2.6), we obtain

$$\sum_{a=1}^{r} g(R(U, \xi_a)V, N_a) = -\frac{c(r+3)}{4}g(U, V) + \frac{3c}{4}g(TU, TV)$$

$$-\frac{c}{4}\sum_{a=1}^{r} g(JU, V)g(J\xi_a, N_a) - \sum_{a=1}^{r} g(A_{N_a}U, h^l(\xi_a, V))$$

$$+ \sum_{a=1}^{r} g(A_{N_a}\xi_a, h^l(U, V)) + \sum_{a=1}^{r} g(D^s(U, N_a), h^s(\xi_a, V))$$

$$- \sum_{a=1}^{r} g(D^s(\xi_a, N_a), h^s(U, V)).$$

(6.2.9)

Using (1.3.11), (1.3.19), (5.2.1) and (6.2.2), we obtain

$$\sum_{b=r+1}^{m} g(R(U, U_b)V, U_b) = -\frac{c}{2}g(\mathbf{\bar{P}}U, \mathbf{\bar{P}}V) - \frac{(m - r)c}{4}g(U, V)$$

$$+ \sum_{b=r+1}^{m} \{ g(h^l(U_b, V), h^s(U_b))$$

$$- g(h^l(U, V), h^s(U_b, U_b))$$

$$+ g(h^s(U_b, V), h^s(U, U_b))$$

$$- g(h^s(U, V), h^s(U_b, U_b)) \}.$$

(6.2.10)
Thus substituting (6.2.9) and (6.2.10) in (6.2.5), we obtain the expression of Ricci tensor of a $GCR$-lightlike submanifold as

\[
Ric(U, V) = -\frac{(m + 3)c}{4} g(U, V) + \frac{3c}{4} g(TU, TV) - \frac{c}{2} g(\bar{P}U, \bar{P}V) \\
- \frac{c}{4} \sum_{a=1}^{r} g(JU, V)g(J\xi_{a}, N_{a}) + \sum_{a=1}^{r} g(A_{N_{a}}\xi_{a}, h^{l}(U, V)) \\
- \sum_{a=1}^{r} g(A_{N_{a}}U, h^{l}(\xi_{a}, V)) + \sum_{a=1}^{r} g(D^{s}(U, N_{a}), h^{s}(\xi_{a}, V)) \\
- \sum_{a=1}^{r} g(D^{s}(\xi_{a}, N_{a}), h^{s}(U, V)) - \sum_{b=r+1}^{m} g(h^{l}(U, V), h^{s}(U_{b}, U_{b})) \\
+ \sum_{b=r+1}^{m} g(h^{l}(U_{b}, V), h^{s}(U, U_{b})) - \sum_{b=r+1}^{m} g(h^{s}(U, V), h^{s}(U_{b}, U_{b})) \\
+ \sum_{b=r+1}^{m} g(h^{s}(U_{b}, V), h^{s}(U, U_{b})).
\]

(6.2.11)

Next, using orthonormal frame fields on $D^{i}$, $D_{0}$, $JD_{2}$ and $Rad(TM)$, we can also define Ricci tensors as

\[
Ric(D^{i}(U, V)) = \sum_{i=1}^{q} g(R(U, F_{i})V, F_{i}) + \sum_{k=1}^{2p} g(R(U, E_{k})V, E_{k}) \\
+ \sum_{l=2s+1}^{r} g(R(U, J\xi_{l})V, JN_{l}) \\
+ \sum_{a=1}^{r} g(R(U, \xi_{a})V, N_{a}),
\]

(6.2.12)

therefore, we have

\[
Ric_{D}(U, V) = \sum_{k=1}^{2p} g(R(U, E_{k})V, E_{k}) + \sum_{l=2s+1}^{r} g(R(U, J\xi_{l})V, JN_{l}) \\
+ \sum_{a=1}^{r} g(R(U, \xi_{a})V, N_{a}),
\]

