CHAPTER 5

THE STOCHASTIC DIFFERENTIAL EQUATION
MODELS I AND II

We propose two SDE models and discuss them in this Chapter. Let us assume that there exists a probability space $\Omega$ and $x = X(t)$ is a stochastic process for real values of $t$. The first model for one space variable is referred to as Model I. This model is developed under the assumption that the value of $x$ at any instant $t$ depends on that at the preceding instant and $X(t)$ satisfies a SDE comprising of a drift term $f \, dt$, a diffusion term $g \, dz$ and jump terms. The jump events are represented as the sum of the positive part and the negative part as elaborated in Chapter 4. Here $dz$ is a Wiener process.

The second model referred to as Model II is an extension to a multi-dimensional (for several space variables) stochastic process $X = (X_1, X_2, \ldots, X_n)$. In both the models, the jump times correspond to a Poisson process and the jump events follow a suitable heavy or long tailed distribution.

5.1 Model I

We make the following assumptions for this model:

Let $f, g, h_1, h_2$ be continuously differentiable functions of $x$ and $t$. Let $Q$ be the random variable denoting the jumps. Define

$$Q^+ = \max (Q, 0) \quad \text{and} \quad Q^- = \max (-Q, 0).$$

Then $Q^+$ and $Q^-$ are respectively the positive and negative parts of $Q$. We can write $Q = Q^+ - Q^-$. 
Let $P^+$ and $P^-$ be the probability measures of $Q^+$ and $Q^-$ respectively in suitable intervals on the real line. Now we construct the stochastic differential equation in the form

$$dX(t) = f(t, X(t))dt + g(t, X(t))dW(t) + h_1dP^+ + h_2dP^- \quad (5.1)$$

The PIDE equivalent of equation (5.1)

Let $v(t, x)$ be continuously differentiable in $x$ and $t$ as many times as required while bounded at $\infty$. Then using Kolmogorov formula, the conditional expectation of the process $V(t, X(t))$ denoted by $v(t, x) = E[V(X(t)|X(t_0)=x_0]$ can be obtained as the solution to the PIDE

$$0 = \frac{\partial v}{\partial t} + f \frac{\partial v}{\partial x} + g \frac{\partial^2 v}{\partial x^2} + \int_{\gamma}^{\infty} [v(t, x + h_1(x, t, q^+)) - v(t, x)] \varphi(q^+)dq^+ - \int_{\beta}^{\infty} [v(t, x + h_2(x, t, q^-)) - v(t, x)] \varphi(q^-)dq^- \quad (5.2)$$

In equation (5.2), $q^+$ and $q^-$ denote respectively the upward and downward jump magnitudes in $v$. $\varphi_1$ and $\varphi_2$ are the respective probability density functions. The domains of $q^+$ and $q^-$ are assumed to be the intervals $[\gamma, \infty]$ and $[\beta, \infty]$. We truncate the domains to the bounded intervals $[\gamma, \delta]$ and $[\alpha, \beta]$ by choosing $\delta$ and $\alpha$ suitably. Equation (5.2) is an extension of the PIDE (3.8) in the case of two jump terms.

Let $P_1, P_2, P_3$ and $N_1, N_2, N_3$ be the parameters of the distributions of $q^+$ and $q^-$ respectively. $\hat{\lambda}, \hat{\mu}$ are the conditional expectations of the jump processes $P^+$ and $P^-$ as in equation (3.9).

The numbers $\alpha, \delta$ are suitably defined and chosen as the upper bounds of $q^+$ and $q^-$ such that $\alpha < N_1 \quad (= \beta)$ and $P_1 \quad (= \gamma) < \delta$. We have explained our choice of the numbers $\alpha, \beta, \gamma$ and $\delta$ in Chapter 4. Let

$$D = f \frac{\partial}{\partial x} + \frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2}.$$
I = \hat{\lambda} \int_{\gamma}^\delta - \hat{\mu} \int_{\beta}^\alpha = I_1 - I_2

Then \quad L = D + I

Now equation 5.1 takes the form

\frac{\partial v}{\partial t} + Lv = 0. \quad \text{(5.3)}

We shall include the initial condition \( v_0 = x_0 \) and the notations

\[ \Delta_{h_1}(v) = [v(t, x + h_1(x, t, q^+)) - v(t, x)] \]

\[ \Delta_{h_2}(v) = [v(t, x + h_2(x, t, q^-)) - v(t, x)] \]

**The explicit implicit scheme for equation (5.3):**

Let the period of time under consideration be \( T \). We introduce a uniform grid in the plane \([0, T] \times [\alpha, \delta]\) as \( t_n = n \Delta t \), for \( n = 0, 1, 2, \ldots, N - 1 \) and \( x_i = \alpha + k \Delta x \) for \( k = 1, 2, \ldots, M \) with \( \Delta t = \frac{T}{N} \) and \( \Delta x = \frac{\delta - \alpha}{M} \).

Let \( \{v_k^n\} \) be the solution of the numerical scheme to be defined. Let \( K_1^+ \) and \( K_2^+ \) be real numbers such that \([P_1, \delta]\) is contained in the interval \([ (K_1^+ - \frac{1}{2}) \Delta x, (K_2^+ + \frac{1}{2}) \Delta x ]\) and \( K_1^-, K_2^- \) be such that the interval \([\alpha, N_1]\) is contained in \([ (K_1^- - \frac{1}{2}) \Delta x, (K_2^- + \frac{1}{2}) \Delta x ]\).

We use an explicit finite difference approximation for the differential operator \( D \) and an implicit difference approximation for the integral operator \( I \) as in R. Cont[2]. The derivatives are discretized as

\[
\left( \frac{\partial v}{\partial x} \right)_k = \frac{v_{k+1} - v_k}{\Delta x} \quad \text{and} \quad \left( \frac{\partial^2 v}{\partial x^2} \right)_k = \frac{v_{k+1} - 2v_k + v_{k-1}}{(\Delta x)^2}.
\]
Also \((J_1(v))_k = \int_{\gamma}^{\delta} \Delta n_1(v) \varphi_1(q^+) dq^+ \approx \sum_{j=K_1^+}^{K_2^+} \varphi_1(q_j^+) (v_{k+j} - v_j)\)

\((J_2(v))_k = \int_{\beta}^{\alpha} \Delta n_2(v) \varphi_2(q^-) dq^- \approx \sum_{j=K_1^-}^{K_2^-} \varphi_2(q_j^-) (v_{k+j} - v_j)\)

We can replace equation (5.3) by the system of equations:

\[
\frac{v_{k+1}^n - v_k^n}{\Delta t} + Dv_k^n + J_1 v_k^n - J_2 v_k^n = 0, \text{ for } k = 1, 2, \ldots, M; v_k^{n+1} = 0, k > M, \text{ for } n = 0, 1, 2, \ldots, N-1 \text{ and } v_0^0 = x_0.
\]

Now the consistency of the scheme is discussed.

