Chapter 6

Distance Based Graph Invariant of Graph Operations

In this paper, the exact formulae for the Harary indices of join, disjunction, symmetric difference, strong product and tensor product of graphs are obtained. Also, the Schultz and modified Schultz indices of join and strong product of graphs are computed. We apply some of our results to compute the Harary, Schultz and modified Schultz indices of fan graph, wheel graph, open fence and closed fence graphs.

6.1 Harary indices of composite graphs

In this section, first we compute the Harary indices of join, disjunction and symmetric difference of two connected graphs. The proof of the following lemma follows easily from the definition of join, disjunction, symmetric difference of two graphs.
Lemma 6.1.1. Let $G$ and $H$ be two graphs. Then

(i) $d_{G+H}(u, v) = \begin{cases} 
0, & \text{if } u = v \\
1, & \text{if } uv \in E(G) \text{ or } uv \in E(H) \text{ or } (u \in V(G) \text{ and } v \in V(H)) \\
2, & \text{otherwise} 
\end{cases}$

(ii) $d_{G+H}(x) = \begin{cases} 
d_G(x) + |V(H)|, & \text{if } x \in V(G) \\
d_H(x) + |V(G)|, & \text{if } x \in V(H) 
\end{cases}$

(iii) $d_{G\vee H}((u, x), (v, y)) = \begin{cases} 
0, & \text{if } u = v \text{ and } x = y \\
1, & \text{if } uv \in E(G) \text{ or } xy \in E(H) \\
2, & \text{otherwise} 
\end{cases}$

(iv) $d_{G\oplus H}((u, x), (v, y)) = \begin{cases} 
0, & \text{if } u = v \text{ and } x = y \\
1, & \text{if } uv \in E(G) \text{ or } xy \in E(H) \text{ but not both} \\
2, & \text{otherwise.} 
\end{cases}$

Theorem 6.1.1. Let $G_1$ and $G_2$ be graphs with $n$ and $m$ vertices, respectively. Then

$$H(G_1 + G_2) = mn + \frac{1}{2}(|E(G_1)| + |E(G_2)|) + \frac{1}{4}(n(n - 1) + m(m - 1)).$$

Proof. Set $V(G_1) = \{u_1, u_2, \ldots, u_n\}$ and $V(G_2) = \{v_1, v_2, \ldots, v_m\}$. Then
by Lemma 6.1.1, we have

\[ H(G_1 + G_2) = \frac{1}{2} \sum_{u,v \in V(G_1+G_2)} \frac{1}{d_{G_1+G_2}(u,v)} \]

\[ = \frac{1}{2} \left( \sum_{uv \in E(G_1)} \frac{1}{d_{G_1+G_2}(u,v)} + \sum_{uv \in E(G_2)} \frac{1}{d_{G_1+G_2}(u,v)} \right) \]

\[ + \sum_{u \in V(G_1), v \in V(G_2)} \frac{1}{d_{G_1+G_2}(u,v)} \]

\[ = \frac{1}{2} \left( |E(G_1)| + \frac{1}{2} \left( \frac{n(n-1)}{2} - |E(G_1)| \right) + |E(G_2)| \right) \]

\[ + \frac{1}{2} \left( \frac{m(m-1)}{2} - |E(G_2)| \right) + mn \]

(by Lemma 6.1.1)

\[ = mn + \frac{1}{2} \left( |E(G_1)| + |E(G_2)| \right) + \frac{1}{4} (n(n-1) + m(m-1)). \]

Using Theorem 6.1.1, we have the following corollary.

**Corollary 6.1.1.** Let \( G \) be graph on \( n \) vertices. Then

\[ H(G + K_m) = mn + \frac{1}{2} |E(G)| + \frac{1}{4} \left( n(n-1) + m(m-1) \right). \]

Using Corollary 6.1.1, we compute the formula for Harary indices of fan and wheel graphs, \( P_n + K_1 \) and \( C_n + K_1 \), see Figure 2.1.

