Chapter 1

Introduction and Review of literature

Graph invariants are properties of graphs that are invariant under graph isomorphisms. There are many examples of graph invariants, especially those based on distances, which are applicable in chemistry. In graph theory, we have many invariants for any graphs, that they have usually integer coefficients. Also, topological indices are real numbers related to a structural graph of a molecule. Such indices based on the distances in graph are widely used for establishing relationships between the structure of a molecular graph and their physiochemical properties. In this thesis we discuss some of the distance based graph invariants (topological indices) of product of graphs.

In this chapter, we present relevant definitions and summary of the results of the thesis. We begin with definitions that we need throughout this thesis.
1.1 Preliminaries

Through out this thesis, we consider simple graphs, that are finite and undirected graphs. Let $G = (V(G), E(G))$ be a connected graph. The distance between two vertices $u$ and $v$ is the length of a shortest path between $u$ and $v$ in $G$, denoted by $d_G(u, v)$. The diameter of $G$, denoted by $D(G)$ is defined as $\max\{d_G(u, v) : u, v \in V(G)\}$. The degree of $u$ is the number of edges incident to $u$ and is denoted by $d_G(u)$. Let $(d_G(u))^r = d_G^r(u)$. For a subset $S \subseteq V(G)$, $\langle S \rangle$ denotes the subgraph of $G$ induced by $S$. Let $P_n$, $C_n$, and $K_n$ are the path, cycle and complete graph on $n$ vertices, respectively. Let $K_{m_0, m_1, \ldots, m_{r-1}}$ be the complete $r$ partite graph with partite set having $m_0, m_1, \ldots, m_{r-1}$ vertices.

A join $G + H$ of two graphs $G$ and $H$ with disjoint vertex sets $V(G)$ and $V(H)$ is the graph on the vertex set $V(G) \cup V(H)$ and the edge set $E(G) \cup E(H) \cup \{uv | u \in V(G), v \in V(H)\}$, see Figure 1.1. Hence, the join of two graphs is obtained by connecting each vertex of one graph to each vertex of the other graph, while keeping all edges of both graphs.

The disjunction $G \lor H$ of graphs $G$ and $H$ is the graph with vertex set $V(G) \times V(H)$ and $(u_1, v_1)$ is adjacent with $(u_2, v_2)$ whenever $u_1u_2 \in E(G)$ or $v_1v_2 \in E(H)$, see Figure 1.2. The symmetric difference $G \oplus H$ of two graphs $G$ and $H$ is the graph with vertex set $V(G) \times V(H)$ and $E(G \oplus H) = \{(u_1, v_1)(u_2, v_2) | u_1u_2 \in E(G) \text{ or } v_1v_2 \in E(H) \text{ but not both}\}$, see Figure 1.3.
Figure 1.1: The graph $G + H = P_2 + P_3$

Figure 1.2: The graph $G \lor H = P_3 \lor P_3$

Figure 1.3: The graph $G \oplus H = P_3 \oplus P_3$
For two simple graphs $G$ and $H$ their tensor product, denoted by $G \times H$, has vertex set $V(G) \times V(H)$ in which $(g_1, h_1)(g_2, h_2)$ is an edge whenever $g_1g_2$ is an edge in $G$ and $h_1h_2$ is an edge in $H$, see Figure 1.4.

The strong product of graphs $G$ and $H$, denoted by $G \boxtimes H$, is the graph with vertex set $V(G) \times V(H) = \{(u, v) : u \in V(G), v \in V(H)\}$ and $(u, x)(v, y)$ is an edge whenever (i) $u = v$ and $xy \in E(H)$, or (ii) $uv \in E(G)$ and $x = y$ or, (iii) $uv \in E(G)$ and $xy \in E(H)$, see Figure 1.5. Similarly, the wreath product of the graphs $G$ and $H$, denoted by $G \circ H$, has vertex set $V(G) \times V(H)$ in which $(g_1, h_1)(g_2, h_2)$ is an edge whenever $g_1g_2$ is an edge in $G$ or, $g_1 = g_2$ and $h_1h_2$ is an edge in $H$, see Figure 1.6.
Let $G$ be a connected graph. Then \textit{Wiener index} of $G$ is defined as

$$W(G) = \sum_{\{u,v\} \subseteq V(G)} d_G(u,v) = \frac{1}{2} \sum_{u,v \in V(G)} d_G(u,v)$$

