Chapter 2

Reciprocal Product Degree Distance of Product Graphs

The motivation for studying the reciprocal product degree distance of a graph comes from the following observations. The sum of distances between all pairs of vertices in a graph $G$ is the Wiener index. Initially, the Wiener index $W(G)$ was considered as a molecular-structure descriptor used in chemical applications, but soon it attracted the interest of pure mathematicians. After the number of modifications in the Wiener index, various graph invariants was introduced and studied, like degree distance or Schulz index (Gutman 1994), product degree distance or Gutman index (Gutman 1994) etc.

In 1990 by Plavsić et al. introduced the another distance based graph invariant is called Harary index. On the extension of this, Su et al. (2012) introduced the reciprocal product degree distance of graphs, which can be seen as a product degree weight version of
Harary index, defined as \( \text{RDD}_*(G) = \frac{1}{2} \sum_{u,v \in V(G)} \frac{d_G(u)d_G(v)}{d_G(u,v)} \). In this chapter, the exact expressions for the reciprocal product degree distance of join and tensor product of two graphs are obtained. Using our results, we have computed the exact value of the reciprocal product degree distance of star, fan and wheel graphs.

### 2.1 Reciprocal product degree distance of join

In this section, first we compute the reciprocal product degree distance of join of two graphs.

**Theorem 2.1.1.** Let \( G_1 \) and \( G_2 \) be graphs with \( n \) and \( m \) vertices \( p \) and \( q \) edges, respectively. Then \( \text{RDD}_*(G_1 + G_2) = M_2(G_1) + M_2(G_2) + mM_1(G_1) + nM_1(G_2) + \frac{1}{2}(\overline{M}_2(G_1) + \overline{M}_2(G_2)) + \frac{m}{2}\overline{M}_1(G_1) + \frac{n}{2}\overline{M}_1(G_2) + 4pq + 2mn(p + q) + \frac{1}{2}(m^2p + n^2q) + \frac{mn}{4}(6mn - m - n).

**Proof.** Set \( V(G_1) = \{u_1, u_2, \ldots, u_n\} \) and \( V(G_2) = \{v_1, v_2, \ldots, v_m\} \). By definition of the join of two graphs, one can see that,

\[
d_{G_1+G_2}(x) = \begin{cases} 
  d_{G_1}(x) + |V(G_2)|, & \text{if } x \in V(G_1) \\
  d_{G_2}(x) + |V(G_1)|, & \text{if } x \in V(G_2) 
\end{cases}
\]

and
\[ d_{G_1+G_2}(u, v) = \begin{cases} 
0, & \text{if } u = v \\
1, & \text{if } uv \in E(G_1) \text{ or } uv \in E(G_2) \text{ or } (u \in V(G_1) \text{ and } v \in V(G_2)) \\
2, & \text{otherwise.} 
\end{cases} \]

Therefore, \( RDD_*(G_1 + G_2) = \frac{1}{2} \sum_{u,v \in V(G_1+G_2)} \frac{d_{G_1+G_2}(u)d_{G_1+G_2}(v)}{d_{G_1}+d_G} \)
\[ = \frac{1}{2} \left( \sum_{uv \in E(G_1)} (d_{G_1}(u) + m)(d_{G_1}(v) + m) + \frac{1}{2} \sum_{uv \notin E(G_1)} (d_{G_1}(u) + m)(d_{G_1}(v) + m) \right) \\
+ \sum_{uv \in E(G_2)} (d_{G_2}(u) + n)(d_{G_2}(v) + n) + \frac{1}{2} \sum_{uv \notin E(G_2)} (d_{G_2}(u) + n)(d_{G_2}(v) + n) \\
+ \sum_{u \in V(G_1), v \in V(G_2)} (d_{G_1}(u) + m)(d_{G_2}(v) + n) \right) \\
= M_2(G_1) + M_2(G_2) + mM_1(G_1) + nM_1(G_2) + \frac{1}{2}((\overline{M}_2(G_1) + \overline{M}_2(G_2)) + \\
m 2M_1(G_1) + \frac{n}{2}M_1(G_2) + 4pq + 2mn(p+q) + \frac{1}{2}(m^2p+n^2q) + \frac{mn}{4}(6mn-m-n).\]

□

Using Theorem 2.1.1, we have the following corollaries.