**Proposition 1:**

The system of equations (5.4) is consistent.

i.e., for all \(v \in C_2^\infty(0,T \times [a,\delta])\) and all \((t_n, x_k) \in [0, T] \times [a, \delta],\)

\[

\left| \left( \frac{v_{k+1}^n - v_k^n}{\Delta t} + Dv_k^n + J_1 v_k^n - J_2 v_k^n \right) - \left( \frac{\partial v}{\partial t} + Lv \right)(t_n, x_k) \right| \leq \varepsilon_k (\Delta t, \Delta x) \to 0 \text{ as } \Delta t, \Delta x \to 0.
\]

Also there exist \(C_1\) and \(C_2\) such that \(|\varepsilon_k| \leq C_1 \Delta t + C_2 \Delta x\).

**Proof:**

Using Taylor’s expansion up to the second order,

\[

\left| \frac{v_{k+1}^n - v_k^n}{\Delta t} - \frac{\partial v}{\partial t} \right| \leq \left| \frac{\Delta t}{2} \right| \left\| \frac{\partial^2 v}{\partial t^2} \right\|_\infty = \frac{1}{2} \left\| \frac{\partial^2 v}{\partial t^2} \right\|_\infty \Delta t \to 0 \text{ as } \Delta t, \Delta x \to 0
\]

\[

\left| Dv_k^n - Dv(t_n, x_k) \right| = \left| Dv_k^n - Dv(t_{n+1}, x_k) - \Delta t Dv(\hat{\varepsilon}, x_k) \right|, \text{ where } \hat{\varepsilon} \in (t_n, t_{n+1})
\]

\[

\leq \left| \Delta t Dv(\hat{\varepsilon}, x_k) \right| + \left| f(t_{n+1}, x_k)(\frac{v_{k+1}^n - v_k^n}{\Delta x}) - f(t_n, x_k) \frac{\partial v}{\partial x} \right| +
\]
\[ \frac{\sigma^2}{2(\Delta x)^2} \left| (\nu_{k+1}^{n+1} - 2\nu_k^{n+1} + \nu_k^{n+1}) - \frac{\partial^2 v}{\partial x^2} (t_{n+1},x_k) (\Delta x)^2 \right| \]

\[ \leq \left| \Delta t Dv(t,x) \right| \frac{1}{\Delta x} \left| f(t_n,x_k) \frac{\partial^2 v}{\partial x^2} \right| \left( \frac{(\Delta x)^2}{2} \right) + \left| f(t_n,x_k) \frac{\partial v}{\partial x} \right| \Delta x \]

\[ + \frac{\sigma^2}{2(\Delta x)^2} \left| (v_{k+1}^{n+1} - v_k^{n+1}) + (v_{k-1}^{n+1} - v_k^{n+1}) \frac{\partial^2 v}{\partial x^2} (t_{n+1},x_k) \right| (\Delta x)^2 \]

\[ \to 0 \text{ as } (\Delta t, \Delta x) \to 0. \]

\[ |J_1 v_k^n - I_1 v(t_n,x_k)| = \left| \lambda \sum_{j=K_1^+}^{K_2^+} [v(t_n,x_k + q_j^+)] - v(t_n,x_k) \varphi_1(q_j^+) \right| \int_{\gamma} \Delta h_1(v) \varphi_1(q_+^+) dq^+ | \]

\[ = \left| \lambda \right| \left| \sum_{j=K_1^+}^{K_2^+} \int_{(j-\frac{1}{2})\Delta x}^{(j+\frac{1}{2})\Delta x} [v(t_n,x_k + q_j^+) \varphi_1(q_j^+)] - v(t_n,x_k + q_j^+) \varphi_1(q_j^+) dq^+ \right|, \]

since \( v_k \sum_{j=K_1^+}^{K_2^+} \varphi_1(q_j^+) \approx \int_{\gamma} v(t_n,x_k) \varphi_1(q_+^+) dq^+ \)

\[ |J_1 v_k^n - I_1 v(t_n,x_k)| = \left| \lambda \right| \left| \sum_{j=K_1^+}^{K_2^+} \int_{(j-\frac{1}{2})\Delta x}^{(j+\frac{1}{2})\Delta x} (q_j^+ - q_+^+) \frac{\partial u}{\partial x} dq^+ \right|, \]

where \( u(t_n,x_k+\theta) = v(t_n,x_k+\theta) \varphi_1(0) \).

So \( |J_1 v_k^n - I_1 v(t_n,x_k)| \leq \left| \lambda \right| \left( \delta - \gamma \right) \left\| \frac{\partial u}{\partial x} \right\| \int_{(j-\frac{1}{2})\Delta x}^{(j+\frac{1}{2})\Delta x} dq^+ \]

\[ \leq \left| \lambda \right| \left( \delta - \gamma \right) \left\| \frac{\partial u}{\partial x} \right\| \infty (K_2^+ - K_1^+) \Delta x \]

\[ \to 0 \text{ as } (\Delta t, \Delta x) \to 0. \]

Similarly,
\[
|J_2v^n_k - I_2v(t_n, x_k)| \leq |\hat{\mu} (\alpha - \beta)| \left\| \frac{\partial w}{\partial x} \right\|_{\infty} (K_2^{-} - K_1^{-}) \Delta x,
\]

where \( w(t_n, x_{i+1}) = v(t_n, x_{i+1}) \varphi_2(0) \).

So
\[
|J_2v^n_k - I_2v(t_n, x_k)| \to 0 \quad \text{as} \quad (\Delta t, \Delta x) \to 0.
\]

Hence the error term \( e_k \) satisfies

\[
|e_k| \leq \frac{1}{2} \left\| \frac{\partial^2 v}{\partial t^2} \right\|_{\infty} \Delta t + \left( \lambda (\delta - \gamma) \right) \left\| \frac{\partial v}{\partial x} \right\|_{\infty} (K_2^+ - K_1^+) + |\hat{\mu}| (\alpha - \beta) \left\| \frac{\partial w}{\partial x} \right\|_{\infty} (K_2^{-} - K_1^{-}) \Delta x
\]

\[
= C_1 \Delta t + C_2 \Delta x, \quad \text{where} \ C_1 \text{ and } C_2 \text{ are independent of } \Delta t \text{ and } \Delta x .
\]

**Stability of the finite difference scheme**

We rewrite the system of equations (5.4) such that at each time step, all the unknown quantities appear on the right hand side and known values (i.e., values of \( v \) at the preceding time step) on the left hand side. If now the system of equations is represented as a matrix equation then the coefficient matrix will be a tri diagonal matrix.