with the summation going over all pairs of distinct vertices of $G$. Similarly, the \textit{Harary index} of $G$ is defined as

$$H(G) = \sum_{\{u,v\} \subseteq V(G)} \frac{1}{d_G(u,v)} = \frac{1}{2} \sum_{u,v \in V(G)} \frac{1}{d_G(u,v)}.$$
The hyper-Wiener index was proposed by Klein et al. [44], as a generalization of the Wiener index of graphs. It is defined as

\[ WW(G) = \frac{1}{2} W(G) + \frac{1}{4} \sum_{u, v \in V(G)} d_G^2(u, v), \]

where \( d_G^2(u, v) = (d_G(u, v))^2 \).

The above definition can be further generalized in the following way:

\[ W_\lambda(G) = \sum_{[u, v] \subseteq V(G)} d_G^\lambda(u, v) = \frac{1}{2} \sum_{u, v \in V(G)} d_G^\lambda(u, v), \]

where \( d_G^\lambda(u, v) = (d_G(u, v))^\lambda \) and \( \lambda \) is a real number [35, 33]. Several particular instances of the invariant \( W_\lambda \) have been previously studied for instance, \( W_{-1} \), \( W_{-2} \), \( \frac{1}{2} W_1 + \frac{1}{3} W_2 \) and \( \frac{1}{6} W_1 + \frac{1}{2} W_2 + \frac{1}{3} W_3 \) are the so-called Harary index [18], reciprocal Wiener index [43], hyper-Wiener index (WW) and Tratch-Stankevich-Zefirov index (TSZ) [32]. In the chemical literature also \( W_{\frac{1}{2}} [72] \) as well as the general case \( W_\lambda \) were examined [27, 30].

The reverse Wiener index of a connected graph \( G \), denoted by \( \Lambda(G) \), is defined as

\[ \Lambda(G) = \frac{|V(G)|(|V(G)| - 1)}{2} D(G) - W(G). \]

The degree distance of a connected graph \( G \), denoted by \( DD(G) \), is defined as
$DD(G) = \sum_{[u,v] \subseteq V(G)} (d_G(u) + d_G(v))d_G(u, v) = \frac{1}{2} \sum_{u,v \in V(G)} (d_G(u) + d_G(v))d_G(u, v)$

with the summation runs over all ordered pairs of vertices of $G$.

The product degree distance of a connected graph $G$, denoted by $DD_*(G)$, is defined as

$DD_*(G) = \sum_{[u,v] \subseteq V(G)} d_G(u)d_G(v)d_G(u, v) = \frac{1}{2} \sum_{u,v \in V(G)} d_G(u)d_G(v)d_G(u, v)$

with the summation runs over all ordered pairs of vertices of $G$.

The reciprocal degree distance of a connected graph $G$, denoted by $RDD(G)$, is defined as

$RDD(G) = \sum_{[u,v] \subseteq V(G)} \frac{(d_G(u) + d_G(v))}{d_G(u, v)} = \frac{1}{2} \sum_{u,v \in V(G)} \frac{(d_G(u) + d_G(v))}{d_G(u, v)}$

with the summation runs over all ordered pairs of vertices of $G$.

The reciprocal product degree distance of a connected graph $G$, denoted by $RDD_*(G)$, is defined as

$RDD_*(G) = \sum_{[u,v] \subseteq V(G)} \frac{d_G(u)d_G(v)}{d_G(u, v)} = \frac{1}{2} \sum_{u,v \in V(G)} \frac{d_G(u)d_G(v)}{d_G(u, v)}$

with the summation runs over all ordered pairs of vertices of $G$. 

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The first Zagreb index $M_1(G)$ and the second Zagreb index $M_2(G)$ of a graph $G$ are defined as

$$M_1(G) = \sum_{u \in V(G)} d_G(u)^2 = \sum_{uv \in E(G)} (d_G(u) + d_G(v))$$

and

$$M_2(G) = \sum_{uv \in E(G)} d_G(u)d_G(v),$$

respectively.

The first Zagreb coindex $\overline{M}_1(G)$ and the second Zagreb coindex $\overline{M}_2(G)$ of a graph $G$ are defined as

$$\overline{M}_1(G) = \sum_{uv \notin E(G)} (d_G(u) + d_G(v))$$

and

$$\overline{M}_2(G) = \sum_{uv \notin E(G)} d_G(u)d_G(v),$$

respectively.