**Corollary 2.1.1.** Let \( G \) be graph on \( n \) vertices and \( p \) edges. Then \( RDD_*(G + K_m) = M_2(G) + mM_1(G) + \frac{1}{2}((\overline{M}_2(G) + mM_1(G)) + \frac{mp}{2}(4n + m) + \frac{mn}{4}(6mn-m-n)+\frac{1}{2}m(m-1)(n^2+m^2+4p-2n-2m+4mn+1).\)

Let \( K_{n,m} \) be the bipartite graph with two partitions having \( n \) and \( m \) vertices. Note that \( K_{n,m} = \overline{K}_n + \overline{K}_m.\)

**Corollary 2.1.2.** \( RDD_*(K_{n,m}) = HM_*(\overline{K}_n + \overline{K}_m) = \frac{nm}{4}(6nm-n-m).\)

By direct calculations, we obtain the following lemma.
Lemma 2.1.1. (i) For $n \geq 3$, $M_1(C_n) = 4n$ and $\overline{M}_1(C_n) = 2n(n - 3)$.

(ii) For $n > 1$, $M_1(P_n) = 4n - 6$ and $\overline{M}_1(P_n) = 2(n - 2)^2$.

(iii) $M_1(K_n) = n(n - 1)^2$ and $\overline{M}_1(K_n) = 0$.

(iv) $M_2(P_n) = 4(n - 2)$ and $\overline{M}_2(P_n) = 2n^2 - 10n + 13$.

(v) $M_2(C_n) = 4n$ and $\overline{M}_2(C_n) = 2n(n - 3)$.

Using Corollary 2.1.1, we compute the formulae for reciprocal product degree distance of star, fan and wheel graphs, $K_1 + K_m, P_n + K_1$ and $C_n + K_1$, see Figure 2.1.

![Fan graph and wheel graph](image)

Figure 2.1: Fan graph and wheel graph

Example 2.1.1. The following can be verified:

(i) $RDD_*(K_1 + K_m) = \frac{m(5m-1)}{4}$.

(ii) $RDD_*(P_n + K_1) = \frac{1}{4}(21n^2 - 11n - 16)$.

(iii) $RDD_*(C_n + K_1) = \frac{1}{4}n(21n^2 + 9)$.

2.2 RPDD of tensor products

Let $G$ be a connected graph with $V(G) = \{v_0, v_1, \ldots, v_n\}$ and let $K_{m_0, m_1, \ldots, m_r}$, $r \geq 3$, be the complete multipartite graph with partite sets $V_0, V_1, \ldots, V_{r-1}$ with $|V_i| = m_i$, $0 \leq i \leq r - 1$. In the graph
Let $B_{ij} = v_i \times V_j, v_i \in V(G)$ and $0 \leq j \leq r - 1$. For our convenience, we write

$$V(G) \times V(K_{m_0, m_1, \ldots, m_{r-1}}) = \bigcup_{i=0}^{n-1} \left( v_i \times \bigcup_{j=0}^{r-1} V_j \right)$$

$$= \bigcup_{i=0}^{n-1} \{ v_i \times V_0 \} \bigcup \{ v_i \times V_1 \} \bigcup \cdots \bigcup \{ v_i \times V_{r-1} \}$$

$$= \bigcup_{i=0}^{n-1} \{ B_{i0} \} \bigcup \bigcup_{i=0}^{n-1} \{ B_{i1} \} \bigcup \cdots \bigcup \{ B_{i(r-1)} \}, \text{ where } B_{ij} = v_i \times V_j$$

$$= \bigcup_{i=0}^{n-1} \bigcup_{j=0}^{r-1} B_{ij}.$$

Let $\mathcal{B} = \{ B_{ij} \}_{i=0,1,\ldots,n-1, \atop j=0,1,\ldots,r-1}$. If $v_i v_k \in E(G)$, then the subgraph $\langle B_{ij} \cup B_{kp} \rangle$ of $G \times K_{m_0, m_1, \ldots, m_{r-1}}$ is isomorphic to $K_{|V_j||V_p|}$ or a totally disconnected graph according to $j \neq p$ or $j = p$. It is used in the proof of the next lemma. If we denote $V(B_{ij}) = \{ x_{i1}, x_{i2}, \ldots, x_{im_j} \}$ and $V(B_{kp}) = \{ x_{k1}, x_{k2}, \ldots, x_{km_p} \}$, then $x_{i\ell}$ and $x_{k\ell}, 1 \leq \ell \leq j$, are called the corresponding vertices of $B_{ij}$ and $B_{kp}$. 

