Chapter 4

Effect of double local quenches on Loschmidt echo and entanglement entropy

We have discussed the fidelity and its dynamical counterpart, i.e., the Loschmidt echo for two spin models in the previous two chapters. In these cases, we have measured the time evolution of the LE in situations where the ground state of the Hamiltonian evolves with another Hamiltonian; the other Hamiltonian is effectively given by the first Hamiltonian with a very small change of tuning parameter. On the other hand, we here study the effect of two simultaneous local quenches (see section 1.3.1) on the evolution of the Loschmidt echo (LE) and the entanglement entropy (EE) of a one-dimensional transverse field Ising model (TFIM) [45]. We consider two independent transverse field Ising spin chains in the ferromagnetic or critical phase. For the times $t < 0$, the two spin chains which are in their respective ground states, are disconnected and are later joined at $t = 0$ ($J$-quenching). Simultaneously, we change the magnitude of the transverse field at a single site of one of the chains ($h$-quenching). This results to a non-equilibrium evolution of the state of the system.

If we consider only $J$-quenching of a critical Ising chain, both the quantities, namely, the LE and the EE show periodic time evolution [135]. However, here, we address the effect of $h$-quenching along with the $J$-quenching in critical and off-critical systems which results to a non-trivial evolution of the LE and the EE when compared to the no $h$-quenching case. Our aim here is to understand this very non-trivial evolution using the
picture of quasiparticles (QPs) generated due to both the local quenches. In the process of understanding the dynamics of the QPs produced at the two sites, we observed the following salient features of the dynamics in the quenches studied here: (i) QPs produced at the site of $h$-quench are produced with a smaller probability as compared to the QPs at the site of $J$-quench. (ii) There occurs a partial/full reflection of QPs at the site of $h$-quench. (iii) It has been well known that the nature of the QPs is different in the ferromagnetic and in the critical phase [150, 223]. The QPs are point like classical objects when in the ferromagnetic phase, but are of extended type in the critical region. We observed the effect of such a difference in the dynamics of the EE. We find some interesting time scales in the evolution of both the LE and the EE when the two quenches are performed simultaneously. These timescales can be explained successfully using the reflection picture of the QPs generated. We have also argued qualitatively how the evolution of the EE changes as the total system moves from deep ferromagnetic phase to critical one.

4.1 Model

In section 1.1.2, we have discussed the exact diagonalization of the one dimensional three-spin interacting TFIM considering both homogeneous and inhomogeneous cases. In this chapter, we consider a one-dimensional Ising chain in a transverse field i.e., the model discussed in section 1.1.2 with zero value of the three-spin interaction strength. The Hamiltonian we consider here is given by

$$H = -\sum_n (J_n \sigma_n^x \sigma_{n+1}^x + h_n \sigma_n^z).$$

(4.1)

As discussed in section 1.1.2, following Jordan-Wigner (JW) transformation, the Hamiltonian in Eq. (4.1) can be expressed as a quadratic form in terms of spinless fermions

$$H = \sum_{m,n} \left[ c_m^\dagger A_{m,n} c_n + \frac{1}{2} (c_m^\dagger B_{m,n} c_n^\dagger + h.c.) \right],$$

(4.2)

where, the elements of the $A$ and $B$ matrices for Eq. (4.1) are given by

$$A_{m,n} = -(J_m \delta_n \alpha_m + J_j \delta_{m,n+1}) - 2h_m \delta_{m,n},$$

$$B_{m,n} = -(J_m \delta_n \alpha_{m+1} - J_j \delta_{m,n+1}).$$

(4.3)
4.2 Loschmidt echo and entanglement entropy for critical chain

We here study double quenches for a critical chain where already some work has been done in Refs. [135, 224] for local $J$-quenches. As discussed before, we consider simultaneous application of two types of local perturbations to the system and study the time evolution of the LE and the EE as a result of such quenches. Initially the spin chain is prepared in the ground state of $H = H_1 + H_2$ where $H_1$ and $H_2$ are the Hamiltonians of two decoupled homogeneous Ising chains of length $L_1$ and $L_2$, respectively, with open boundary conditions ($J_{L_1} = J_{L_2} = 0$). Two simultaneous quenches are performed at $t = 0$, namely, (i) the two spin-1/2 chains are suddenly connected together resulting to a chain of total length $L = L_1 + L_2$, and, (ii) the transverse field at a particular site $L'$ belonging to either the chain 1 or 2 is changed from $h$ to $h + \delta$. The system then evolves with the final Hamiltonian

$$H_f = H_1 + H_2 + H_{12}^I - \delta \sigma^z_{L'},$$

(4.4)