1.2 Review of Literature

In this thesis, we have obtained the exact values of the topological indices, namely, Wiener index, hyper-Wiener index, Harary index, reciprocal degree distance and reciprocal product degree distance of some classes of product graphs.
A graph invariant is a real number related to a graph which is invariant under graph isomorphism, that is, it does not depend on the labeling or the pictorial representation of a graph. In chemistry, graph invariants are known as topological indices. Topological indices have many applications as tools for modeling chemical and other properties of molecules. The topological indices and graph invariants based on distances between vertices of a graph are widely used in mathematical chemistry [60]. These indices may be used to derive quantitative structure-property or structure-activity relationships (QSPR/QSAR). The Wiener index is one of the most used topological indices with high correlation with many physical and chemical indices of molecular compounds. It is used in the study of paraffin boiling points [63].

Gutman and Yeh [34] examined operations on a connected graph that have been studied by Yan et al. [66]. H.B. Walikar et al. [64] obtained the upper and lower bounds for $W(G)$, in terms of its radius, diameter, order, size, independence number, connectivity and chromatic number of $G$. Very recently Balakrishnan et al. [9] have given a sharp lower bound for the Wiener index of a graph $G$ in terms of its order, size and diameter. The Wiener index of four new sums of two graphs are studied in [22]. The Wiener index of the $k$th power of a given graph is computed in [4]. Characterization of trees which minimize the Wiener index among all trees of given order and maximum degree and the trees which maximize the Wiener index among all
trees of given order that have only vertices of two different degrees are computed in [24]. Balakrishan et. al. [10] have given an expression for $W(G)$ in terms of the Wiener indices of the blocks of $G$ and other quantities.

Motivated by the Wiener index, Randic [55] introduced an extension of the Wiener index for trees, and this has come to be known as the hyper-Wiener index. Then, as a generalization of the Wiener index, Klein et al. [47] generalized Randic’s definition for all connected graphs. Applications of the hyper-Wiener index as well as its calculation are well explained in [46, 49, 50, 51]. The hyper-Wiener index of four new sums of two graphs are studied in [21]. Gutman [34] discuss the Wiener index of $G \Box H$. Khalifeh et. al. [45] have obtained the exact hyper-Wiener indices of the cartesian product, composition, join and disjunction of graphs. The Wiener and hyper-Wiener indices of the tensor products of graphs are studied by Vumar et. al. [37]. Very recently Feng et. al. [23] have obtained a sharp lower bound for the hyper-Wiener index of graphs with a given matching number.

Relationship between the Hosoya polynomial and the hyper-Wiener index of a graph is studied in [14]. The exact formula for the hyper-Wiener index of the generalized hierarchical product of two graphs is obtained in [21]. Relations between Wiener, hyper-Wiener and Zagreb indices of a graph have been studied by Zhou and Gutman [70]. Bounds on the hyper-Wiener index of the $k$th power of a given graph are given by Zhang [67]. The reverse Wiener index was proposed by
Balaban et al. in 2000 [8]. It is important for a reverse problem and it is also found applications in modeling of structure property relations. Some mathematical properties of the reverse Wiener index may be found in [42, 12, 48, 71].

In this thesis, we have obtained the the Wiener-type indices of strong and wreath product of graphs are obtained. As an immediate consequence, the formulae for Wiener index, Harary index, reciprocal Wiener index, hyper-Wiener index and Tratch-Stankevich-Zefirov index of strong and wreath product of graphs are computed. Also, the Wiener, hyper Wiener and reverse Wiener indices of some more classes of graphs are obtained.

The Harary index of a graph $G$ has been introduced independently by Plavsic et al. [54] and by Ivanciuc et al. [40] in 1993. Its applications and mathematical properties are well studied in [15, 23, 68, 52]. Zhou et al. [69] have obtained the lower and upper bounds of the Harary index of a connected graph. Very recently, Xu et al. [65] have obtained lower and upper bounds for the Harary index of a connected graph in relation to $\chi(G)$, chromatic number of $G$ and $\omega(G)$, clique number of $G$. and characterized the extremal graphs that attain the lower and upper bounds of Harary index. Also, Feng et. al. [23] have given a sharp upper bound for the Harary indices of graphs based on the matching number, that is, the size of a maximum matching.
In this thesis, the exact formulae for the Harary indices of join, disjunction, symmetric difference, strong product and tensor product of graphs are obtained. We apply some of our results to compute the Harary indices of fan graph, wheel graph, open fence and closed fence graphs.