**Example 6.1.1.** (i) \( H(P_n + K_1) = \frac{1}{4} (n^2 + 5n - 2) \)

(ii) \( H(C_n + K_1) = \frac{1}{4} (n^2 + 5n) \).
Theorem 6.1.2. Let $G_1$ and $G_2$ be graphs with $n$ and $m$ vertices, respectively. Then

$$H(G_1 \vee G_2) = \frac{m^2}{2} |E(G_1)| + \frac{m^2}{2} |E(G_2)| - |E(G_1)||E(G_2)| + \frac{1}{4}mn(mn - 1).$$

Proof. Set $V(G_1) = \{u_1, u_2, \ldots, u_n\}$ and $V(G_2) = \{v_1, v_2, \ldots, v_m\}$. Let $x_{ij}$ denote the vertex $(u_i, v_j)$ of $G_1 \vee G_2$. Then by Lemma 6.1.1, we have

$$H(G_1 \vee G_2)$$

$$= \frac{1}{2} \sum_{x_{ij}, x_{kp} \in V(G_1 \vee G_2)} \frac{1}{d_{G_1 \vee G_2}(x_{ij}, x_{kp})}$$

$$= \frac{1}{2} \sum_{x_{ij} \in V(G_1 \vee G_2)} \left\{ \left( \frac{1}{2}md(u_i) + \frac{1}{2}nd(v_j) - d(u_i) + d(v_j) \right) \right.$$

$$\left. + \frac{1}{2} \left( mn - md(u_i) - nd(v_j) + d(u_i)d(v_j) - 1 \right) \right\}$$

$$= \frac{1}{2} \sum_{x_{ij} \in V(G_1 \vee G_2)} \left( \frac{1}{2}md(u_i) + \frac{1}{2}nd(v_j) - \frac{1}{2}d(u_i)d(v_j) + \frac{1}{2}(mn - 1) \right)$$

$$= \frac{m^2}{2} |E(G_1)| + \frac{m^2}{2} |E(G_2)| - |E(G_1)||E(G_2)| + \frac{1}{4}mn(mn - 1).$$

□

Using similar argument as Theorem 6.1.2, one can prove the following result:

Theorem 6.1.3. Let $G_1$ and $G_2$ be graphs with $n$ and $m$ vertices, respectively. Then

$$H(G_1 \oplus G_2) = \frac{m^2}{2} |E(G_1)| + \frac{m^2}{2} |E(G_2)| - |E(G_1)||E(G_2)| + \frac{1}{4}mn(mn - 1).$$
6.2 Harary index of strong product of graphs

In this section, we obtain the Harary index of \( G \bowtie K_r \).

**Theorem 6.2.1.** Let \( G \) be a connected graph with \( n \) vertices and \( m \) edges. Then \( H(G \bowtie K_r) = r^2H(G) + \frac{1}{2}nr(r - 1) \).

**Proof.** Set \( V(G) = \{u_1, u_2, \ldots, u_n\} \) and \( V(K_r) = \{v_1, v_2, \ldots, v_r\} \). Let \( x_{ij} \) denote the vertex \((u_i, v_j)\) of \( G \bowtie K_r \). One can see that for any pair of vertices \( x_{ij}, x_{kp} \in V(G \bowtie K_r) \), \( d_{G \bowtie K_r}(x_{ij}, x_{ip}) = 1 \) and \( d_{G \bowtie K_r}(x_{ij}, x_{kp}) = d_G(u_i, u_k) \).

\[
H(G \bowtie K_r) = \frac{1}{2} \sum_{x_{ij}, x_{kp} \in V(G \bowtie K_r)} \frac{1}{d_{G \bowtie K_r}(x_{ij}, x_{kp})}
\]

\[
= \frac{1}{2} \left( \sum_{i=0}^{n-1} \sum_{j=0}^{r-1} \frac{1}{d_{G \bowtie K_r}(x_{ij}, x_{ip})} + \sum_{i,k=0}^{n-1} \sum_{j=0}^{r-1} \frac{1}{d_{G \bowtie K_r}(x_{ij}, x_{kj})} \right)
\]

\[
+ \sum_{i,k=0}^{n-1} \sum_{j,p=0}^{r-1} \frac{1}{d_{G \bowtie K_r}(x_{ij}, x_{kp})} \right)
\]

\[
= \frac{1}{2} \left( nr(r - 1) + 2rH(G) + 2r(r - 1)H(G) \right)
\]

\[
= r^2H(G) + \frac{1}{2}nr(r - 1).
\]

One can see that \( H(P_n) = n \left( \sum_{i=1}^{n} \frac{1}{i} \right) - n \). By using Theorem 6.2.1, \( H(C_n) \) and \( H(P_n) \), we obtain the exact Harary indices of the open and closed fence graphs, see Figures 3.3 and 6.1.
Example 6.2.1. The following are true:

(i) \( H(P_n \boxtimes K_2) = n \left( 4 \sum_{i=1}^{n} \frac{1}{i} - 3 \right) \)

(ii) \( H(C_n \boxtimes K_2) = \begin{cases} 
  n \left( 1 + 4 \sum_{i=1}^{\frac{n}{2}} \frac{1}{i} \right) - 4 & \text{n is even} \\
  n \left( 1 + 4 \sum_{i=1}^{\frac{n-1}{2}} \frac{1}{i} \right) & \text{n is odd}.
\end{cases} \)

6.3 Harary index of tensor product of graphs

In this section, we compute the Harary index of \( G \times K_r \).