![Diagram](image-url)
In the above Figure 2.2, if $v_iv_k$ is an edge but not a triangle of $G$, then a shortest path of length 3 a vertex of $B_{ij}$ to a vertex of $B_{kj}$ is shown in solid edges. If $v_iv_k$ is on a triangle $v_iv_kv_t$ of $G$, then a shortest path of length 2 from a vertex of $B_{ij}$ from a vertex of $B_{kj}$ is shown in broken edges.

The proof of the following lemma follows easily from the properties, structure of $G \times K_r$ and the paths as shown in Figures 2.2 and 2.3.

**Lemma 2.2.1.** Let $G$ be a connected graph on $n \geq 2$ vertices. For any pair of vertices $x_{ij}, x_{kp} \in V(G \times K_r)$, $r \geq 3$, $i, k \in \{1, 2, \ldots, n\}$, $j, p \in \{1, 2, \ldots, r\}$. Then

(i) If $v_iv_k \in E(G)$, then

$$d_{G \times K_r}(x_{ij}, x_{kp}) = \begin{cases} 
1, & \text{if } j \neq p, \\
2, & \text{if } j = p \text{ and } u_iu_k \text{ is on a triangle of } G, \\
3, & \text{if } j = p \text{ and } u_iu_k \text{ is not on a triangle of } G.
\end{cases}$$

(ii) If $v_iv_k \notin E(G)$, then $d_{G \times K_r}(x_{ij}, x_{kp}) = d_G(v_i, v_k)$.

(iii) $d_{G \times K_r}(x_{ij}, x_{ip}) = 2$.

**Proof.** Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $V(K_r) = \{u_1, u_2, \ldots, u_r\}$. Let $x_{ij}$ denote the vertex $(v_i, u_j)$ of $G \times K_r$. We only prove the case when $v_iv_k \notin E(G)$, $i \neq k$ and $j = p$. The proofs for other cases are similar.

We may assume $j = 1$. Let $P = v_iv_{s_1}v_{s_2} \ldots v_{s_p}v_k$ be the shortest
path of length $p+1$ between $v_i$ and $v_k$ in $G$. From $P$ we have a $(x_{i1}, x_{k1})$-path $P_1 = x_{i1}x_{i2} \ldots x_{s_p-1}2x_{s_p}3x_{k1}$ if the length of $P$ is odd, and $P_1 = x_{i1}x_{i2} \ldots x_{s_p-1}2x_{s_p}2x_{k1}$ if the length of $P$ is even.

Obviously, the length of $P_1$ is $p+1$, and thus $d_{G \times K_r}(x_{i1}, x_{k1}) \leq p+1 \leq d_G(v_i, v_k)$. If there were a $(x_{i1}, x_{k1})$-path in $G \times K_r$ that is shorter than $p+1$ then it is easy to find a $(v_i, v_k)$-path in $G$ that is also shorter than $p+1$ in contrast to $d_G(v_i, v_k) = p + 1$. □

The proof of the following lemma follows easily from Lemma 2.2.1 and is used in the proof of the main theorem of this section.

**Lemma 2.2.2.** Let $G$ be a connected graph on $n \geq 2$ vertices and let $B_{ij}, B_{kp} \in \mathcal{B}$ of the graph $G' = G \times K_{m_0, m_1, \ldots, m_{r-1}}$, where $r \geq 3$.

(i) If $v_i v_k \in E(G)$, then

$$d^H_{G'}(B_{ij}, B_{kp}) = \begin{cases} m_jm_p, & \text{if } j \neq p, \\ \frac{m^2_j}{2}, & \text{if } j = p \text{ and } v_i v_k \text{ is on a triangle of } G, \\ \frac{m^3_j}{3}, & \text{if } j = p \text{ and } v_i v_k \text{ is not on a triangle of } G. \end{cases}$$

(ii) If $v_i v_k \notin E(G)$, then $d^H_{G'}(B_{ij}, B_{kp}) = \begin{cases} \frac{m_jm_p}{d_G(v_i, v_k)}, & \text{if } j \neq p, \\ \frac{m_j}{d_G(v_i, v_k)}, & \text{if } j = p. \end{cases}$