(6.2.13)

\[
Ric_{D'}(U, V) = \sum_{i=1}^{q} g(R(U, F_{i})V, F_{i}).
\]

(6.2.14)
Using (1.3.13), (1.3.19), (5.2.1) and (6.2.3), we obtain
\[
\sum_{k=1}^{2p} g(R(U, E_k)V, E_k) = -\frac{c}{2}g(QU, QV) - \frac{pc}{2}g(U, V) + \sum_{k=1}^{2p} g(h^l(E_k, V), h^*(U, E_k)) \\
- \sum_{k=1}^{2p} g(h^l(U, V), h^*(E_k, E_k)) + \sum_{k=1}^{2p} g(h^*(E_k, V), h^*(U, E_k)) \\
- \sum_{k=1}^{2p} g(h^*(U, V), h^*(E_k, E_k)) \quad (6.2.15)
\]

and
\[
\sum_{l=2s+1}^{r} g(R(U, J_{\xi_l})V, J_{N_l}) = -\frac{c(r - 2s)}{4}g(U, V) + \frac{c}{4} \sum_{l=2s+1}^{r} g(J_{\xi_l}, V)g(U, J_{N_l}) \\
- \sum_{l=2s+1}^{r} g(A_{h^l(U,V)}J_{\xi_l}, J_{N_l}) + \sum_{l=2s+1}^{r} g(A_{h^*(J_{\xi_l},V)}U, J_{N_l}) \\
- \sum_{l=2s+1}^{r} g(A_{h^*(U,V)}J_{\xi_l}, J_{N_l}) \\
+ \sum_{l=2s+1}^{r} g(A_{h^*(J_{\xi_l},V)}U, J_{N_l}) \quad (6.2.16)
\]
Using (6.2.9), (6.2.15) and (6.2.16) in (6.2.13), we obtain

\[ \text{Ric}_D(U, V) = \frac{3c}{4} g(TU, TV) - \frac{c(2p + 2r - 2s + 3)}{4} g(U, V) - \frac{c}{2} g(QU, QV) \]

\[- \frac{c}{4} \sum_{a=1}^{r} g(JU, V)g(J\xi_a, N_a) + \frac{c}{4} \sum_{l=2s+1}^{r} g(J\xi_l, V)g(U, JN_l) \]

\[+ \sum_{a=1}^{r} g(A_{N_a} \xi_a, h^l(U, V)) - \sum_{a=1}^{r} g(A_{N_a} U, h^l(\xi_a, V)) \]

\[- \sum_{a=1}^{2p} g(D^a(\xi_a, N_a), h^s(U, V)) + \sum_{a=1}^{2p} g(D^a(U, N_a), h^s(\xi_a, V)) \]

\[+ \sum_{k=1}^{2p} g(h^l(E_k, V), h^s(U, E_k)) - \sum_{k=1}^{2p} g(h^l(U, V), h^s(E_k, E_k)) \]

\[+ \sum_{k=1}^{2p} g(h^s(E_k, V), h^s(U, E_k)) - \sum_{k=1}^{2p} g(h^s(U, V), h^s(E_k, E_k)) \]

\[ - \sum_{l=2s+1}^{r} g(A_{h^l(U, V)} J\xi_l, JN_l) + \sum_{l=2s+1}^{r} g(A_{h^l(J\xi_l, V)} U, JN_l) \]

\[- \sum_{l=2s+1}^{r} g(A_{h^s(U, V)} J\xi_l, JN_l) \]

\[+ \sum_{l=2s+1}^{r} g(A_{h^s(J\xi_l, V)} U, JN_l). \quad (6.2.17) \]