The system of equations (5.4) may be written in the form,

\[
\frac{v_{k+1}^{n+1} - v_k^n}{\Delta t} + \Gamma \frac{v_{k+1}^{n+1} - v_{k-1}^{n+1}}{\Delta x} + \frac{\sigma^2 v_{k+1}^{n+1} - 2v_k^{n+1} + v_{k-1}^{n+1}}{\Delta x^2} = J_2v_k^n - J_1v_k^n, \quad k = 0, 1, \ldots, N-1,
\]

\( v_k^0 = x_0, \ v_k^{n+1} = 0, \quad \text{for} \ k > N. \)

or equivalently,

\[
c \Delta t v_{k+1}^{n+1} + (1 - a \Delta t)v_k^{n+1} + b \Delta t v_{k+1}^{n+1} = v_k^n + (J_2v_k^n - J_1v_k^n) \Delta t, \quad k = 0, 1, \ldots, N-1
\]

where
\[ a = \frac{m}{\Delta x} + \frac{s^2}{(\Delta x)^2}, \quad b = \frac{m}{\Delta x} + \frac{s^2}{2(\Delta x)^2} \quad \text{and} \quad c = \frac{s^2}{(\Delta x)^2}, \]

and \( v_{k+1}^n = 0, \) for \( k > N. \) \( (5.5) \)

It can be seen that
\[ a = b + c. \]

\( \Delta x \) may be chosen to satisfy \( b > 0. \) Obviously \( c \geq 0. \)

Hence we have \( a \geq 0. \)

Thus the system of equations is of the matrix form

\[ AX = B \]

where the matrices in the equation are

\[
A = \begin{pmatrix}
1 - a\Delta t & b\Delta t & 0 & \cdots & 0 \\
c\Delta t & 1 - a\Delta t & b\Delta t & 0 & \vdots \\
0 & c\Delta t & 1 - a\Delta t & b\Delta t & 0 & \ddots \\
& & & \ddots & \ddots \\
& & & & 0 & b\Delta t \\
0 & 0 & c\Delta t & 1 - a\Delta t \\
\end{pmatrix}
\]

\[
B = \begin{pmatrix}
v_0^n + \Delta t(v_0^n + h_0^{+n}) \\
v_1^n + \Delta t(v_1^n + h_1^{+n}) \\
\vdots \\
v^n + \Delta t(v_{N-1}^n + h_{N-1}^{+n})
\end{pmatrix}
\]

and \( X = \begin{pmatrix}
x_0^{n+1} \\
x_1^{n+1} \\
\vdots \\
x_{N-1}^{n+1}
\end{pmatrix} \)
It is seen that A is a tri diagonal matrix.

We determine the condition for the stability of the finite difference scheme and also deduce the uniqueness of the solution to system (5.4)

**Proposition 2:**

The solution of the system of equations (5.5) is unique and stable if \( \Delta t < \frac{1}{a} \).

**Proof:**

Suppose \( \Delta t < \frac{1}{2a} \).

Then \( 1 - a \Delta t > \frac{1}{2} \) whereas \((b+c)\Delta t = a \Delta t < \frac{1}{2}\). Thus the matrix A is diagonally dominant and the system has a unique solution when

\[
\Delta t < \frac{1}{2a} \quad (i)
\]

To prove the stability of the finite difference scheme we shall obtain the condition for the spectral radius \( \rho \) of the matrix A to be less than unity.

The eigen values of the tri diagonal matrix A are given by

\[
\lambda_k = 1 - a \Delta t + 2\sqrt{bc} \Delta t \cos\frac{k\pi}{N+1}, \quad k = 0,1,2,3,\ldots,N.
\]

The absolute value of the maximum among the eigen values is

\[
|1 - a \Delta t + 2\sqrt{bc} \Delta t|
\]

Now \( \rho < 1 \) requires \( |1 - a \Delta t + 2\sqrt{bc} \Delta t| < 1 \).

i.e., \(-1 < 1 - a \Delta t + 2\sqrt{bc} \Delta t < 1\)

i.e., \(0 < (a - 2\sqrt{bc}) \Delta t < 2\)

or \(\Delta t < \frac{2}{a - 2\sqrt{bc}} \quad (ii)\)
Consider
\[
\frac{2}{a-2\sqrt{bc}} - \frac{1}{2a} = \frac{3a + 2\sqrt{bc}}{2a(a-2\sqrt{bc})} > 0 \quad (iii)
\]

In the above difference, the numerator is non-negative when the numbers a, b, c are non-negative. Moreover the denominator is also non-negative because of the fact that the arithmetic mean of two quantities is greater than their geometric mean, here a being equal to the sum of b and c.

Thus from inequalities (i), (ii) and (iii) we see that
\[
\min \left(\frac{2}{a-2\sqrt{bc}}, \frac{1}{2a}\right) = \frac{1}{2a} < \frac{1}{a}
\]

Hence the condition \( \Delta t < \frac{1}{a} \) is sufficient for stability of the solution.

It should be noted that \( b \geq 0 \) requires \( f + \frac{\sigma^2}{2(\Delta x)^2} \geq 0 \). This is true if \( f \geq 0 \).

If \( f < 0 \), we select \( \Delta x < -\frac{\sigma^2}{2f} \).

5.2 Multi dimensional SDE model

Some physical problems may involve multiple state variables and multiple sources of random disturbances. In such cases the multi dimensional stochastic differential equation is useful.