(iii) $d^H_{G'}(B_{ij}, B_{ip}) = \begin{cases} \frac{m_j(m_j-1)}{2}, & \text{if } j = p, \\ \frac{m_jm_p}{2}, & \text{if } j \neq p. \end{cases}$
Lemma 2.2.3. Let $G$ be a connected graph and let $B_{ij}$ in $G' = G \times K_{m_0, m_1, \ldots, m_{r-1}}$. Then the degree of a vertex $x_{ij} = (v_i, u_j) \in B_{ij}$ in $G'$ is $d_{G'}(x_{ij}) = d_G(v_i)(n_0 - m_j)$, where $n_0 = \sum_{j=0}^{r-1} m_j$.

Lemma 2.2.4. Let $x_{it} \in B_{ij}$, then $d_{G'}(B_{ij}) = \sum_{t=1}^{m_j} d_G(x_{it})$, where $m_j$ is the number of vertices in the $j$th partite set of $K_{m_0, m_1, \ldots, m_{r-1}}$.

Remark 2.2.1. (i) The sums $\sum_{j=0}^{r-1} m_j^2 = n_0^2 - 2q, \sum_{j=0}^{r-1} m_j m_p = n_0^2 - 2n_0 q - \sum_{j=0}^{r-1} m_j - \sum_{j=0}^{r-1} m_j^2 = \sum_{j=0}^{r-1} m_j^3$, where $n_0 = \sum_{j=0}^{r-1} m_j$ and $q$ is the number of edges of $K_{m_0, m_1, \ldots, m_{r-1}}$.

(ii) Let $x_{it} \in B_{ij}$. Then $d_{G'}(B_{ij}) = \sum_{t=1}^{m_j} d_G(x_{it})$, where $m_j$ is the number of vertices in the $j$th partite set of $K_{m_0, m_1, \ldots, m_{r-1}}$.

Next we determine the reciprocal product degree distance of $G \times K_{m_0, m_1, \ldots, m_{r-1}}$ in the following Figure 2.3, if a $(v_i, v_k)$-shortest path is of even (resp. odd $\geq 3$) length in $G$, then a shortest path from a vertex of $B_{ij}$ to a vertex of $B_{kj}$ is shown in solid edges (resp. broken edges).
Theorem 2.2.1. Let $G$ be a connected graph with $n \geq 2$ vertices and $m$ edges and let $E_2$ be the set of edges of $G$ which do not lie on any $C_3$ of it. If $n_0$ and $q$ are the numbers of vertices and edges of $K_{m_0,m_1,...,m_{r-1}}$, $r \geq 3$, respectively, then

$$RDD_s(G \times K_{m_0,m_1,...,m_{r-1}}) = 4q^2RDD_s(G) + \frac{M_1(G)}{4}(n_0^3 + 4q^2 - 4qn_0 - \sum_{j=0}^{r-1} m_j^3) - \left(\frac{M_3(G)}{2} + \sum_{v,v_k \in E_2} \frac{d_G(v_j)d_G(v_k)}{12}\right)\left(n_0^4 - 2qn_0^2 - 2n_0 \sum_{j=0}^{r-1} m_j^3 + \sum_{j=0}^{r-1} m_j^4\right).$$

Proof. Let $G' = G \times K_{m_0,m_1,...,m_{r-1}}$. Clearly,

$$RDD_s(G') = \frac{1}{2} \sum_{B_{ij}, B_{kp} \in B} d_{G'}(B_{ij})d_{G'}(B_{kp})d^H_{G'}(B_{ij}, B_{kp}) = \frac{1}{2} \sum_{i=0}^{n-1} \sum_{j,p=0}^{r-1} d_{G'}(B_{ij})d_{G'}(B_{ip})d^H_{G'}(B_{ij}, B_{ip}).$$
\[
\begin{align*}
&+ \sum_{i,k=0}^{n-1} \sum_{j,p=0}^{r-1} d_G'(B_{ij})d_G'(B_{kp})d_G^H(B_{ij}, B_{kp}) \\
&+ \sum_{i=0}^{n-1} \sum_{j=0}^{r-1} d_G'(B_{ij})d_G'(B_{ij})d_G^H(B_{ij}, B_{ij}) \\
= \frac{1}{2}(A_1 + A_2 + A_3 + A_4), \quad (2.1)
\end{align*}
\]

where \(A_1\) to \(A_4\) are the sums of the above terms, in order.