where $H_{12}^I$ defines the connection between the two spin chains of length $L_1$ and $L_2$ and is of the form $J \sigma^x_{L'} \sigma^x_{L'+1}$. At the same time, the term $-\delta \sigma^z_{L'}$ in the Hamiltonian corresponds to the $h-$quenching which changes the magnitude of transverse field at site $L'$ from $h$ to $h + \delta$. We incorporate these quenches numerically by considering a single spin chain of total length $L = (L_1 + L_2)$ with the first $L_1$ spins forming the system 1 and the remaining system 2. They are disconnected at $t < 0$ by putting $J_{L_1} = 0$ which at $t = 0$ is then increased to $J$, also the interaction strength at all the other sites. For all our calculations, we have set $J = 1$. The details of the numerical calculations for the LE and the EE are outlined in the Appendix C, see also Refs. [54, 132].

As we switch on the two local perturbations discussed above, there is a local increase in energy of the system at the site of local perturbations [152, 224]. These sites then become the source of QPs production. Henceforth, we shall call the QPs created due to the $h$-quenching at $L'$ as $QP^1$, and the corresponding left and right moving QPs as $QP^1_L$ and $QP^1_R$, respectively. Similarly, the left and right moving QPs created at the site $L_1$ of $J$-quenching shall be called as $QP^2_L$ and $QP^2_R$. Below, we present our results for the evolution of the LE and the EE for various cases or geometry and discuss these results in the light of QPs propagating in the system.

4.2.1 LE and EE for $L_1 = L_2$

In this section, we consider $J$-quenching at $L_1 = L_2 = L/2$ and the $h$-quenching at some site $L'$ of the total spin chain of length $L$ at time $t = 0$. Let us first discuss the dynamics of the LE. In general, we shall assume $L' < L_1 \leq L_A$ ($L_A$ explained later in the context
Chapter 4

4.2. Loschmidt echo and entanglement entropy for critical chain

of the EE) throughout this analysis but the case with $L' > L_1$ is also presented and discussed in the caption of various figures. Due to the propagation of the generated QPs, we expect four time scales which are the times of come back of the QPs at the source point after getting reflected from boundaries of the chain, thus removing the effect of their dynamics. When this happens, the overlap in the definition of the LE increases and shows a peak. These time scales are determined by the fastest moving QPs with maximal group velocity $v_{\text{max}} = \max_k v_g(k)$ (see also the discussion in sections 1.3.1 and 1.3.3) and are given by $t_1 = (2L')/v_{\text{max}} \quad \text{(time of come back of } QP_L^1 \text{)}, \quad t_2 = 2(L - L')/v_{\text{max}} \quad \text{(time of come back of } QP_L^1 \text{)}, \quad t_3 = (2L_1)/v_{\text{max}} \quad \text{(time of come back of } QP_R^1 \text{)}$ and $t_4 = (2L_2)/v_{\text{max}} \quad \text{(time of come back of } QP_R^1 \text{)}. \quad \text{With } L_1 = L_2, \quad \text{the values of } t_3 \quad \text{and } t_4 \quad \text{are equal, but it is not the case in general. Other than the above mentioned obvious time scales, two more time scales are observed numerically, } \quad t' \quad \text{given by } 2|L_1 - L'|/v_{\text{max}}, \quad \text{which appears to be the time taken by } QP_L^2 \quad \text{(for } L_1 > L' \text{)} \quad \text{to reach } L' \quad \text{and get partially reflected at } L' \quad \text{where it finds a change in potential from } h \quad \text{to } h + \delta. \quad \text{The second time scale is same as } t_2 \quad \text{in magnitude, but it is due to } QP^2 \quad \text{and is given by } t'' = 2(L - L')/v_{\text{max}}, \quad \text{which is the time taken by the } QP_L^2 \quad \text{(} QP_R^2 \text{)} \quad \text{to get reflected at } L' \quad \text{(right boundary)} \quad \text{and come back to } L_1 \quad \text{after getting fully (partially) reflected at the right boundary } (L'). \quad \text{It is to be noted that } v_{\text{max}} \quad \text{for the homogeneous TFIM with elements as defined in Eq. (4.3) is 2 at the critical point and also in the paramagnetic phase. We shall use the same value of } v_{\text{max}} \quad \text{in our case also since the numerically obtained value of } v_{\text{max}} \quad \text{by differentiating the eigenvalues is also close to 2.}