Let \( \bar{X}(t) = [X_i(t)] \) be a vector stochastic process of dimension \( n_x \) and \( \bar{W}(t) = [W_i(t)] \) be a Wiener process of dimension \( n_w \). If \( f \) and \( g \) are functions depending on the vector stochastic process \( \bar{X}(t) \) then a multi dimensional SDE is of the form

\[
d\bar{X}(t) = f(\bar{X}(t),t)dt + g(\bar{X}(t),t)d\bar{W}(t)
\]

The multi dimensional SDE with jump is of the form
\[d\vec{X}(t) = f(\vec{X}(t), t) dt + g(\vec{X}(t), t) d\vec{W}(t) + h(\vec{X}(t), t, \vec{Q}) d\hat{P}(\vec{X}(t), t, \vec{Q}) \]  (5.6)

where \(\hat{P}(\vec{X}(t), t, \vec{Q})\) is a \(n_o\) dimensional vector state dependent jump process and \(\vec{Q} = [Q_i]\) is the vector notation for the jump amplitude mark. The coefficient \(h\) of the jump amplitude process has the form \(h(\vec{X}(t), t, \vec{Q}) = [h_{ij}(\vec{X}(t), t, Q_i)]\) such that the \(j^{th}\) component depends only on the \(j^{th}\) jump mark \(Q_j\).

**Ito’s chain rule in multi dimensions:**

If \(\vec{Y}(t)\) is twice continuously differentiable in the space variables and once in \(t\) then the chain rule is:

\[d\vec{Y}(t) = \vec{F}(t) + \vec{J} \nabla_x[\vec{F}] + \frac{1}{2} \vec{g} \vec{R} \vec{g}^T \nabla_x \left[ \nabla_x^T [\vec{F}] \right] \nabla_x(\vec{X}(t), t) dt + \int Q \nabla^T [\vec{F}] \tilde{P}.\]

Here \(\nabla_x[\vec{F}] = \left[ \frac{\partial F_i}{\partial x_i} \right]\) is the state-space gradient and \(\nabla_x^T [\vec{F}]\) is its transpose[8].

Also \(\nabla_x [\nabla_x^T [\vec{F}]] = \left[ \frac{\partial^2 F_i}{\partial x_i \partial x_j} \right]\) and \(\vec{R}'\) is a correlation matrix \(\vec{R}' = \left[ p_{ij}^{nw \times nw} \right]\) and \(A:B\) is the double product

\[A: B = \sum_{i=1}^n \sum_{j=1}^n A_{ij}B_{ij} = \text{tr}(AB^T).\]

As in the case of one dimension, the equivalent form of the PDE is derived from equation (1) and Ito’s chain rule. Let \(\vec{X}(t_o) = \vec{x}_o\) and \(u(\vec{x}_o, t_o) = \mathbb{E}(v(\vec{X}(t)|\vec{X}(t_o) = \vec{x}_o)).\) Then \(u\) satisfies the partial differential equation:

\[0 = \frac{\partial u}{\partial t}(\vec{x}_o, t_o) + f(\vec{x}_o, t^T) \left[ \nabla_x(\vec{x}_o, t_o) \right] + \frac{1}{2} \vec{g} \vec{R} \vec{g}^T : \left[ \nabla_x \nabla_x^T (\vec{x}_o, t_o) \right] + \sum_{j=1}^n \tilde{\lambda}_j(\vec{x}_o, t_o) \int Q \left[ u \left( x + h(\vec{x}_o, t, q_j) \right) - u(\vec{x}_o, t) \right] \varphi(q_j) dq_j.\]  (5.7)
5.3 The proposed multi dimensional SDE model (Model II)

Assumptions for the model:
We make the following assumptions for this model:
Suppose a stochastic process \( X \) depends on several mutually independent variables \( X_1, X_2, \ldots, X_k \). Then \( X \) is a vector process having \( k \) components. This process is written as \( \bar{X} \). We assume that \( \bar{X} \) satisfies a SDE with jumps. Let \( \bar{Q} \) be the jump vector. Each jump component is represented in the form \( Q_i = Q_i^+ - Q_i^- \). We define

\[
Q_i^+ = \max(Q_i, 0) \quad \text{and} \quad Q_i^- = \max(-Q_i, 0).
\]

Then \( Q_i^+ \) and \( Q_i^- \) are respectively the positive and negative parts of \( Q_i \). We shall assume that the components \( Q_i^+ \) and \( Q_i^- \) are independent.

Now we consider a stochastic differential equation in the form

\[
d\bar{X}(t) = f(\bar{X}(t), t)dt + g(\bar{X}(t), t)d\bar{W}(t) + h_1(\bar{X}(t), t, \bar{Q}^+)d\bar{P}^+ + h_2(\bar{X}(t), t, \bar{Q}^-)d\bar{P}^- \quad (5.8)
\]

where the components \( Q_i^+ \) and \( Q_i^- \) of \( \bar{Q} \) have probability distributions in suitable domains. Using the link to PIDE, the SDE (5.8) can be written in its equivalent form as a PIDE as

\[
0 = \frac{\partial u}{\partial t}(\bar{x}_o, t_o) + f^T(\bar{x}_o, t_o) [\nabla_x(\bar{x}_o, t_o)] + \frac{1}{2} g^T g : [\nabla^T_x(\bar{x}_o, t_o) ] + 

\hat{\lambda} \int_{\bar{q}^+} [\bar{u}(x + h(\bar{x}_o, t, \bar{q}), t) - \bar{u}(\bar{x}_o, t)] \Phi^+(\bar{q}^+)d\bar{q}^+ + 

\hat{\mu} \int_{\bar{q}^-} [(\bar{u}(x + h(\bar{x}_o, t, \bar{q}), t) - \bar{u}(\bar{x}_o, t)] \Phi^-(\bar{q}^-)d\bar{q}^- \quad (5.9)
\]

Here \( \bar{q}^+ \) and \( \bar{q}^- \) denote the jump amplitude marks of \( \nu(\bar{X}(t), t) \) in suitable domains and \( \Phi^+ \) and \( \Phi^- \) are the probability density functions of \( \bar{q}^+ \) and \( \bar{q}^- \) respectively. Also \( u(\bar{x}_o, t_o) = E(\nu(\bar{X}(t), t) \mid \bar{X}(t_o) = \bar{x}_o) \). The other notations have
the same meaning as in equation (5.8). We use the symbol \( \hat{Q} \) in the integral terms to denote \( k \)-dimensional integrals and the symbol \( d\bar{q} \) is used to denote \( dq_1 dq_2 \ldots dq_k \). The partial derivative terms having coefficients \( f \) and \( g \) have the following meaning:

Let \( f = (f_1, f_2, \ldots, f_k) \). Then

\[
\begin{align*}
\mathbf{f}^T(\bar{x}_n, t_n^T) [\nabla_x u] &= f_{1,1} \frac{\partial u}{\partial x_1} + f_{1,2} \frac{\partial u}{\partial x_2} + \ldots + f_{1,k} \frac{\partial u}{\partial x_k}.
\end{align*}
\]

Taking \( g = (g_{i1}, g_{i2}, \ldots, g_{ik}) \) and \( \mathbf{R}' = [\rho_{ij}] \) a correlation matrix of order \( k \times k \), the product \( g\mathbf{R}'\mathbf{g}^T \) is the scalar \( \sum_{i=1}^{k} g_i \sum_{j=1}^{k} g_j \rho_{ij} \).