We shall calculate \(A_1\) to \(A_4\) of (2.1) separately.

First we compute \(A_1\). By Lemmas 2.2.2 and 2.2.3, we obtain

\[
\sum_{j,p=0}^{r-1} \left( (n_0 - m_j)(n_0 - m_p)(d_G(v_i))^2 \right) \frac{m_jm_p}{2} \quad (2.2)
\]

By summing (2.2) over \(i = 0, 1, \ldots, n - 1\), we get

\[
A_1 = \sum_{i=0}^{n-1} \left( d_G(v_i) \right)^2 \sum_{j,p=0}^{r-1} \left( n_0^2 - n_0m_j - n_0m_p + m_jm_p \right) \frac{m_jm_p}{2}
\]

Now by Remark 2.2.1, we have,

\[
A_1 = \frac{M_1(G)}{2} \left( 2qn_0^2 - n_0^4 + 4q^2 + 2n_0 \sum_{j=0}^{r-1} m_j^2 - \sum_{j=0}^{r-1} m_j^4 \right), \quad (2.3)
\]

where \(M_1(G)\) is the first Zagreb index of \(G\).

Next we compute \(A_2\).

That is initially we calculate

\[
\sum_{i,k=0}^{n-1} \sum_{j \neq k}^{r-1} d_G'(B_{ij})d_G'(B_{kj})d_G^H(B_{ij}, B_{kj}).
\]

Let \(E_1 = \{uv \in E(G) | uv \text{ is on a } C_3 \text{ in } G\} \) and \(E_2 = E(G) - E_1\).
\[
\sum_{i, k = 0}^{n-1} d_G'(B_{ij})d_G'(B_{kj})d_H'(B_{ij}, B_{kj})
\]

\[
\sum_{i, k = 0}^{n-1} d_G'(B_{ij})d_G'(B_{kj})d_H'(B_{ij}, B_{kj}) + \sum_{i, k = 0}^{n-1} d_G'(B_{ij})d_G'(B_{kj})d_H'(B_{ij}, B_{kj})
\]

\[
+ \sum_{i, k = 0}^{n-1} d_G'(B_{ij})d_G'(B_{kj})d_H'(B_{ij}, B_{kj})
\]

\[
\sum_{i, k = 0}^{n-1} (n_0 - m_j)^2 d_G(v_i)d_G(v_k) \frac{m_j^2}{d_G(v_i, v_k)} + \sum_{i, k = 0}^{n-1} (n_0 - m_j)^2 d_G(v_i)d_G(v_k) \frac{m_j^2}{2}
\]

\[
+ \sum_{i, k = 0}^{n-1} (n_0 - m_j)^2 d_G(v_i)d_G(v_k) \frac{m_j^2}{3}, \text{ by Lemmas 2.2.2 and 2.2.3}
\]

\[
\sum_{i, k = 0}^{n-1} (n_0 - m_j)^2 d_G(v_i)d_G(v_k) \frac{m_j^2}{d_G(v_i, v_k)} +
\]

\[
\sum_{i, k = 0}^{n-1} (n_0 - m_j)^2 d_G(v_i)d_G(v_k) \left( \frac{m_j^2}{2} + m_j^2 - m_j^2 \right)
\]
\[
\begin{align*}
\sum_{i, k = 0}^{n-1} (n_0 - m_j)^2 d_G(v_i)d_G(v_k) & \cdot \frac{m_j^2}{d_G(v_i, v_k)} + \\
\sum_{i, k = 0}^{n-1} (n_0 - m_j)^2 d_G(v_i)d_G(v_k) & \cdot \frac{m_j}{d_G(v_i, v_k)} \\
+ & \sum_{i, k = 0}^{n-1} (n_0 - m_j)^2 d_G(v_i)d_G(v_k) \cdot \frac{m_j}{d_G(v_i, v_k)} \\
- & \sum_{i, k = 0}^{n-1} (n_0 - m_j)^2 d_G(v_i)d_G(v_k) \cdot \frac{m_j}{2} - \sum_{i, k = 0}^{n-1} (n_0 - m_j)^2 d_G(v_i)d_G(v_k) \cdot \frac{2m_j}{3},
\end{align*}
\]