Also using (1.3.11), (1.3.19), (5.2.1) and (6.2.4), we obtain

\[ \text{Ric}_D'(U, V) = -\frac{c}{2} g(PU, PV) - \frac{qc}{4} g(U, V) - \sum_{i=1}^{q} g(h^l(U, V), h^s(F_i, F_i)) \]

\[+ \sum_{i=1}^{q} g(h^l(F_i, V), h^s(U, F_i)) - \sum_{i=1}^{q} g(h^s(F_i, F_i), h^s(U, V)) \]

\[+ \sum_{i=1}^{q} g(h^s(F_i, V), h^s(U, F_i)). \quad (6.2.18) \]
Let $X, Y \in \Gamma(D)$ and $Z, W \in \Gamma(D')$, then particularly, we have

$$Ric_{D'}(X, Y) = -\frac{qC}{4}g(X, Y) - \sum_{i=1}^{q} g(h^l(X, Y), h^*(F_i, F_i))$$

$$+ \sum_{i=1}^{q} g(h^l(F_i, Y), h^*(X, F_i)) - \sum_{i=1}^{q} g(h^*(F_i, F_i), h^*(X, Y))$$

$$+ \sum_{i=1}^{q} g(h^*(F_i, Y), h^*(X, F_i)), \quad (6.2.19)$$

$$Ric_D(X, Y) = \frac{c(s - r - p - 1)}{2}g(X, Y) - \frac{c}{4} \sum_{a=1}^{r} g(JX, Y)g(J\xi_a, N_a)$$

$$+ \sum_{a=1}^{r} g(A_{N_a}\xi_a, h^l(X, Y)) - \sum_{a=1}^{r} g(A_{N_a}X, h^l(\xi_a, Y))$$

$$- \sum_{a=1}^{r} g(D^s(\xi_a, N_a), h^s(X, Y)) + \sum_{a=1}^{r} g(D^s(X, N_a), h^s(\xi_a, Y))$$

$$+ \sum_{k=1}^{2p} g(h^l(E_k, Y), h^s(X, E_k)) - \sum_{k=1}^{2p} g(h^l(X, Y), h^s(E_k, E_k))$$

$$+ \sum_{k=1}^{2p} g(h^*(E_k, Y), h^*(X, E_k)) - \sum_{k=1}^{2p} g(h^*(X, Y), h^*(E_k, E_k))$$

$$- \sum_{l=2s+1}^{r} g(A_{h^l(X, Y)}J\xi_l, JN_l) + \sum_{l=2s+1}^{r} g(A_{h^l(J\xi_l, Y)}X, JN_l)$$

$$- \sum_{l=2s+1}^{r} g(A_{h^*(J\xi_l, Y)}X, JN_l)$$

$$+ \sum_{l=2s+1}^{r} g(A_{h^*(J\xi_l, Y)}X, JN_l), \quad (6.2.20)$$
\[ \text{Ric}_{D'}(X, Z) = \sum_{i=1}^{q} g(h^l(X, Z), h^s(F_i, F_i)) + \sum_{i=1}^{q} g(h^l(F_i, Z), h^s(X, F_i)) \]
\[ - \sum_{i=1}^{q} g(h^s(F_i, F_i), h^s(X, Z)) \]
\[ + \sum_{i=1}^{q} g(h^s(F_i, Z), h^s(X, F_i)), \quad (6.2.21) \]

\[ \text{Ric}_D(X, Z) = \frac{c}{4} \sum_{l=2s+1}^{r} g(J_{\xi_l}, Z)g(X, J_{N_l}) + \sum_{a=1}^{r} g(A_{N_a} \xi_a, h^l(X, Z)) \]
\[ - \sum_{a=1}^{r} g(A_{N_a} X, h^l(\xi_a, Z)) - \sum_{a=1}^{r} g(D^s(\xi_a, N_a), h^s(X, Z)) \]
\[ + \sum_{a=1}^{r} g(D^s(X, N_a), h^s(\xi_a, Z)) + \sum_{k=1}^{2p} g(h^l(E_k, Z), h^s(X, E_k)) \]
\[ - \sum_{k=1}^{2p} g(h^s(X, Z), h^s(E_k, E_k)) - \sum_{k=1}^{2p} g(h^s(E_k, Z), h^s(X, E_k)) \]
\[ - \sum_{k=1}^{2p} g(h^s(X, Z), h^s(E_k, E_k)) - \sum_{l=2s+1}^{r} g(A_{h^l(X,Z)} J_{\xi_l}, J_{N_l}) \]
\[ + \sum_{l=2s+1}^{r} g(A_{h^l(J_{\xi_l}, Z)} X, J_{N_l}) - \sum_{l=2s+1}^{r} g(A_{h^s(X,Z)} J_{\xi_l}, J_{N_l}) \]
\[ + \sum_{l=2s+1}^{r} g(A_{h^s(J_{\xi_l}, Z)} X, J_{N_l}) \quad (6.2.22) \]