Thus \( g\mathbf{R}'\mathbf{g}^T : [\nabla_x [\nabla^T x u]] \) is the expression

\[
\sum_{i=1}^{k} g_i \sum_{j=1}^{k} g_j \rho_{ij} \left( \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \ldots + \frac{\partial^2 u}{\partial x_k^2} \right).
\]

Let \( H = \sum_{j=1}^{k} g_j \rho_{ij} \).

Let \( \mathbf{D} = f_{1,1} \frac{\partial u}{\partial x_1} + f_{2,2} \frac{\partial u}{\partial x_2} + \ldots + f_{k,k} \frac{\partial u}{\partial x_k} + \frac{h_1}{2} \frac{\partial^2 u}{\partial x_1^2} + \frac{h_2}{2} \frac{\partial^2 u}{\partial x_2^2} + \ldots + \frac{h_k}{2} \frac{\partial^2 u}{\partial x_k^2} \)

and \( \mathbf{I} = I_1 - I_2 \).

Now we can use a finite difference scheme to represent the PIDE similar to Model I.

The finite difference scheme

The implicit – explicit scheme is used in which the derivative terms are replaced by finite difference approximations at time step \( n+1 \) and the integral terms by approximate summations at time step \( n \), for any \( n = 0, 1, 2, \ldots, N-1 \).

Let \( C_0^\infty \) be the space of continuously differentiable functions with a norm \( \| \cdot \| \).

Let \( u \in C_0^\infty (R^k \times [0, T]) \) be continuously differentiable as many times as required for \( t \in [0, T] \) and \( \bar{x} \in R^k \). The first and second order partial derivatives of \( u \) with respect to each component \( x_i \) of \( \bar{x} \) are discretized as follows:

For \( (x_i, t_{n+1}) \in R \times [0, T] \),
\[ \left( \frac{\partial u}{\partial x} \right)_i = \frac{(u_{i+1} - u_i)}{\Delta x} ; \]

\[ \left( \frac{\partial^2 u}{\partial x^2} \right)_i = \frac{(u_{i+1} - 2u_i + u_{i-1})}{(\Delta x)^2} . \]

Moreover an integral of the form \( \int_\gamma^\infty [u(t,x + h(x,t,q)) - u(t,x)] \varphi(q) \, dq \) is approximated to a sum \( \sum_{j=\gamma}^{k} \varphi_1(q)(u_{k+j} - u_j) \) using trapezoidal quadrature formula after replacing the limits of integration by a suitable bounded interval, say, \([\gamma, \delta]\). The following notations are used to write the partial derivative approximations when \( u \in C_0^\infty (\mathbb{R}^k \times [0,T]) \).

The cartesian product of the intervals of the form \([a_i, \delta_i]\) is denoted as \([a, \delta]\) and the region \([a, \delta] \times [0,T]\) is divided into sub regions by introducing meshes of width \( \Delta t \) for \([0,T]\) and \( \Delta x_i \) for each interval \([a_i, \delta_i]\).

Further,

for \( \bar{x}_{il}^n = (t_n, x_{i1}, x_{i2}, \ldots, x_{ik}) \in [a, \delta] \times [0,T] \)

\[ \left( \frac{\partial u}{\partial x_{il}} \right)_{\bar{x}_{il}^n} = \frac{(u_{i_1 \ldots i_l+1 \ldots -i_k}^n - u_{i_1 \ldots i_l-1 \ldots -i_k}^n)}{\Delta x_{il}} \] where \( i_l = i_1, i_2, \ldots, i_k \).

\[ \left( \frac{\partial^2 u}{\partial x_{il}^2} \right)_{\bar{x}_{il}^n} = \frac{(u_{i_1 \ldots i_l+1 \ldots -i_k}^n - 2u_{i_1 \ldots i_l-1 \ldots -i_k}^n + u_{i_1 \ldots i_l-1 \ldots -i_k}^n)}{(\Delta x_{il})^2} \] and

\[ \left( \frac{\partial u}{\partial t} \right)_{\bar{x}_{il}^n} = \frac{(u_{i_1 \ldots i_k+1}^n - u_{i_1 \ldots i_k-1})}{\Delta t} . \]
Let \( \{u^N_{i,t_1,...,t_k}\} \) be the solution of the numerical scheme to be defined. Let \( K_{1,t}^+ \) and 
\( K_{2,t}^+ \) be real numbers such that \([\gamma, \delta]\) is contained in 
\([K_{1,t}^+, K_{2,t}^+; t/2) \Delta x, (K_{2,t}^+; t/2) \Delta x]\) and 
\( K_{1,t}^- \), \( K_{2,t}^- \) be such that \([\alpha, \beta]\) is contained in 
\([(K_{1,t}^-; t/2) \Delta x, (K_{2,t}^-; t/2) \Delta x]\).

Now the partial integro differential equation gives rise to the system of linear

equations

\[
\begin{align*}
\left( \frac{u^{n+1}_{i,t_1,...,t_k} - u^n_{i,t_1,...,t_k}}{\Delta t} \right) + \sum_{l=1}^{k} f_l \left( \frac{u^{n+1}_{i+1,t_1+1,...,t_{k}-1} - u^{n+1}_{i,t_1,...,t_{k}}}{{\Delta x}_{i,l}} \right) + \sum_{l=1}^{k} \frac{H_l}{2} \\
\left( \frac{u^{n+1}_{i,t_1,...,t_{k}-1} + 2u^{n+1}_{i+1,t_1,...,t_{k}} + u^{n+1}_{i,t_1,...,t_{k}-1}}{2} \right) = J_2 - J_1 , \ldots (5.10)
\end{align*}
\]

\( n= 0,1,\ldots N-1; i_t = 1,2,\ldots,k; u_{i_t}^{n+1} = 0 \) for \( i_t > M \) and with initial condition
\( u_0^0 = x_0 \).