\[= \sum_{i, k = 0}^{n-1} (n_0 - m_j)^2 d_G(v_i)d_G(v_k) \cdot \frac{m_j^2}{d_G(v_i, v_k)} - \left( \sum_{i, k = 0}^{n-1} (n_0 - m_j)^2 d_G(v_i)d_G(v_k) \cdot \frac{m_j}{2} \right) \\
+ \sum_{i, k = 0}^{n-1} (n_0 - m_j)^2 d_G(v_i)d_G(v_k) \cdot \frac{m_j}{2} - \sum_{i, k = 0}^{n-1} (n_0 - m_j)^2 d_G(v_i)d_G(v_k) \cdot \frac{m_j^2}{6} \\
= 2RDD_\star(G) (n_0 - m_j)^2 m_j^2 - 2M_2(G) (n_0 - m_j)^2 \frac{m_j^2}{2} - \\
\sum_{i, k = 0}^{n-1} d_G(v_i)d_G(v_k)(n_0 - m_j)^2 \frac{m_j^2}{6},
\]

(2.4)

where $M_2(G)$ is the second Zagreb index of $G$. Note that each
edge $v_iv_k$ of $G$ is being counted twice in the sum, namely, $v_iv_k$ and $v_kv_i$.

Now summing (2.4) over $j = 0, 1, \ldots, r - 1$, we get,

$$A_2 = \left(2\text{RDD}_t(G) - M_2(G) - \sum_{v_iv_k \in E_2} \frac{d_G(v_i)d_G(v_k)}{6} \right) \sum_{j=0}^{r-1} (n_0 - m_j)^2m_j^2$$

$$= \left(2\text{RDD}_t(G) - M_2(G) - \sum_{v_iv_k \in E_2} \frac{d_G(v_i)d_G(v_k)}{6} \right) \sum_{j=0}^{r-1} \left(n_0^2m_j^2 + m_j^4 - 2n_0m_j^3\right).$$

By Remark 2.2.1, we have

$$A_2 = \left(2\text{RDD}_t(G) - M_2(G) - \sum_{v_iv_k \in E_2} \frac{d_G(v_i)d_G(v_k)}{6} \right) \left(n_0^4 - 2qn_0^2 - 2n_0 \sum_{j=0}^{r-1} m_j^3 + \sum_{j=0}^{r-1} m_j^4\right). \quad (2.5)$$

Next we compute $A_3$. For this, first we calculate

$$\sum_{j,p=0, j \neq p}^{r-1} d_{G'}(B_{ij})d_{G'}(B_{kp})d_{G'}^H(B_{ij}, B_{kp}).$$

$$\sum_{j,p=0, j \neq p}^{r-1} d_{G'}(B_{ij})d_{G'}(B_{kp})d_{G'}^H(B_{ij}, B_{kp}) = \sum_{j,p=0, j \neq p}^{r-1} \left((n_0 - m_j)d_G(v_i)(n_0 - m_p)d_G(v_k)\right) \frac{m_jm_p}{d_G(v_i, v_k)},$$

by Lemmas 2.2.2 and 2.2.3(2.6)
Using 2.6 we have,

\[ A_3 = \sum_{i,k=0}^{n-1} \sum_{j,p=0}^{r-1} \frac{(n_0 - m_j) d_G(v_i)(n_0 - m_p)d_G(v_k)}{d_G(v_i,v_k)} \frac{m_j m_p}{d_G(v_i,v_k)} \]

\[ = \sum_{i,k=0}^{n-1} \sum_{j,p=0}^{r-1} \left( n_0^2 m_j m_p - n_0 m_j m_p^2 - n_0 m_j m_p + m_j^2 m_p^2 \right) \frac{d_G(v_i)d_G(v_k)}{d_G(v_i,v_k)} \]

\[ = \left( n_0^2 (2q) - 2n_0(n_0^3 - 2q n_0 - \sum_{j=0}^{r-1} m_j^3) + (n_0^2 - 2q)^2 - \sum_{j=0}^{r-1} m_j^4 \right) 2 \text{RDD}_1(G), \]

by Remark 2.2.1 and definition of reciprocal product degree distance

\[ = 2 \text{RDD}_1(G) \left( 2q n_0^2 - n_0^4 + 4q^2 + 2n_0 \sum_{j=0}^{r-1} m_j^3 - \sum_{j=0}^{r-1} m_j^4 \right). \tag{2.7} \]