Theorem 6.3.1. Let \( G \) be a connected graph with \( n \geq 2 \) vertices and \( m \) edges and let \( \mu \) be the number of edges of \( G \) which do not lie on any \( C_3 \) of it. Then

\[
H(G \times K_r) = r^2 H(G) + \frac{nr(r-1)}{4} - \frac{r}{2} \left( m + \frac{\mu}{3} \right), \text{ where } r \geq 3.
\]
Proof. Set $V(G) = \{u_1, u_2, \ldots, u_n\}$ and $V(K_r) = \{v_1, v_2, \ldots, v_r\}$. Let $x_{ij}$ denote the vertex $(u_i, v_j)$ of $G \times K_r$. By the definition of Harary index

$$H(G \times K_r) = \frac{1}{2} \sum_{x_{ij}, x_{kp} \in V(G \times K_r)} \frac{1}{d_{G \times K_r}(x_{ij}, x_{kp})} = \frac{1}{2} \left( \sum_{i=0}^{n-1} \sum_{j=0}^{r-1} \frac{1}{d_{G \times K_r}(x_{ij}, x_{ip})} + \sum_{i=0}^{n-1} \sum_{j=0}^{r-1} \frac{1}{d_{G \times K_r}(x_{ij}, x_{kj})} + \sum_{i=0}^{n-1} \sum_{j=0}^{r-1} \frac{1}{d_{G \times K_r}(x_{ij}, x_{kp})} \right).$$

(6.1)

First we compute

$$\sum_{i=0}^{n-1} \sum_{j=0}^{r-1} \sum_{j \neq p} \frac{1}{d_{G \times K_r}(x_{ij}, x_{ip})}.$$  

Let $E_1 = \{uv \in E(G) \mid uv \text{ is on a } C_3 \text{ in } G\}$ and $E_2 = E(G) - E_1$. 

Next we compute $\sum_{j=0}^{r-1} \sum_{i=0}^{n-1} \sum_{i \neq k} \frac{1}{d_{G \times K_r}(x_{ij}, x_{kj})}$. 

Let $E_1 = \{uv \in E(G) \mid uv \text{ is on a } C_3 \text{ in } G\}$ and $E_2 = E(G) - E_1$. 

96
\[
\sum_{i, k = 0}^{n-1} \frac{1}{d_{G \times K_r}(x_{ij}, x_{kj})}
\]
\[
= \sum_{i, k = 0}^{n-1} \frac{1}{d_{G \times K_r}(x_{ij}, x_{kj})} + \sum_{i, k = 0}^{n-1} \frac{1}{d_{G \times K_r}(x_{ij}, x_{kj})} + \sum_{i, k = 0}^{n-1} \frac{1}{d_{G \times K_r}(x_{ij}, x_{kj})}
\]
\[
= \left( \sum_{i, k = 0}^{n-1} \frac{1}{d_G(u_i, u_k)} + \sum_{i, k = 0}^{n-1} \frac{1}{2} + \sum_{i, k = 0}^{n-1} \frac{1}{3} \right) (\text{by Lemma 2.2.1})
\]
\[
= \left( \sum_{i, k = 0}^{n-1} \frac{1}{d_G(u_i, u_k)} + \sum_{i, k = 0}^{n-1} \left(1 - \frac{1}{2}\right) + \sum_{i, k = 0}^{n-1} \left(1 - \frac{2}{3}\right) \right)
\]
\[
= \left( \sum_{i, k = 0}^{n-1} \frac{1}{d_G(u_i, u_k)} + \sum_{i, k = 0}^{n-1} \frac{1}{d_G(u_i, u_k)} + \sum_{i, k = 0}^{n-1} \frac{1}{d_G(u_i, u_k)} \right)
\]
\[
- \sum_{i, k = 0}^{n-1} \frac{2}{3}, \text{ since } d_G(u_i, u_k) = 1, \text{ in the second and third sums}
\]
\[
= \sum_{i, k = 0}^{n-1} \frac{1}{d_G(u_i, u_k)} - \sum_{i, k = 0}^{n-1} \frac{2}{3} - \sum_{i, k = 0}^{n-1} \left(\frac{1}{2} + \frac{1}{6}\right)
\]
\[
= 2H(G) - \sum_{i, k = 0}^{n-1} \frac{1}{2} - \sum_{i, k = 0}^{n-1} \frac{1}{6}, \text{ by the definition of Harary index}
\]
\[
= 2H(G) - \left(m + \frac{\mu}{3}\right), \text{ where } \mu \text{ and } m \text{ are the numbers of edges of } G
\]