Here \( I_1 = \hat{\lambda} \int_{Q^+} [\bar{u}(x + h(x_o, t, \bar{q}), t) - \bar{u}(x_o, t)] \Phi^+(\bar{q}+) d\bar{q}^+ \) and

\( I_2 = \mu \int_{Q^-} [\bar{u}(x + h(x_o, t, \bar{q}), t) - \bar{u}(x_o, t)] \Phi^-(\bar{q}^-) d\bar{q}^- \),

where the symbols \( \bar{Q}^+ \) and \( \bar{Q}^- \) denote the respective region of integration as the
intervals \([\gamma, \delta]\) and \([\alpha, \beta]\). Also \( \hat{\lambda} \) and \( \mu \) are parameters of the distributions of
\( \bar{Q}^+ \) and \( \bar{Q}^- \) respectively are the actual integral terms in the PIDE. These two terms
are replaced by the approximate sums \( J_1 \) and \( J_2 \) respectively as given below:

\( J_1 = \sum_{i_1=K_{1,t}}^{K_{2,t}} \sum_{i_2=K_{1,t}}^{K_{2,t}} \sum_{i_k=K_{1,t}}^{K_{2,t}} u(x_{i_1} + q_{i_1}^+, \ldots, x_{i_k} + q_{i_k}^+, t_n) - u(x_{i_1}, \ldots, x_{i_k}, t_n) \Phi^+(\bar{q}+) \)

\( J_2 = \sum_{i_1=K_{1,t}}^{K_{2,t}} \sum_{i_2=K_{1,t}}^{K_{2,t}} \sum_{i_k=K_{1,t}}^{K_{2,t}} u(x_{i_1} + q_{i_1}^-, \ldots, x_{i_k} + q_{i_k}^-, t_n) - u(x_{i_1}, \ldots, x_{i_k}, t_n) \Phi^-(\bar{q}^-) \)
The quantities $\Delta t$ and $\Delta x_{i_l}$ are the mesh width defined as:

$$\Delta x_{i_l} = \frac{\delta_{i_l} - \alpha_{i_l}}{M}, \quad i_l = 1, 2, \ldots, k \quad \text{and} \quad \Delta t = \frac{T}{N}, \quad T \text{ being the time period under consideration.}$$

In order to reduce writing the subscripts in the finite difference approximation of the derivative terms, the following notation is introduced:

$$u(x_{i_1}, x_{i_2}, \ldots, x_{i_l}, \ldots, x_{i_k}, t_n) = u_{i_l}^n \quad \text{and} \quad u(x_{i_1}, x_{i_2}, \ldots, x_{i_l}, \ldots, x_{i_k}, t_{n+1}) = u_{i_l}^{n+1}.$$

Now the system of equations (5.10) take the form

$$c_{i_1} \Delta t \ u_{i_1}^{n+1} + (1 - a \Delta t) u_{i_1}^{n+1} + b_{i_1} \Delta t \ u_{i_{1+1}}^{n+1} + c_{i_2} \Delta t \ u_{i_{2-1}}^{n+1} + b_{i_2} \Delta t \ u_{i_{2+1}}^{n+1} + \ldots + c_{i_k} \Delta t \ u_{i_{k+1}}^{n+1} + b_{i_k} \Delta t \ u_{i_{k-1}}^{n+1} = J_2 - J_1,$$

for $n = 0, 1, \ldots, N-1$ and $u_{i_l}^{n+1} = 0$ for $i_l > M$ and with initial condition $u_0^0 = x_0$ (5.11)

The coefficients in the above equation are defined as

$$c_{i_1} = \frac{H_{i_1}}{2(\Delta x_{i_1})^2},$$

$$b_{i_l} = \frac{f_{i_l}}{\Delta x_{i_l}} \Delta t + c_{i_l}, \quad \text{for} \quad i_l = 1, 2, \ldots, k$$

$$a = \sum_{l=1}^{k} (b_{i_l} + c_{i_l}).$$

With these notations, equations (5.11) can be represented in the matrix form

$$S\bar{U}^n = J_2 - J_1 \quad \ldots \quad (5.12)$$
The column vector $\hat{u}^n$ has its $k(M-1)$ components in the order

$$u^n_{1,i_2,...i_k}, u^n_{2,i_2,...i_k}, \ldots u^n_{M-1,i_2,...i_k}, u^n_{i_1,i_2,...i_k}, \ldots u^n_{i_1,i_2,...M-1}.$$ 

$D$ is a block matrix of order $kM \times kM$. It has tri diagonal matrices along the main diagonal and diagonal matrices off the main diagonal. Each block is a square matrix of order $M \times M$.

$$S = \begin{pmatrix}
A & B_2 & B_3 & \cdots & B_{M-1} & 0 \\
C_2 & A & B_3 & \cdots & B_{M-1} & \\
C_3 & C_2 & A & \ddots & \vdots & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \\
0 & C_{M-1} & \cdots & A
\end{pmatrix}$$

0 denotes the zero matrix of order $(M-1) \times (M-1)$.

The diagonal block and each of the blocks in the super diagonal and sub diagonal are the square matrices given by,

$$A = \begin{pmatrix}
1 - a_1 \Delta t & b_1 \Delta t & 0 & 0 \\
c_1 \Delta t & 1 - a_1 \Delta t & b_1 \Delta t & 0 & \cdots & 0 \\
0 & c_1 \Delta t & 1 - a_1 \Delta t & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & \cdots & 1 - a_1 \Delta t
\end{pmatrix}$$

For $i = 2, 3, \ldots, M-1$ the matrices $B_i$ and $C_i$ are the diagonal matrices as shown below:

$$B_i = \begin{pmatrix}
b_1 \Delta t & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & b_1 \Delta t
\end{pmatrix}$$
C_i = \begin{pmatrix}
    c_i \Delta t & 0 & \cdots & 0 \\
    \vdots & \ddots & \vdots \\
    0 & \cdots & c_i \Delta t 
\end{pmatrix}

The matrix \( J_2 - J_1 \) has the representation

\[
J_2 - J_1 = \begin{pmatrix}
    v_0^n + \Delta t(v_0^n + h_0^{+n}) \\
    v_1^n + \Delta t(v_1^n + h_1^{+n}) \\
    \vdots \\
    v_N^n + \Delta t(v_N^{-1}n + h_{N-1}^{+n})
\end{pmatrix}
\]

Let \( E \) be all of the difference terms in each equation of the system.

Now we verify the consistency and obtain the condition for the stability of the finite difference scheme used.