Finally, we compute \( A_4 \).

\[ A_4 = \sum_{i=0}^{n-1} \sum_{j=0}^{r-1} (d_G(B_{ij}))^2 d_G^H(B_{ij}, B_{ij}) \]

\[ = \sum_{i=0}^{n-1} \sum_{j=0}^{r-1} (n_0 - m_j)^2 (d_G(v_i))^2 \frac{m_j(m_j - 1)}{2}, \] by Lemmas 2.2.2 and 2.2.3

\[ = \sum_{i=0}^{n-1} (d_G(v_i))^2 \sum_{j=0}^{r-1} \left( n_0^2 + 2n_0 m_j^2 - n_0 m_j + m_j^4 - (2n_0 + 1)m_j^3 \right) \]

\[ = \frac{M_1(G)}{2} \left( n_0^4 + 2n_0(n_0^3 - 2q) - n_0^3 + \sum_{j=0}^{r-1} m_j^4 - (2n_0 + 1) \sum_{j=0}^{r-1} m_j^3 \right), \]

by Remark 2.2.1

\[ = \frac{M_1(G)}{2} \left( n_0^4 - 2q n_0^2 + n_0^3 - 4q n_0 + \sum_{j=0}^{r-1} m_j^4 - (2n_0 + 1) \sum_{j=0}^{r-1} m_j^3 \right). \tag{2.8} \]

Using (2.1) and the sums \( A_1, A_2, A_3 \) and \( A_4 \) in 2.3,2.5,2.7 and 2.8 respectively, we have,
\[
\text{RDD}_s(G') = 4q^2\text{RDD}_s(G) + \frac{M_1(G)}{4}(n_0^3 + 4q^2 - 4qn_0 - \sum_{j=0}^{r-1} m_j^3)
\]
\[
-\left(\frac{M_2(G)}{2} + \sum_{v_i v_k \in E_2} \frac{d_G(v_i)d_G(v_k)}{12}\right)(n_0^4 - 2qn_0^2 - 2n_0 \sum_{j=0}^{r-1} m_j^3 + \sum_{j=0}^{r-1} m_j^4).
\]

□

Using above theorem, we have the following corollaries.

**Corollary 2.2.1.** Let \( G \) be a connected graph with \( n \geq 2 \) vertices and \( m \) edges. If each edge of \( G \) is on a \( C_3 \), then

\[
\text{RDD}_s(G \times K_{m_0, m_1, \ldots, m_{r-1}}) = 4q^2\text{RDD}_s(G) + \frac{M_1(G)}{4}(n_0^3 + 4q^2 - 4qn_0 - \sum_{j=0}^{r-1} m_j^3) - \frac{M_2(G)}{2}(n_0^4 - 2qn_0^2 - 2n_0 \sum_{j=0}^{r-1} m_j^3 + \sum_{j=0}^{r-1} m_j^4), \quad r \geq 3.
\]

For a triangle free graph, \( E_2 = E(G) \) and hence

\[
\sum_{v_i v_k \in E_2} d_G(v_i)d_G(v_k) = M_2(G).
\]

**Corollary 2.2.2.** If \( G \) is a connected triangle free graph on \( n \geq 2 \) vertices and \( m \) edges, then

\[
\text{RDD}_s(G \times K_{m_0, m_1, \ldots, m_{r-1}}) = 4q^2\text{RDD}_s(G) + \frac{M_1(G)}{4}(n_0^3 + 4q^2 - 4qn_0 - \sum_{j=0}^{r-1} m_j^3) - \frac{2M_2(G)}{3}(n_0^4 - 2qn_0^2 - 2n_0 \sum_{j=0}^{r-1} m_j^3 + \sum_{j=0}^{r-1} m_j^4), \quad r \geq 3.
\]