The degree distance of a graph was introduced by Dobrynin and Kochetova [19] and the weighted version of the Wiener index is given by Gutman [28]. The degree distance is a structure descriptor based on molecular topology, of quantitative relations between structure and activity. Its physiochemical applications range from the prediction of boiling points to the calculation of velocity of ultrasound in organic materials [6]. In [28], Gutman showed that if G is a tree on n vertices, then $DD^*_s(G) = 4W(G) - n(n-1)$, where $DD^*_s(G)$ and $W(G)$ denote the degree distance and Wiener index of $G$. In [13] it has been demonstrated that the Wiener index and the degree distance are closely related for certain classes of molecular graphs. In [57], Tomescu proved one of the conjectures and disproved the other made by Dobrynin and Kochetova [19] on the minimum and maximum values of the degree distance of a graph. Hou and Chang [38] have obtained the maximum degree distance among unicyclic graphs on n vertices. In [11], Bucicovschi and Cioaba, have studied the minimum degree distance of graphs with given order and size. In [58], Tomescu has obtained the minimum degree distance of the connected unicyclic and bicyclic graphs in terms of the degree sequence. In [16], Dankel-
mann et al. have presented an asymptotically sharp upper bound of the degree distance of graphs with given order and diameter. In [59], Tomescu has deduced some properties of connected graphs having minimum degree distance.

In [28], Gutman defined the Schultz index of the second kind, which is now known as the product degree distance. If G is a tree on n vertices, then Wiener index $W(G)$ and product degree distance $DD^*_x(G)$ are related by $DD^*_x(G) = 4W(G) - (2n - 1)(n - 1)$; see [28].

In [25], Feng and Liu have determined that among all bicyclic graphs, the graph formed from two triangles linked by a path has maximal product degree distance. Andova et al. have shown that among all graphs on n vertices, the star graph $S_n$ has minimal product degree distance and $P_n$ has maximal product degree distance. They have also obtained upper and lower bounds on product degree distance for graphs with minimal and graphs with maximum product degree distance [7]. In [26], Feng has characterized unicyclic graphs with given girth having minimal product degree distance.

In this thesis, we have obtained the degree distance and product degree distance of join, tensor and strong product of graphs are computed. We apply some of our results to compute the degree distance and product degree distance of fan graph, wheel graph, open fence and closed fence graphs.
Hua and Zhang [39] introduced a new graph invariant named reciprocal degree distance, which can be seen as a degree weight version of Harary index. Hua and Zhang [39] have obtained lower and upper bounds for the reciprocal degree distance of graph in terms of other graph invariants including the degree distance, Harary index, the first Zagreb index, the first Zagreb coindex, pendent vertices, independence number, chromatic number and vertex and edge connectivity. The chemical applications and mathematical properties of the reciprocal degree distance are well studied in [5, 53, 56]. In Su et.al. [36] introduce the reciprocal product degree distance or or multiplicatively weighted Harary index of graphs, which can be seen as a product degree weight version of Harary index. The Zagreb index have been introduced more than thirty years ago by Gutman and Trinajestic [29]. The Zagreb indices are found to have applications in QSPR and QSAR studies as well, see [17]. Noticing that contribution of nonadjacent vertex pair should be taken into account when computing the weighted wiener polynomials of certain composite graphs, see [20], Ashrafi et al. [1, 2] was introduced the Zagreb coindex. In this thesis, the exact formulae for the reciprocal product degree distance of $G + H$, $G \times K_{m_0, m_1, \ldots, m_r-1}$ and $G \boxtimes K_{m_0, m_1, \ldots, m_r-1}$ are obtained.
1.3 Results obtained

The thesis consists of six chapters. In chapter 1, we give the notation and definitions required throughout the thesis.

In chapter 2, we study the reciprocal product degree distance of join and tensor product of graphs. We have proved the following result in this chapter.

