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4.10 Variation of \(f_q\) with \(\varphi\) for a critical chain. The dashed line or the lowest curve corresponds to the single \(h\)-quench case (no \(J\)-quench) which clearly is an order of magnitude smaller than the single \(J\)-quench (no \(h\) quench), the dotted line. The case of double quench is shown by solid line which more or less overlaps with the case when only \(J\)-quench is performed. Similar behavior is also observed for a ferromagnetic chain.

5.1 Phase diagram of the one-dimensional \(p\)-wave superconducting system (see Eq. (5.3)). Phases I and II are topologically non-trivial while phase III is topologically trivial. Quenching paths A and B are shown in the phase diagram.

5.2 Two isolated Majorana states are localized in two edges of a 100-site open Majorana chain in phase I \((\Delta = 0.1\) and \(\mu = 0.0\)). \(j\) labels as Majorana sites 1, 2, ...200. One can see that here probability is non-zero only if \(j\) is odd (even) for the left (right) end of Majorana chain. Here, the odd and even \(j\) sites represent \(a\) and \(b\) type of Majoranas respectively.

5.3 Energy spectrum of the Majorana chain of system size \(N = 100\) as a function of \(\xi = \Delta/w\) with \(w = 1\) and \(\mu = 0\) using (a) open boundary condition and (b) periodic boundary condition, respectively. Note that two zero-energy Majorana modes are present in case (a) but not in case (b). We mention that the energy is scaled by a factor \(1/4\) in Eq. (5.4) in comparison to Eq. (5.2).
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5.5 The probabilities of an end mode ($\Delta = 1.0$ and $\mu = 1.8$) after a quench (a) at a point ($\Delta = 1.0$ and $\mu = 2.2$) of phase III and (b) at a QCP ($\Delta = 1.0$ and $\mu = 2.0$) along the path B show same behavior as of the path A.

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5.7 Two zero energy Majorana modes of a system Hamiltonian defined in Eq. (5.10) exist in two edges of a 100-site open Majorana chain in phase I ($\Delta = 0.1$ and $\mu = 0.0$). Similarly to Fig. 5.2, here also $j$ labels as Majorana sites 1, 2, ...200. The probabilities are non-zero only if $j$ is odd (even) for the left (right) end of Majorana chain.

5.8 Energy spectrum of the Majorana chain (with only next-nearest-neighbor hopping and interaction as defined in Eq. (5.8)) as a function of $\Delta$ with $\mu = 0$ and $N = 100$ sites for open boundary condition. In this case also an energy scale difference exists (see caption of Fig. 5.3).

5.9 $P_m(t)$ of a right end zero energy Majorana mode after quenching along different paths. (a) MSP decays rapidly and stays minimum with some noisy fluctuation of small amplitude when the system (see Eq.(5.8)) is quenched from phase I ($\Delta = 0.1$) to the phase II ($\Delta = -0.1$) along the path A. (b) For quenching to QCP $P_m(t)$ is nearly perfect oscillatory function of time $t$ with a interesting fact that its time period becomes half of the earlier case (see Fig. 5.4) and scales linearly with the system size $N$. (c) Time variation of MSP with $\Delta = 1.0$ and $\mu = 1.8$ following a quench to a point ($\Delta = 1.0$ and $\mu = 2.2$) and (d) the QCP ($\Delta = 1.0$ and $\mu = 2.0$) along the path B show same behavior as of the path A($\mu = 0$).
5.10 System changes suddenly from a phase with two Majorana fermions to a phase which contains only one Majorana mode at each end of the chain at time $t = 0$. We have set here the parameter values $\Delta = 0.1$ and $\mu = 0.0$. (a) MSP of a left end ($a_2$) Majorana mode shows collapse and revival with time, but at each revival there are fluctuations and also the peaks of revivals decrease rapidly. (b) While on the other hand when the system is quenched reversely the Majorana ($a_1$) decoheres rapidly with time (with no prominent revival) and fluctuates around zero mean.

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5.12 Phase diagram of the model Hamiltonian (5.12) for different phases of hopping parameter (a) $\phi = 0$, (b) $\phi = \pi / 4$, (c) $\phi = 2\pi / 5$ and (d) $\phi = \pi / 2$. Here, I and II are two distinct topological phases and III is the non-topological phase. The quenching path is shown using the vertical arrow.

5.13 Variation of energy levels as a function of parameter $\xi = \frac{\Delta}{w_0}$ (with $w_0 = 1$) for periodic (a) and open (b) boundary conditions with $\phi = \pi / 10$ and $N = 100$. The inverted energy levels within the gapless region of the system is denoted by the red color.

5.14 Variation of energy levels as a function of parameter $\xi = \frac{\Delta}{w_0}$ (with $w_0 = 1$) for (a) $\phi = \pi / 4$ and $N = 100$ and (b) $\phi = \pi / 10$ and $N = 150$. The inverted energy levels within the gapless region of the system is denoted by the red color.

5.15 (a) The logarithm of defect density $\ln n$ with the logarithm of quench time $\ln \tau$ for $\phi = \pi / 4$ and $\pi / 10$ are plotted. (b) The plot shows the variation of $\ln n$ with $\ln \cos \phi$ for a quench time $\tau = 200$ which confirms the $\phi$ dependence of $n$ given in Eq. (5.21).

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5.17 (a) Plots for $P_{\text{def}}$, $P_{\text{neg}}$ and $P_m$ with $\tau$ for $\phi = \pi/10$ show that all of them add up to unity. (b) The plot shows a linear variation of $\ln(\tau_c)$ as a function of $\ln(\sin \phi)$ with slope (-0.9) nearly equal to -1.

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5.19 The plot shows $P_{\text{def}}$ as a function of $\tau$ considering overlaps between quenched Majorana state at final time and different number of positive bulk energy states (close to zero-energy) at the final parameter value. We consider the case $\phi = \pi/10$ and $N = 100$, where the number of inverted levels in both positive and negative sides of the zero energy level is 18 (see Appendix). The plot signifies that in the limit of $\tau \leq \tau_c$ ($\Delta t < \Delta t_{\text{th}}$) the initial Majorana state interacts with all the energy levels. For the other regime, $\tau \geq \tau_c$, the $P_{\text{def}}$ calculated using only the inverted positive energy levels nearly coincides with the exact $P_{\text{def}}$ obtained considering all positive energy bands. This clearly confirms that time evolved Majorana states mix only with the inverted levels for a transit time $\Delta t \geq \Delta t_{\text{th}}$.

6.1 (a) Overlap of instantaneous wave function with classical spin glass ground state with $\Gamma(t) = 3/\sqrt{t}$ and $h_l(t) = -0.5/\sqrt{t}$. For comparison, we have shown the result for the same system with a fixed longitudinal field $h_l = 0.1$ and same transverse annealing schedule. (b) Time variation of $P(t)$ for 5 different sets of exchange interactions which have been taken from a fixed Gaussian distribution. Annealing with $\Gamma(t) = 4/t$ and $h_l(t) = -1/t$.

D.1 (a) Plots of $P_m$ for a non-linear time variation with $\alpha = 1.5$ as a function of $\tau$ with different values of $\phi$. (b) Plots of $P_m$ as a function of $\tau$ for different values of $\phi$ with $\alpha = 2$. (c) The plot shows a linear variation of $\ln \tau_c$ as a function of $\ln \sin \phi$ with slope (=0.77) nearly equal to -0.75 for $\alpha = 2$. It justifies that $\tau_c$ is proportional to $\sin \phi^{-\alpha+1)/2\alpha}$ for a fixed $\Delta t_{\text{op}}$. Here $N = 100$. ...........