If \( m_i = s, \quad 0 \leq i \leq r - 1 \), in Theorem 2.2.1 and Corollaries 2.2.1, 2.2.2, we have the following corollaries:
Corollary 2.2.3. Let $G$ be a connected graph with $n \geq 2$ vertices and $m$ edges. Let $E_2$ be the set of edges of $G$ which do not lie on a triangle. Then

$$RDD_*(G \times K_{r(s)}) = r^2(r-1)^2s^4 RDD_*(G) + \frac{M_1(G)}{4}rs^3(r^2(r-1)^2-r^2+2r-1) - \left(\frac{M_2(G)}{2} + \sum_{v_i, v_k \in E_2} \frac{d_G(v_i)d_G(v_k)}{12}\right)rs^4(r-1)^2, \quad r \geq 3.$$ 

Corollary 2.2.4. Let $G$ be a connected graph with $n \geq 2$ vertices and $m$ edges. If each edge of $G$ is on a $C_3$, then

$$RDD_*(G \times K_{r(s)}) = r^2(r-1)^2s^4 RDD_*(G) + \frac{M_1(G)}{4}rs^3(r^2(r-1)^2-r^2+2r-1) - \frac{M_2(G)}{2}rs^4(r-1)^2, \quad r \geq 3.$$ 

Corollary 2.2.5. If $G$ is a connected triangle free graph on $n \geq 2$ vertices and $m$ edges, then $RDD_*(G \times K_{r(s)}) = r^2(r-1)^2s^4 RDD_*(G) + \frac{M_1(G)}{4}rs^3(r^2(r-1)^2-r^2+2r-1) - \frac{2M_2(G)}{3}rs^4(r-1)^2, \quad r \geq 3.$

Taking $s = 1$ in Corollaries 2.2.3, 2.2.4, 2.2.5, we have the following corollaries.

Corollary 2.2.6. Let $G$ be a connected graph with $n \geq 2$ vertices and $m$ edges. Let $E_2$ be the set of edges of $G$ which do not lie on a triangle. Then

$$RDD_*(G \times K_r) = r(r-1)^2\left(rRDD_*(G) + \frac{1}{4}(r-1)M_1(G) - \frac{1}{2}M_2(G) - \frac{1}{12} \sum_{v_i, v_k \in E_2} d_G(v_i)d_G(v_k)\right), \quad r \geq 3.$$
Corollary 2.2.7. Let $G$ be a connected graph on $n \geq 2$ vertices with $m$ edges. If each edge of $G$ is on a $C_3$, then

$$\text{RDD}_*(G \times K_r) = r(r-1)^2 \left( r\text{RDD}_*(G) + \frac{1}{4}(r-1)M_1(G) - \frac{1}{2}M_2(G) \right),$$

where $r \geq 3$.

Corollary 2.2.8. If $G$ is a connected triangle free graph on $n \geq 2$ vertices and $m$ edges, then

$$\text{RDD}_*(G \times K_r) = r(r-1)^2 \left( r\text{RDD}_*(G) + \frac{1}{4}(r-1)M_1(G) - \frac{2}{3}M_2(G) \right),$$

for $r \geq 3$.

By direct calculations, we obtain the following lemma.

Lemma 2.2.5. (i) $H(K_n) = \frac{n(n-1)}{2}$.

(ii) $H(C_n) = n \left( \sum_{i=1}^{\frac{n}{2}} \frac{1}{i} \right) - 1$ when $n$ is even, and $n \left( \sum_{i=1}^{\frac{n-1}{2}} \frac{1}{i} \right)$ otherwise.

(iii) $\text{RDD}_*(K_n) = \frac{n(n-1)^3}{2}$ and $\text{RDD}(K_n) = n(n-1)^2$.

(iv) $\text{RDD}_*(C_n) = \text{RDD}(C_n) = 4H(C_n)$.

Using Corollaries 2.2.7 and 2.2.8, we obtain the reciprocal product degree distance of the graphs $K_n \times K_r$ and $C_n \times K_r$.

Example 2.2.1. (i) $\text{RDD}_*(K_n \times K_r) = \frac{n^2}{12}(n-1)^2(r-1)^2(6nr-4n-3r+1)$.

(ii) $\text{RDD}_*(C_n \times K_r) = \begin{cases} r(r-1)^2 \left( 4rH(C_n) + n(r-3) \right), & \text{if } n = 3, \\ r(r-1)^2 \left( 4rH(C_n) + \frac{n}{3}(3r-11) \right), & \text{if } n > 3. \end{cases}$