1. Let $G_1$ and $G_2$ be graphs with $n$ and $m$ vertices $p$ and $q$ edges, respectively. Then

$$\text{RDD}^*(G_1 + G_2) = M_2(G_1) + M_2(G_2) + mM_1(G_1) + nM_1(G_2) + \frac{1}{2}(M_2(G_1) + M_2(G_2)) + \frac{m}{2}M_1(G_1) + \frac{n}{2}M_1(G_2) + 4pq + 2mn(p + q) + \frac{1}{2}(m^2p + n^2q) + \frac{mn}{4}(6mn - m - n).$$

2. Let $G$ be a connected graph with $n \geq 2$ vertices and $m$ edges and let $E_2$ be the set of edges of $G$ which do not lie on any $C_3$ of it. If $n_0$ and $q$ are the numbers of vertices and edges of $K_{m_0, m_1, \ldots, m_{r-1}}$, $r \geq 3$, respectively, then

$$\text{RDD}^*(G \times K_{m_0, m_1, \ldots, m_{r-1}}) = 4q^2\text{RDD}^*(G) + \frac{M_1(G)}{4}(n_0^3 + 4q^2 - 4qn_0 - \sum_{j=0}^{r-1} m_j^3) - \left(\frac{M_2(G)}{2} + \sum_{v_i, v_k \in E_2} \frac{d_{G(v_i)}d_{G(v_k)}}{12}\right)(n_0^4 - 2qn_0^2 - 2n_0 \sum_{j=0}^{r-1} m_j^3 + \sum_{j=0}^{r-1} m_j^4).$$

3. Let $G$ be a connected graph with $n \geq 2$ vertices and $m$ edges. If each edge of $G$ is on a $C_3$, then

$$\text{RDD}^*(G \times K_{m_0, m_1, \ldots, m_{r-1}}) = 4q^2\text{RDD}^*(G) + \frac{M_1(G)}{4}(n_0^3 + 4q^2 - 4qn_0 - \sum_{j=0}^{r-1} m_j^3) - \frac{M_2(G)}{2}(n_0^4 - 2qn_0^2 - 2n_0 \sum_{j=0}^{r-1} m_j^3 + \sum_{j=0}^{r-1} m_j^4), \ r \geq 3.$$
4. If $G$ is a connected triangle free graph on $n \geq 2$ vertices and $m$ edges, then
\[
RDD_*(G \times K_{m_0, m_1, \ldots, m_{r-1}}) = 4q^2RDD_*(G) + \frac{M_t(G)}{4}(n_0^3 + 4q^2 - 4qn_0 - \sum_{j=0}^{r-1} m_j^3) - \frac{2M_t(G)}{3}(n_0^4 - 2qn_0^2 - 2n_0 \sum_{j=0}^{r-1} m_j^3 + \sum_{j=0}^{r-1} m_j^4), \quad r \geq 3.
\]

5. Let $G$ be a connected graph with $n \geq 2$ vertices and $m$ edges. Let $E_2$ be the set of edges of $G$ which do not lie on a triangle. Then
\[
RDD_*(G \times K_{r(s)}) = r^2(r - 1)^2s^4 \quad RDD_*(G) + \frac{M_t(G)}{4}rs^3(r - 1)^2 - r^2 + 2r - 1) - \left(\frac{M_t(G)}{2} + \sum_{v_i \in E_2} d_G(v_i)d_G(v_i)\right)rs^4(r - 1)^2, \quad r \geq 3.
\]

In chapter 3, we discuss about the reciprocal product degree distance of strong product of graphs. We have proved the following result in this chapter.

1. Let $G$ be a connected graph with $n$ vertices and $m$ edges. Then
\[
RDD_*(G \times K_{m_0, m_1, \ldots, m_{r-1}}) \text{ is obtained.}
\]

2. Let $G$ be a connected graph with $n$ vertices and $m$ edges. Then
\[
RDD_*(G \times K_{r(s)}) = RDD_*(G \times K_{r(s)}) = RDD_*(G)\left(r^4s^4 - r^3s^4 + 2r^3s^3 - 2r^2s^3 + r^2s^2 + rs^3 + r^2s^4(r - 1)^2\right) + \frac{M_t(G)}{4}\left(5r^2s^2 - 3r^2s^3 - 10rs^2 + 6rs - 5s - 1 + 6rs + 5s^2 - 3s^3\right) + \frac{m}{2}\left(3r^3s^3 - 3r^3s^4 - 4r^2s^3 - 2r^2s^2 + 2rs^2 + 4r62s^4(r - 1)^2 + (6rs + 1)rs^3 - 3rs^4\right) + \frac{2}{3}\left(2r^2s^4(r - 1)^2 - r^3s^3 - r^2s^3 + 2r^2s^4 - 2rs^3 - rs^4\right) + (r^2s^4(r - 1)^2)H(G) + RDD(G)\left(r^4s^4 - r^3s^4 + r^3s^3 - 2r^2s^3 + r^2(r - 1)^2 + rs^3\right) + \frac{M_t(G)}{2}\left(r^3s^3 + rs - r^3s^4 - 4r^2s^3 + 2r^2s^2 - 2rs^2 - 2r^2s^4 - rs^4\right), \quad r \geq 2.
\]