which do not on any $C_3$ and edges of $G$, respectively.  \( (6.3) \)
Now summing (6.3) over \( j = 0, 1, \ldots, r - 1 \), we get,

\[
\sum_{j=0}^{r-1} \left( \sum_{i, k=0, i \neq k}^{n-1} \frac{1}{d_{G \times K_r}(x_{ij}, x_{kj})} \right) = 2rH(G) - (m + \frac{\mu}{3})r. \tag{6.4}
\]

Finally, we compute

\[
\sum_{i, k=0, i \neq k}^{n-1} \sum_{j=0, j \neq p}^{r-1} \frac{1}{d_{G \times K_r}(x_{ij}, x_{kp})} = \sum_{i, k=0, i \neq k}^{n-1} \sum_{j=0, j \neq p}^{r-1} \frac{1}{d_G(u_i, u_k)}, \text{ by Lemma 2.2.1}
\]

\[
= r(r - 1) \sum_{i, k=0, i \neq k}^{n-1} \frac{1}{d_G(u_i, u_k)}
\]

\[
= 2r(r - 1)H(G). \tag{6.5}
\]

Using (6.1), and the sums (6.2), (6.4) and (6.5), respectively, we have,

\[
H(G \times K_r) = r^2H(G) + \frac{nr(r-1)}{4} - \frac{r}{2}(m + \frac{\mu}{3}).
\]

Using Theorem 6.3.1, we have the following corollaries.

**Corollary 6.3.1.** Let \( G \) be a connected graph on \( n \geq 2 \) vertices with \( m \) edges. If each edge of \( G \) is on a \( C_3 \), then \( H(G \times K_r) = r^2H(G) + \frac{nr(r-1)}{4} - \frac{mr}{2} \), where \( r \geq 3 \).
Corollary 6.3.2. If $G$ is a connected triangle free graph on $n \geq 2$ vertices and $m$ edges, then $H(G \times K_r) = r^2 H(G) + \frac{nr(r-1)}{4} - \frac{2mr}{3}$, where $r \geq 3$.

By using Corollary 6.3.2, $H(P_n)$ and $H(C_n)$, we obtain the exact Harary indices of the following graphs.

Example 6.3.1. The following are true:
(i) If $n \geq 2$ and $r \geq 3$, then $H(P_n \times K_r) = nr^2 \left( \sum_{i=1}^{n} \frac{1}{i} \right) - \frac{r}{12}(11n + 9rn - 8)$

$(ii)$ $H(C_n \times K_r) = \begin{cases} 
\frac{3r(5r-3)}{4}, & \text{if } n = 3 \\
r^2 \left( \sum_{i=1}^{n} \frac{1}{i} \right) + \frac{nr}{12}(3r - 11), & \text{if } n > 3 \text{ is odd.}
\end{cases}$

6.4 Degree distance and product degree distance

In this section, first we obtain the degree distance and product degree distance of $G_1 + G_2$ and $G \boxtimes K_r$.

Theorem 6.4.1. Let $G_1$ and $G_2$ be graphs with $n$ and $m$ vertices, respectively. Then

$$DD(G_1 + G_2) = M_1(G) + M_1(H) + 2(M_1(G) + M_1(H)) + nm(3n + 3m - 4).$$

Proof. Set $V(G) = \{u_1, u_2, \ldots, u_n\}$ and $V(H) = \{v_1, v_2, \ldots, v_m\}$. By
Lemma 6.1.1, we have

\[
DD(G + H) = \frac{1}{2} \sum_{u,v \in V(G+H)} \left( d_{G+H}(u) + d_{G+H}(v) \right) d_{G+H}(u, v)
\]

\[
= \frac{1}{2} \left( \sum_{uv \in E(G)} (d_G(u) + m + d_G(v) + m) + 2 \sum_{uv \notin E(G)} (d_G(u) + m + d_G(v) + m)
+ \sum_{uv \in E(H)} (d_H(u) + n + d_H(v) + n)
+ 2 \sum_{uv \in E(H)} (d_H(u) + n + d_H(v) + n)
+ \sum_{u \in V(G), v \in V(H)} (d_G(u) + m + d_H(v) + n) \right)
\]

\[
= M_1(G) + M_1(H) + 2(\overline{M}_1(G) + \overline{M}_1(H))
+ nm(3n + 3m - 4).
\]

\[\square\]

Using similar arguments as Theorem 6.4.1, one can prove the following result.