and

\[ \text{Ric}_{D'}(Z, W) = -\frac{(q + 2)c}{4} g(Z, W) - \sum_{i=1}^{q} g(h^l(Z, W), h^s(F_i, F_i)) \]
\[ + \sum_{i=1}^{q} g(h^l(F_i, W), h^s(Z, F_i)) - \sum_{i=1}^{q} g(h^s(F_i, F_i), h^s(Z, W)) \]
\[ + \sum_{i=1}^{q} g(h^s(F_i, W), h^s(Z, F_i)). \quad (6.2.23) \]

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Thus from (6.2.19) to (6.2.24), we have the following results.

**Theorem 6.2.1.** Let $M$ be a totally geodesic GCR-lightlike submanifold of an indefinite complex space form $\bar{M}(c)$, then for any $X,Y \in \Gamma(D)$ and $Z,W \in \Gamma(D')$

$$Ric_D(X,Y) = -\frac{c(2p + 2r - 2s + 3)}{4} g(Z,W) + \frac{c}{4} \sum_{l=2s+1}^{r} g(J\xi_l, W) g(Z, JN_l)$$

$$Ric_D(X,Z) = \frac{c}{4} \sum_{l=2s+1}^{r} g(J\xi_l, Z) g(X, JN_l),$$

$$Ric_D(Z,W) = -\frac{c(2p + 2r - 2s + 3)}{4} g(Z,W) + \frac{c}{4} \sum_{l=2s+1}^{r} g(J\xi_l, W) g(Z, JN_l)$$

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and
\[ \text{Ric}_D(X, Y) = -\frac{qc}{4} g(X, Y), \]
\[ \text{Ric}_D(X, Z) = 0, \]
\[ \text{Ric}_D(Z, W) = -\frac{(q + 2)c}{4} g(Z, W). \]

**Theorem 6.2.2.** Let \( M \) be a \( D \)-geodesic GCR-lightlike submanifold of an indefinite complex space form \( \bar{M}(c) \), then for any \( X, Y \in \Gamma(D) \)
\[ \text{Ric}_D(X, Y) = \frac{c(s - r - p - 1)}{2} g(X, Y) - \frac{c}{4} \sum_{a=1}^{r} g(JX, Y) g(J\xi_a, N_a) \]
and
\[ \text{Ric}_D(X, Y) = -\frac{qc}{4} g(X, Y) + \sum_{i=1}^{q} g(h^l(F_i, Y), h^s(X, F_i)) \]
\[ + \sum_{i=1}^{q} g(h^s(F_i, Y), h^s(X, F_i)). \]