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3. Let $G$ be a connected graph with $n$ vertices and $m$ edges. Then
\[
RDD_*(G \boxtimes K_r) = r^2 (r^2 RDD_*(G) + r(r-1)RDD(G) + (r-1)^2 H(G)) + r(r-1) \left( \frac{r^2 M_1(G)}{2} + \frac{n(r-1)^2}{2} + 2mr(r-1) \right), \ r \geq 2.
\]

In chapter 4, we discuss about the some distance based topological indices of strong and wreath product of graphs. We have proved the following result in this chapter.

1. Let $G$ be a connected graph with $n$ vertices. Then
\[
W_\lambda(G \boxtimes K_{m_0, m_1, \ldots, m_{r-1}}) = n_0^2 W_\lambda(G') = n_0^2 W_\lambda(G) + m(n_0 + 2q)(1 - 2^4) + n_0^2 (2^4 - 1) + \frac{n}{2} (n_0(n_0 - 1)2^4 + 2q(1 - 2^4)), \ r \geq 2.
\]

2. Let $G$ be a connected graph with $n$ vertices and $m$ edges. Then
\[
W_\lambda(G \boxtimes K_{r(s)}) = r^2 s^2 W_\lambda(G) + m(1 - 2^4)rs(1 - s) + \frac{nrs}{2} (rs - s + 2^3(s - 1)), \ r \geq 2.
\]

3. Let $G$ be a connected graph with $n$ vertices and $m$ edges. Then
\[
W_\lambda(G \boxtimes K_r) = r^2 W_\lambda(G) + \frac{nr(r-1)}{2}, \ r \geq 2.
\]

4. Let $G$ be connected graph with $n$ vertices and $m$ edges. Then

(i) $W(G \boxtimes K_{m_0, m_1, \ldots, m_{r-1}}) = n_0^2 W(G) + m(n_0^2 - n_0 - 2q) + n(n_0^2 - n_0 - q)$.

(ii) $W_{-1}(G \boxtimes K_{m_0, m_1, \ldots, m_{r-1}}) = n_0^2 W_{-1}(G) - m(n_0^2 - n_0 - 2q) + \frac{n}{4}(n_0^2 - n_0 + 2q)$.

(iii) $W_{-2}(G \boxtimes K_{m_0, m_1, \ldots, m_{r-1}}) = n_0^2 W_{-2}(G) - \frac{3m}{4}(n_0^2 - n_0 - 2q) + \frac{n}{8}(n_0^2 - n_0 + 6q)$.
(iv) \( WW(G \boxtimes K_{m_0, m_1, \ldots, m_{r-1}}) = n_0^2 WW(G) + 2m(n_0^2 - n_0 - 2q) + \frac{n}{2}(3n_0^2 - 3n_0 - 4q) \).

(v) \( W_1(G \boxtimes K_{m_0, m_1, \ldots, m_{r-1}}) = n_0^2 W_1(G) + m(1 - \sqrt{2})(n_0 + q - n_0^2) + \frac{n}{2}(n_0(n_0 - 1) \sqrt{2} + 2q(1 - \sqrt{2})) \).

(vi) \( TSZ(G \boxtimes K_{m_0, m_1, \ldots, m_{r-1}}) = n_0^2 TSZ(G) + 4m(n_0^2 - n_0 - 2q) + \frac{n}{6}(15n_0^2 - 15n_0 - 24q) \).

5. Let \( G \) and \( G' \) be two connected graphs with \( |V(G)| = n \) and \( |V(G')| = m \). Then \( W_{\lambda}(G \circ G') = m^2 W_{\lambda}(G) + 2^{1-1}nm(m-1) + (1 - 2^1)n |E(G')| \).

6. (i) \( W(G \circ G') = m^2 W(G) + nm(m - 1) - n |E(G')| \).