**Convergence, Consistency and Stability of the FD scheme.**

**Proposition**

The finite difference scheme defined above is consistent or the error term \( e^n \) satisfies

\[
\left| \left( \frac{\partial u}{\partial t} + Du + Iu \right) - \left( \frac{u^{n+1} - u^n}{\Delta t} + Eu + J_1 - J_2 \right) \right| = \left| \overline{e^n}(x_{i1}, x_{i2}, \ldots, x_{ik}, t_n) \right| \to 0 \quad \text{as} \quad (\Delta x_1, \Delta x_2, \ldots, \Delta x_k, \Delta t) \to 0 \quad \text{at each point} \quad (x_{i1}, x_{i2}, \ldots, x_{ik}, t_n) \quad \text{in the mesh under consideration.}
\]

**Proof:**

By Taylor’s theorem, \( u^{n+1}_{i_1i_2\ldots i_k} = u^n + \frac{\partial u}{\partial t} + \frac{(\Delta t)^2}{2!} \frac{\partial^2 u}{\partial t^2} \) to the second order.

\[
\left| \frac{\partial u}{\partial t} \left( \frac{u^{n+1}_{i_1i_2\ldots i_k} - u^n_{i_1i_2\ldots i_k}}{\Delta t} \right) \right| = \left| \frac{\partial u}{\partial t} \frac{u^{n+1} - u^n}{\Delta t} \right|
\]

\[
= \left| \frac{\partial u}{\partial t} \frac{1}{\Delta t} \left( \frac{\partial u}{\partial t} + \frac{(\Delta t)^2}{2!} \frac{\partial^2 u}{\partial t^2} \right) \right| 
\]
\[ \frac{(\Delta t)^2}{2!} \| \frac{\partial^2 u}{\partial t^2} \| \rightarrow 0 , \text{ as } (\Delta x_1, \Delta x_2, \ldots, \Delta x_k, \Delta t) \rightarrow 0 \quad \text{.....(i)} \]

\[ |\mathbf{D}u^n - \mathbf{E}u(x_i, x_{i+1}, \ldots, x_k)\| = |\mathbf{D}u^{n+1} + \Delta t \mathbf{D}(x_i, x_{i+1}, \ldots, x_k, \hat{t}) - \mathbf{E}u(x_i, x_{i+1}, \ldots, x_k, \hat{t})\| , \]

where \( \hat{t} \) lies in the interval \([t_n, t_{n+1}]\).

For any one of the components \( x_i \),

\[ |f_{ii} \frac{\partial u}{\partial x_i} - f_{ii} \left( \frac{u_{i+1}^{n+1} i_{i+1} i_{i+2} \ldots i_k - u_{i+2}^{n+1} i_{i+2} i_{i+3} \ldots i_k}{\Delta t} \right) | = |f_{ii} \frac{\partial u}{\partial x_i} - f_{ii} \frac{u_{i+1}^{n+1} - u_i^{n+1}}{\Delta x_i} | \]

\[ = \frac{\left( u_{i+1}^{n+1} - u_i^{n+1} \right) + (u_{i-1}^{n+1} - u_i^{n+1})}{(\Delta x_i)^2} \]

\[ = \left( \frac{\partial u}{\partial x_{i+1}} + \frac{\Delta x_i}{2!} \frac{\partial^2 u}{\partial x_i^2} + \frac{\Delta x_i^2}{3!} \frac{\partial^3 u}{\partial x_i^3} + \ldots \right) + \left( - \frac{\partial u}{\partial x_{i-1}} + \frac{\Delta x_i}{2!} \frac{\partial^2 u}{\partial x_i^2} - \frac{\Delta x_i^2}{3!} \frac{\partial^3 u}{\partial x_i^3} + \ldots \right) \]

\[ = \Delta x_i \frac{\partial^2 u}{\partial x_i^2} + 2\frac{\Delta x_i^3}{4!} \frac{\partial^4 u}{\partial x_i^4} + \ldots \]

so \( |\mathbf{D}u^{n+1} + \Delta t \mathbf{D}(x_i, x_{i+1}, \ldots, x_k) - \mathbf{E}u(x_i, x_{i+1}, \ldots, x_k, \hat{t})\| \)

\[ \leq \Delta t \left| \frac{\partial u}{\partial t} \right| + \sum_{i=1}^{k} |f_{ii} \left( \frac{\Delta x_i}{2!} \frac{\partial^2 u}{\partial x_i^2} \right) + H_{ii} \left( \frac{\Delta x_i}{2} \frac{\partial^2 u}{\partial x_i^2} + 2 \frac{\Delta x_i^3}{4!} \frac{\partial^4 u}{\partial x_i^4} \right) | \]

\[ \leq \Delta t \left| \frac{\partial u}{\partial t} \right| + \sum_{i=1}^{k} \left[ |f_{ii} \left| \frac{\Delta x_i}{2!} \right| \frac{\partial^2 u}{\partial x_i^2} \right| + H_{ii} \left| \frac{\Delta x_i}{2} \left| \frac{\partial^2 u}{\partial x_i^2} \right| \right| \]

\[ \rightarrow 0 \text{ as } (\Delta x_1, \Delta x_2, \ldots, \Delta x_k, \Delta t) \rightarrow 0 \quad \text{.....(ii)} \]
For the integral term $I_1$ and its approximation $J_1$,

$$|I_1 - J_1| =$$

$$
\left| \hat{\mathcal{A}} \left( \int_{\mathcal{Q}^+} [\bar{u}(x + h(x, t, \bar{q}), t) - \bar{u}(x, t)] \Phi^+(q^+) dq^+ \right) - 
\left( \sum_{i_1=K_1^+}^{K_2^+} \sum_{i_2=K_1^+}^{K_2^+} \cdots \sum_{i_k=K_1^+}^{K_2^+} u(x_{i_1} + q_{i_1}^+, \ldots x_{i_k} + q_{i_k}^+, t_n) - u(x_{i_1}, \ldots x_{i_k}, t_n) \Phi^+(\bar{q}^+) \right) \right|
$$

For the sake of simplicity of notations considering $k=2$ and taking $\bar{x} = (x, y)$,

$$\bar{q} = (q_x, q_y),$$

$$|I_1 - J_1| =$$

$$\left| \left| \hat{\mathcal{A}} \right| \sum_{i=K_1^+}^{K_2^+} \sum_{j=K_1^+}^{K_2^+} [u_{a+i,b+j}^n - u_{i,j}^n] \Phi^+(q_i^+, q_j^+) \right| -$$

$$\left| \left\{ \int_{\bar{Q}} [u(x_i + q_x^+, y_j + q_y^+, t_n) - u(x_i, y_j, t_n)] \Phi^+(\bar{q}^+) dq^+ \right\} \right|
$$