**Theorem 6.4.2.** Let \(G_1\) and \(G_2\) be graphs with \(n\) and \(m\) vertices \(p\) and \(q\) edges, respectively. Then

\[
DD_*(G_1 + G_2) = M_2(G) + M_2(H) + mM_1(G) + nM_1(H) + 2\overline{M}_2(G) + 2\overline{M}_2(H) + 2m\overline{M}_1(G) + 2n\overline{M}_1(H) + mn(3mn - m - n) + 4pq - m^2p - n^2q + 2mn(p + q).
\]
Using Theorems 6.4.1 and 6.4.2, we have the following corollaries.

**Corollary 6.4.1.** Let $G$ be graph on $n$ vertices. Then

$$DD(G + K_m) = M_1(G) + 2M_1(G) + m(m - 1)^2 + nm(3n + 3m - 4).$$

**Corollary 6.4.2.** Let $G$ be graph on $n$ vertices and $p$ edges. Then

$$DD_s(G + K_m) = M_2(G) + mM_1(G) + 2M_2(G) + 2mM_1(G) + mn(3mn - m - n) + mp(2n - m) + \frac{1}{2}m(m - 1)(4p - n^2 + 2mn + 3m^2 - 4m + 1).$$

Using Corollaries 6.4.1 and 6.4.2, we compute the formulae for degree distance and product degree distance of fan and wheel graphs.

**Example 6.4.1.** The following are true:

(i) $DD(P_n + K_1) = 7n^2 - 13n + 10$

(ii) $DD(C_n + K_1) = n(7n - 9)$

(iii) $DD_s(P_n + K_1) = 12n^2 - 32n + 29$

(iv) $DD_s(C_n + K_1) = 12n^2 - 18n$.

**Theorem 6.4.3.** Let $G$ be a connected graph with $n$ vertices and $m$ edges. Then

$$DD(G \boxtimes K_r) = r\left(r^2W_+(G) + 2r(r - 1)W(G) + 2r(r - 1)m + n(r - 1)^2\right).$$

**Proof.** Set $V(G) = \{u_1, u_2, \ldots, u_n\}$ and $V(K_r) = \{v_1, v_2, \ldots, v_r\}$. Let $x_{ij}$ denote the vertex $(u_i, v_j)$ of $G \boxtimes K_r$. The degree of the vertex $x_{ij}$ in $G \boxtimes K_r$ is $d_G(u_i) + d_{K_r}(v_j) + d_G(u_i)d_{K_r}(v_j)$.
That is \( d_{G \otimes K_r}(x_{ij}) = rd_G(u_i) + (r - 1) \). One can observe that for any pair of vertices \( x_{ij}, x_{kp} \in V(G \otimes K_r) \), \( d_{G \otimes K_r}(x_{ij}, x_{ip}) = 1 \) and \( d_{G \otimes K_r}(x_{ij}, x_{kp}) = d_G(u_i, u_k) \).

\[
DD(G \otimes K_r) = \frac{1}{2} \sum_{x_{ij}, x_{kp} \in V(G \otimes K_r)} \left( d_{G \otimes K_r}(x_{ij}) + d_{G \otimes K_r}(x_{kp}) \right) d_{G \otimes K_r}(x_{ij}, x_{kp})
\]

\[
= \frac{1}{2} \left( \sum_{i=0}^{n-1} \sum_{j=0}^{r-1} \sum_{p=0}^{r-1} \left( d_{G \otimes K_r}(x_{ij}) + d_{G \otimes K_r}(x_{kp}) \right) d_{G \otimes K_r}(x_{ij}, x_{ip}) \right)
\]

\[
+ \sum_{i=0}^{n-1} \sum_{j=0}^{r-1} \sum_{k=0}^{r-1} \left( d_{G \otimes K_r}(x_{ij}) + d_{G \otimes K_r}(x_{kj}) \right) d_{G \otimes K_r}(x_{ij}, x_{kj})
\]

\[
+ \sum_{i=0}^{n-1} \sum_{j=0}^{r-1} \sum_{k=0}^{r-1} \sum_{p=0}^{r-1} \left( d_{G \otimes K_r}(x_{ij}) + d_{G \otimes K_r}(x_{kp}) \right) d_{G \otimes K_r}(x_{ij}, x_{kp}) \right).
\]