(ii) \( W_{-1}(G \circ G') = m^2 W_{-1}(G) + \frac{nm(m-1)}{4} + \frac{n}{2} |E(G')| \).

(iii) \( WW(G \circ G') = m^2 WW(G) + \frac{n}{2}(3m(m - 1) - 4 |E(G')|) \).

(iv) \( W_{-2}(G \circ G') = m^2 W_{-2}(G) + \frac{nm(m-1)}{8} + \frac{3n}{4} |E(G')| \).

(v) \( W_{\frac{1}{2}}(G \circ G') = m^2 W_{\frac{1}{2}}(G) + \frac{nm(m-1)}{\sqrt{2}} + n(1 - \sqrt{2}) |E(G')| \).

7. Let \( G \) be a connected graph with \( n \) vertices. Then
\[
\Lambda(G \boxtimes K_{m_0,m_1,\ldots,m_{r-1}}) = \frac{n_0(n_0 - 1)}{2} D(G) - n_0^2 W(G) - m(n_0^2 - n_0 - 2q) - n(n_0^2 - n_0 - q), \text{ for } r \geq 2.
\]

8. Let \( G \) be a connected graph with \( n \) vertices. Then
\[
\Lambda(G \boxtimes K_{r(s)}) = \frac{r(s)(s-1)}{2} D(G) - r^2 s^2 W(G) - mrs(s - 1) - \frac{n}{2}(rs + s - 2), \text{ for } r \geq 2.
\]

9. Let \( G \) be a connected graph with \( n \) vertices. Then \( \Lambda(G \boxtimes K_r) = \frac{r(r-1)}{2}(D(G) - n) - r^2 W(G), \text{ for } r \geq 2.\)
10. Let $G$ and $G'$ be two connected graphs with $|V(G)| = n$ and $|V(G')| = m$. Then
\[
\Lambda(G \circ G') = \frac{mn}{2}(mn - 2m + 1) - m^2W(G) + n|E(G')|.
\]

In chapter 5, we study about the Wiener, hyper-Wiener and reverse Wiener indices of some graph operations. We have proved the following result in this chapter.

1. Let $G$ and $H$ be graphs with $n_1$ and $n_2$ vertices and let the copies of $H$ used in the construction of $G\{H\}$ be rooted in vertex $h$. Then
\[
W(G\{H\}) = n_2^2W(G) + n_1W(H) + n_1(n_1 - 1)n_2D_H(h),
\]
where $D_H(h) = \sum_{u \in V(H)} d_H(u, h)$.

2. Let $G$ be a connected graph with $n$ vertices. Then $W(K_2 \bullet G) = 4W(G) + 2n$.

3. Let $G$ be a connected graph with $n$ vertices. Then $W(K_2 \star G) = n(3n - 2)$.

4. If $G$ is a graph on $n$ vertices with each edge of $G$ is on a triangle, then
\[
W(G \times K_{a,b}) = (a + b)^2W(G) + (a^2 + b^2)\ell_1 + ab(3n + 2\ell_2) + n(a^2 + b^2 - a - b).
\]

5. Let $G$ and $H$ be graphs with $n_1$ and $n_2$ vertices and let the copies of $H$ used in the construction of $G\{H\}$ be rooted in vertex $h$. 
Then $WW(G\{H\}) = n_2^2 WW(G) + n_1 WW(H) + 2n_2 W(G)D_H(h) + \frac{n_1(n_1-1)}{2}(n_2D_H^2(h) + (D_H(h))^2)$,

where $D_H(h) = \sum_{u \in V(H)} d_H(u, h)$ and

$D_H^2(h) = \sum_{u \in V(H)} (d_H(u, h) + d_H^2(u, h))$.

6. Let $G$ be a connected graph with $n$ vertices. Then

$WW(K_2 \bullet G) = 4WW(G) + 3n$.

7. Let $G$ be a connected graph with $n$ vertices. Then

$WW(K_2 \star G) = \frac{n(8n-5)}{2}$.

8. If $G$ is a graph on $n$ vertices with each edge of $G$ is on a triangle, then

$WW(G \times K_{a,b}) = (a+b)^2 WW(G) + (a^2+b^2)\ell_1 + 2ab\ell_2 + \frac{3n}{2}(a^2+b^2-a-b+4ab) + (a^2+b^2) \sum_{i,k=0}^{n-1} d_G(v_i, v_k) + 2ab \sum_{i,k=0}^{n-1} d_G(v_i, v_k)$.