**Theorem 6.2.3.** Let \( M \) be a \( D' \)-geodesic GCR-lightlike submanifold of an indefinite complex space form \( \bar{M}(c) \), then for any \( Z, W \in \Gamma(D') \)
\[ \text{Ric}_D(Z, W) = -\frac{c(2p + 2r - 2s + 3)}{4} g(Z, W) + \frac{c}{4} \sum_{l=2s+1}^{r} g(J\xi_l, W) g(Z, JN_l) \]
\[ - \sum_{a=1}^{r} g(A_{N_a} Z, h^l(\xi_a, W)) + \sum_{a=1}^{r} g(D^s(Z, N_a), h^s(\xi_a, W)) \]
\[ + \sum_{k=1}^{2p} g(h^l(E_k, W), h^s(Z, E_k)) + \sum_{k=1}^{2p} g(h^s(E_k, W), h^s(Z, E_k)) \]
\[ + \sum_{l=2s+1}^{r} g(A_h(J\xi_l, W) Z, JN_l) + \sum_{l=2s+1}^{r} g(A_h(J\xi_l, W) Z, JN_l) \]
and
\[ \text{Ric}_{D'}(Z,W) = -\frac{(g + 2)c}{4} g(Z,W) + \sum_{i=1}^{q} g(h^i(F_i,W), h^s(Z,F_i)) + \sum_{i=1}^{q} g(h^s(F_i,W), h^s(Z,F_i)). \]

**Theorem 6.2.4.** Let \( M \) be a mixed-geodesic GCR-lightlike submanifold of an indefinite complex space form \( \bar{M}(c) \), then for any \( X \in \Gamma(D) \) and \( Z \in \Gamma(D') \)
\[ \text{Ric}_D(X,Z) = \frac{c}{4} \sum_{l=2s+1}^{r} g(J\xi_l,Z)g(X,JN_l) \]
and
\[ \text{Ric}_{D'}(X,Z) = \sum_{i=1}^{q} g(h^i(F_i,Z), h^s(X,F_i)) + \sum_{i=1}^{q} g(h^s(F_i,Z), h^s(X,F_i)). \]

**Theorem 6.2.5.** Let \( M \) be a proper totally umbilical GCR-lightlike submanifold of an indefinite Kaehler manifold \( \bar{M} \), then \( H^l = 0 \).

**Proof.** Since \( M \) is a totally umbilical GCR-lightlike submanifold then, by direct calculation, using (1.3.7), (1.3.8) and (1.2.1) and then taking tangential parts of the resulting equation, we obtain
\[ A_wZ + T\nabla_Z Z + Bh^l(Z,Z) + Bh^s(Z,Z) = 0, \]
where \( Z \in \Gamma(JL_2) \). Hence for \( \xi \in \Gamma(D_2) \), we obtain
\[ g(A_wZ, J\xi) + g(h^l(Z,Z), \xi) = 0. \]
Using (3.2.11), we get
\[ g(h^s(Z,J\xi), wZ) + g(h^l(Z,Z), \xi) = 0, \]
therefore using (3.2.2), we get \( g(Z,Z)g(H^l, \xi) = 0 \), then the non-degeneracy of \( JL_2 \) implies that \( H^l = 0 \), which completes the proof. \( \square \)
Lemma 6.2.6. Let $M$ be a totally umbilical $GCR$-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$, then $\nabla_X X \in \Gamma(D)$, for any $X \in \Gamma(D)$.

Proof. Since $D' = J(L_1 \perp L_2)$, therefore $\nabla_X X \in \Gamma(D)$, if and only if, $g(\nabla_X X, J\xi) = 0$ and $g(\nabla_X X, JW) = 0$, for any $\xi \in \Gamma(D_2)$ and $W \in \Gamma(L_2)$. Using $M$ is a totally umbilical $GCR$-lightlike submanifold, we obtain

$$g(\nabla_X X, J\xi) = -\bar{g}(\nabla_X JX, \xi) = -\bar{g}(h^l(X, JX), \xi) = -\bar{g}(H^l, \xi)g(X, JX) = 0 \quad (6.2.25)$$

and

$$g(\nabla_X X, JW) = -\bar{g}(\nabla_X JX, W) = -\bar{g}(h^s(X, JX), W)$$
$$= -\bar{g}(H^s, W)g(X, JX) = 0. \quad (6.2.26)$$

Thus using (6.2.25) and (6.2.26), the result follows. \hfill \Box

Theorem 6.2.7. Let $M$ be a proper totally umbilical $GCR$-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$, then $H^s \in \Gamma(L_2)$.