Since

$$\sum_{i=K_1^+}^{K_2^+} \sum_{j=K_1^+}^{K_2^+} u_{i,j}^n \Phi^+(q_x^+, q_y^+) \approx \int_{\bar{Q}} \left[ u(x_i, y_j, t_n) \Phi^+(\bar{q}^+) dq^+ \right],$$

$$\left| \left| \hat{\mathcal{A}} \right| \sum_{i=K_1^+}^{K_2^+} \sum_{j=K_1^+}^{K_2^+} u_{i,j}^n \Phi^+(q_i^+, q_j^+) \right| -$$

$$\left| \left\{ \int_{\bar{Q}} [u(x_i + q_x^+, y_j + q_y^+, t_n)] \Phi^+(\bar{q}^+) dq^+ \right\} \right|
$$

$$= \left| \left| \hat{\mathcal{A}} \right| \sum_{i=K_1^+}^{K_2^+} \sum_{j=K_1^+}^{K_2^+} \int_{(i-\frac{1}{2})\Delta x}^{(i+\frac{1}{2})\Delta x} \int_{(j-\frac{1}{2})\Delta y}^{(j+\frac{1}{2})\Delta y} [u_{a+i,b+j}^n \Phi^+(q_i^+, q_j^+)] -$$

$$\int_{\bar{Q}} [u(x_i + q_x^+, y_j + q_y^+, t_n)] \Phi^+(\bar{q}^+) dq^+ \right|.$$
\[ \leq |\lambda| [\Delta x] |\frac{\partial u}{\partial x}| + \Delta y |\frac{\partial u}{\partial y}| \sum_{i=K1x}^{K2x} \sum_{j=K1y}^{K2y} \int_{(i-\frac{1}{2})\Delta x}^{(i+\frac{1}{2})\Delta x} \int_{(j-\frac{1}{2})\Delta y}^{(j+\frac{1}{2})\Delta y} dq_x dq_y \]

\[ \leq |\lambda| [\Delta x] |\frac{\partial u}{\partial x}| + \Delta y |\frac{\partial u}{\partial y}| (K2x^+ - K1x^+) (K2y^+ - K1y^+) \rightarrow 0 \text{ as } (\Delta x_1, \Delta x_{i2}, \ldots, \Delta x_{ik}, \Delta t) \rightarrow 0 \text{ ....(iii)} \]

Similarly it can be proved \(|I_2 - J_2| \rightarrow 0 \text{ as } (\Delta x_1, \Delta x_{i2}, \ldots, \Delta x_{ik}, \Delta t) \rightarrow 0\)

From (i), (ii) and (iii) the required result follows.

**To find the condition for the stability of the scheme:**

For any square matrix \(A\) of order \(n\), the norm of the matrix is defined as the maximum of the sum of the absolute values of any row (column) and is written as \(\|A\|\). Also the spectral radius of \(A\) is defined as \(\rho(A) = \max_{1 \leq i \leq n} |\lambda_i|\), where \(\lambda_i\) (\(i = 1, 2, \ldots, n\)) are the eigenvalues of the matrix \(A\). Moreover it is known that \(\rho(A) \leq \|A\|\) and the condition \(\rho(A) \leq 1\) ensures the stability of the scheme.

So the finite difference scheme is stable if the spectral radius is not greater than 1.

For the block matrix \(S\) in equation (5.12) its norm is given by,

\[ \|S\| = \|1 - a\Delta t\| + \sum_{i=2}^{M-1} (|b_i| + |c_i|)\Delta t \]

\(\|S\| \leq 1\) gives the condition,

\[ \|1 - a\Delta t\| + \sum_{i=2}^{M-1} (|b_i| + |c_i|)\Delta t \leq 1 \text{ ....(5.13)} \]

But \(|a\Delta t| -1 \leq |1 - a\Delta t|\)

So inequality (8) can be written as \(|a\Delta t| -1 \leq |1 - a\Delta t|\)
Thus the condition for stability of the scheme is $\Delta t \leq \frac{2}{|a|}$.

**Convergence**

According to Lax Richtmeyer theorem the numerical solution of any well-posed problem converges to the actual solution provided that the scheme is consistent and stable. Thus the finite difference scheme that has been defined for the system of equations (5.13) is convergent.

**To verify diagonal dominance for the matrix $S$:**

The system of equations (5.13) will have a solution and the solution will be unique if the entries of the matrix exhibit diagonal dominance.

For diagonal dominance it should be true that

$$\sum_{i=1}^{M-1}(|b_i| + |c_i|)\Delta t \leq |1 - a\Delta t|$$

$$\leq 1 + |a\Delta t|$$

$$\leq 1 + \sum_{i=1}^{M-1}(|b_i| + |c_i|)\Delta t$$

i.e $\sum_{i=1}^{M-1}(|b_i| + |c_i|)\Delta t \leq 1 + \sum_{i=1}^{M-1}(|b_i| + |c_i|)\Delta t$,

which is always true.

Hence the solution is unique.
Thus we conclude that the finite difference scheme is consistent and stable if the step size satisfies the condition \( \Delta t \leq \frac{2}{|a|} \).

We illustrate the applications of these two models in the next chapter.

### 5.4 The case of dependent components

In the above discussion we assumed that the different components of the \( k \) dimensional stochastic process are independent and so they have zero correlation. Now suppose \( X \) is a two dimensional process having the components \( X_1 \) and \( X_2 \) that are correlated. Let the coefficient of correlation be \( \rho \).

Then the term \( gR'g^T : [\nabla X][\nabla^T u] \) in equation (5.9) will have an additional term as given below:

Writing \( g = (g_1, g_2) \) and the matrix of correlation coefficient \( R = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \) we get

\[
gR'g^T = g_1^2 + 2 \rho \ g_1 \ g_2 + g_2^2.
\]

This results in adding the quantity \( 2 \rho \ g_1 \ g_2 \) to each of the second order partial derivative coefficients in equation (5.9) and (5.10) as well. The middle term was equal to zero in the previous section. Thus we have

\[
c_i = \frac{1}{2} (g_i^2 + 2 \rho \ g_1 \ g_2) \quad \text{and} \quad b_i = \frac{f_i}{\Delta x_i} + \frac{c_i}{(\Delta x_i)^2}.
\]

The coefficient \( a \) has the same expression as before.

Thus the special forms of the matrices in equation (5.12) remain the same. Hence Model II can be applied even when some or all the components of the \( k \) dimensional process are correlated.