9. Let $G$ and $H$ be graphs with $n_1$ and $n_2$ vertices and let the copies of $H$ used in the construction of $G\{H\}$ be rooted in vertex $h$. Then

$\Lambda(G\{H\}) = \frac{n_1n_2}{2} \left((n_1n_2 - 1)(2e(h) + D(G)) - 2(n_1 - 1)D_H(h)\right) - n_2^2 W(G) - n_1 W(H)$, where $D_H(h) = \sum_{u \in V(H)} d_H(u, h)$.

10. Let $G$ be a connected graph with $n$ vertices. Then

$\Lambda(K_2 \bullet G) = n(2n-1)D(G) - 4W(G) - 2n$.

11. Let $G$ be a connected graph with $n$ vertices. Then $\Lambda(K_2 \star G) = n^2$. 
12. If $G$ is a graph on $n$ vertices with each edge of $G$ is on a triangle, then

$$\Lambda(G \times K_{a,b}) = \frac{n}{2} \left( (n - 1)D(G) - 2(a^2 + b^2 - a - b + 3ab) \right) - (a + b)^2 W(G) - (a^2 + b^2) \ell_1 - 2ab\ell_2.$$ 

In chapter 6, we discuss about the Harary index and degree distance and product degree distance of some graph operations. We have proved the following result in this chapter.

1. Let $G_1$ and $G_2$ be graphs with $n$ and $m$ vertices, respectively. Then

$$H(G_1 + G_2) = mn + \frac{1}{2}(|E(G_1)| + |E(G_2)|) + \frac{1}{4}(n(n - 1) + m(m - 1)).$$

2. Let $G_1$ and $G_2$ be graphs with $n$ and $m$ vertices, respectively. Then

$$H(G_1 \lor G_2) = \frac{m^2}{2} |E(G_1)| + \frac{n^2}{2} |E(G_2)| - |E(G_1)| |E(G_2)| + \frac{1}{4}mn(mn - 1).$$

3. Let $G_1$ and $G_2$ be graphs with $n$ and $m$ vertices, respectively. Then

$$H(G_1 \oplus G_2) = \frac{m^2}{2} |E(G_1)| + \frac{n^2}{2} |E(G_2)| - |E(G_1)| |E(G_2)| + \frac{1}{4}mn(mn - 1).$$

4. Let $G$ be a connected graph with $n$ vertices and $m$ edges. Then

$$H(G \boxtimes K_r) = r^2 H(G) + \frac{1}{2}nr(r - 1).$$

5. Let $G$ be a connected graph with $n \geq 2$ vertices and $m$ edges and let $\mu$ be the number of edges of $G$ which do not lie on any $C_3$ of it. Then
\[ H(G \times K_r) = r^2 H(G) + \frac{nr(r-1)}{4} - \frac{r}{2}(m + \frac{\mu}{3}), \text{ where } r \geq 3. \]

6. Let \( G_1 \) and \( G_2 \) be graphs with \( n \) and \( m \) vertices, respectively. Then
\[ DD(G_1 + G_2) = M_1(G) + M_1(H) + 2(M_1(G) + M_1(H)) + nm(3n + 3m - 4). \]

7. Let \( G_1 \) and \( G_2 \) be graphs with \( n \) and \( m \) vertices \( p \) and \( q \) edges, respectively. Then
\[ DD_*(G_1 + G_2) = M_2(G) + M_2(H) + mM_1(G) + nM_1(H) + 2M_2(G) + 2M_2(H) + 2mM_1(G) + 2nM_1(H) + nm(3mn - m - n) + 4pq - m^2p - n^2q + 2mn(p + q). \]

8. Let \( G \) be a connected graph with \( n \) vertices and \( m \) edges. Then
\[ DD(G \boxtimes K_r) = r\left(r^2W_+(G) + 2r(r-1)W(G) + 2r(r-1)m + n(r-1)^2\right). \]

9. Let \( G \) be a connected graph with \( n \) vertices and \( m \) edges. Then
\[ DD_*(G \boxtimes K_r) = r\left(r^3W_+(G) + r^2(r-1)W_+(G) + r(r-1)^2W(G) + \frac{1}{2}r^2(r-1)^2M_1(G) + \frac{1}{2}(r-1)^3n + 2r(r-1)^2m\right). \]