Proof. Since $M$ is a totally umbilical $GCR$-lightlike submanifold, therefore for any $X \in \Gamma(D_0)$, using (2.2.7), we get $g(X, JX)H^s = wP_2\nabla_X X + g(X, X)CH^s$. Using the lemma (6.2.6), we have $g(X, X)CH^s = 0$, then the non-degeneracy of $D_0$ implies that, $CH^s = 0$. Hence $H^s \in \Gamma(L_2)$. \hfill \Box

Theorem 6.2.8. Let $M$ be a totally umbilical $GCR$-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$, then $\nabla_X JX = J\nabla_X X$, for any $X \in \Gamma(D_0)$.

Proof. For any $X \in \Gamma(D_0)$, using (2.2.10) and (2.2.12), we have $w\nabla_X X = h(X, JX) - Ch(X, X)$. Since $M$ is totally umbilical, therefore using (3.2.1), we have $w\nabla_X X = Hg(X, JX) - CH^l g(X, X) - CH^s g(X, X)$, then using the theorems
(6.2.5) and (6.2.7), we get $w\nabla_X X = 0$. Hence $\nabla_X X \in \Gamma(D)$. Let $X, Y \in \Gamma(D_0)$ then using (1.2.1), we have $g(\nabla_X JX, Y) = g(\nabla_X JX, Y) = g(J\nabla_X X, Y) = g(J\nabla_X X, Y)$ which implies $g(\nabla_X JX - J\nabla_X X, Y) = 0$, then the non-degeneracy of $D_0$ implies that $\nabla_X JX - J\nabla_X X = 0$, that is, $\nabla_X JX = J\nabla_X X$ for any $X \in \Gamma(D_0)$. \hfill \Box

**Theorem 6.2.9.** Let $M$ be a proper totally umbilical GCR-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$, then $H^* = 0$.

**Proof.** For any $W \in \Gamma(S(TM^\perp))$ and $X \in \Gamma(D_0)$, using (1.2.1), (3.2.1) and the theorem (6.2.8), we have

$$
g(J\nabla_X X, JW) = g(\nabla_X JX, JW)$$
$$= g(\nabla_X JX, JW) + g(h(X, JX), JW)$$
$$= g(\nabla_X JX, JW) + g(X, JX)g(H, JW)$$
$$= g(J\nabla_X X, JW) = g(\nabla_X X, W) = 0. \hspace{1cm} (6.2.27)$$

Now using (3.2.2), we have

$$
g(J\nabla_X X, JW) = g(\nabla_X X, W) = g(h^*(X, X), W)$$
$$= g(X, X)g(H^*, W). \hspace{1cm} (6.2.28)$$

From (6.2.27) and (6.2.28), we have $g(X, X)g(H^*, W) = 0$, then the non-degeneracy of $D_0$ and $S(TM^\perp)$ implies that $H^* = 0$. \hfill \Box

**Theorem 6.2.10.** Let $M$ be a proper totally umbilical GCR-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$, then $M$ is a totally geodesic GCR-lightlike submanifold.

**Proof.** Proof follows from the theorems (6.2.5) and (6.2.9). \hfill \Box

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In [17], Duggal and Sahin proved a theorem for non-existence of totally umbilical proper $GCR$-lightlike submanifolds in a complex space form as

**Theorem 6.2.11.** ([17]). There exist no totally umbilical proper $GCR$-lightlike submanifold of an indefinite complex space form $M(c)$, such that $c \neq 0$.

Thus finally using the Theorems (6.2.10) and (6.2.11) in (6.2.11), we obtain

**Theorem 6.2.12.** Let $M$ be a totally umbilical $GCR$-lightlike submanifold of an complex space form $M(c)$, then

$$\text{Ric}(U, V) = 0,$$

for any $U, V \in \Gamma(TM)$. 