We shall obtain the individual sums in the above expression separately.

\[
\sum_{i=0}^{n-1} \sum_{j=0}^{r-1} \sum_{p=0}^{r-1} \left( d_{G \otimes K_r}(x_{ij}) + d_{G \otimes K_r}(x_{ip}) \right) d_{G \otimes K_r}(x_{ij}, x_{ip})
\]

\[
= \sum_{i=0}^{n-1} \sum_{j=0}^{r-1} \sum_{p=0}^{r-1} \left( 2d_G(u_i) + 2(r - 1) + 2(r - 1)d_G(u_i) \right)
\]

\[
= 4r^2(r - 1)m + 2nr(r - 1)^2 \tag{6.6}
\]
\[
\sum_{j=0}^{r-1} \sum_{i,k=0}^{n-1} d_{G \boxtimes K_r}(x_{ij}) d_{G \boxtimes K_r}(x_{kj})
\]
\[
= \sum_{j=0}^{r-1} \sum_{i,k=0}^{n-1} d_G(u_i)
\]
\[
+ (r-1)d_G(u_i) + d_G(u_k) + (r-1)d_G(u_k) + 2(r-1)d_G(u_i, u_k)
\]
\[
= r \sum_{j=0}^{r-1} \sum_{i,k=0}^{n-1} (d_G(u_i) + d_G(u_k)) d_G(u_i, u_k) + \sum_{j=0}^{r-1} \sum_{i,k=0}^{n-1} 2(r-1)d_G(u_i, u_k)
\]
\[
= 2r^2 W_+(G) + 4r(r-1)W(G).
\] (6.7)

\[
\sum_{i,k=0}^{n-1} \sum_{j=0}^{r-1} (d_{G \boxtimes K_r}(x_{ij}) + d_{G \boxtimes K_r}(x_{kp})) d_{G \boxtimes K_r}(x_{ij}, x_{kp})
\]
\[
= r^2(r-1) \sum_{i,k=0}^{n-1} (d_G(u_i) + d_G(u_k)) d_G(u_i, u_k) + 2r(r-1)^2 \sum_{i,k=0}^{n-1} d_G(u_i, u_k)
\]
\[
= 2r^2(r-1)W_+(G) + 4r(r-1)^2W(G).
\] (6.8)

Using (6.6), (6.7) and (6.8), we have

\[
DD(G \boxtimes K_r) = r \left( r^2 W_+(G) + 2r(r-1)W(G) + 2r(r-1)m + n(r-1)^2 \right).
\]

□

Using similar arguments as Theorem 6.4.3, one can prove the following result:
**Theorem 6.4.4.** Let $G$ be a connected graph with $n$ vertices and $m$ edges. Then

$$DD_*(G \boxtimes K_r) = r \left( r^3 W_*(G) + r^2 (r-1) W_*(G) + r(r-1)^2 W(G) + \frac{1}{2} r^2 (r-1)^2 M_1(G) + \frac{1}{2} (r-1)^3 n + 2r(r-1)^2 m \right).$$

**Lemma 6.4.1.** The following are true:

(i) $DD(P_n) = \frac{1}{3} n(n-1)(2n-1)$

(ii) $DD_*(P_n) = \frac{1}{3} (n-1)(2n^2 - 4n + 3)$

(iii) $DD(C_n) = DD_*(C_n) = 4 W(C_n)$.

Using Theorems 6.4.3, 6.4.4 and Lemma 6.4.1, we compute the exact formulae for degree distance and product degree distance of open and closed fence graphs.

**Example 6.4.2.**

(i) $DD(P_n \boxtimes K_2) = \frac{1}{3} (4n(n-1)(5n-1) + 30n - 24)$

(ii) $DD(C_n \boxtimes K_2) = \begin{cases} 
5n(n^2 + 2) & \text{n is even} \\
5n(n^2 + 1) & \text{n is odd}
\end{cases}$

(iii) $DD_*(P_n \boxtimes K_2) = \frac{1}{3} \left( 2(n-1)(25n^2 - 35n + 24) + 75n - 96 \right)$

(vi) $DD_*(C_n \boxtimes K_2) = \begin{cases} 
\frac{25}{2} n(n^2 + 2) & \text{n is even} \\
\frac{25}{2} n(n^2 + 1) & \text{n is odd}.
\end{cases}$