We show the evolution of the LE in Fig. 4.1 for different values of $\delta$. Let us first concentrate on the results of LE with $\delta = 0$. Since the LE is the overlap of two wave-functions which had unity overlap initially, it starts decreasing from one at $t = 0$ till both the QPs reach the boundary at $t = L/4$ (since $L_1 = L_2 = L/2$ and $v_{\text{max}} = 2$), where it gets reflected. Intuitively, during its return path, QPs will undo their effect of dynamics. Thus, we expect to see a decrease in LE till the reflection of the first QP (i.e., till $t = L/4$) after which there is an increase till the QP reaches its origin or till $t = L/2(=2L_1/2)$. After this time, the initially left (right) moving QPs will move to the system 2 (system 1) and eventually come back to its origin showing a peak at $t = T = L$, which is also the time period of the QPs, see Fig. 4.1. When the second quench or the h-quench is also performed simultaneously, we expect to see some structures close to the time scales mentioned before, i.e., at $t_1, t_2, t', t''$ along with the $\delta = 0$ structure. The numerical results of such double quenches are shown in Fig. 4.1 for various coupling strengths $\delta$ and fixed $L'$. As expected, the decay in the LE is stronger with increasing strength of the local h-quench or $\delta$. On the other hand, Fig. 4.2 shows the variation of the LE($t$) when the h-quench is performed at different positions of the chain with fixed $\delta = 1$ demonstrating the above mentioned time scales more clearly, especially the variation of $t''$ with $L'$. We find good agreement between the timescales proposed above
Figure 4.1: The plot shows the LE as a function of time for different values of $\delta$ with $J-$quench at $L_1 = L/2$ and the $h-$quench at site $L' = L/3$. The transverse field at $L'$ is changed from 1 to $1 + \delta$. For the $J$ quenching alone (i.e., $\delta = 0$ case), the LE shows peak at $t_3 = L/v_{\text{max}} = 150$ and $T = 2t_3$ where $v_{\text{max}} = 2$ and $L = 300$. By applying two local perturbations simultaneously at time $t = 0$, we observe a small peak at $t' = 50$ and comparatively a stronger peak at $t'' = 200$. We also note small fluctuations near $t_1 = 100$ which is more clearly seen for $\delta = 1$ curve.

and the numerics, see the caption for more details.

In conclusion, we find that the dominant peaks are due to $QP^2$ at times $t', t'', t_3$ and $t_4$, where $t_3 = t_4$ in this case. We also note that the peak at $t''$ is a strong peak which may be due to the fact that at this time three different QPs return to their origin after reflections at various points as discussed below: (i) $QP^1_R$ after reflection from the right boundary, (ii)$QP^2_R$ after reflection from right boundary and a second reflection at $L'$ causing it to return to $L_1$ (iii) $QP^2_L$ after reflection at $L'$ and a second reflection at right boundary resulting to its return to $L_1$. Finally, a peak is also observed at $t = L$ which is the return time of all the fastest QPs back to their origin when there is no reflection at $L'$, thus giving us a hint that there may be a transmitted component of the QP also. We shall comment more on it after discussing the results in the ferromagnetic phase.

It is to be noted that the presence of time scale $t_1$ due to $QP^1$ is almost negligible in these figures, though we do observe some perturbation at this time. On the other hand, it is too early to discard the presence of $QP^1$ as its effect is very clearly observed in the evolution of the EE as explained in the next paragraph, thus ruling out the possibility of absence of such QPs. We shall try to argue about the absence of $t_1$ scale in the evolution of the LE in section 4.4.

We now focus on the EE as a function of time for the above scenario. In this case, another parameter is the size $L_A$ of system $A$ of which we calculate the EE with the remaining system of size $L - L_A$. Let us first consider the simplest case where $L_1 = L_A$,
Figure 4.2: The plot shows the LE as a function of time when $h$-quenching of strength $\delta = 1.0$ is performed at different sites $L'$ of the total chain with $J$-quenching fixed at $L_1 = L/2$. Here $L' = 0$ corresponds to the case of $J$-quenching alone where $t_3 = t_4 = 150$. For $L' = L/3$, $t' = 50$ and $t'' = 200$ whereas $t' = 90$ and $t'' = 240$ for $L' = L/5$. All these timescales are clearly seen in the above figure. We also observe small perturbations at $t_1$ which is not very clear.

i.e., the location of $J$-quenching also determines the size $L_A$ of system A. Interestingly, the bipartite EE of two critical transverse Ising chains can also detect the response of $h$-quenching (see Fig. 4.3, 4.4). We see that for $J$ quenching alone or for the single quench, the EE shows perfect periodic oscillations with dips at $t_3 = t_4 = 2L_1/v_{\text{max}}$, also discussed in Ref. [135] using conformal field theory. This can also be explained using the QP picture. A pair of QP will increase the entanglement between the system A and the rest if one of them is in system A and the other is in B. The QP pairs are generated at $L_1 = L_A = L/2$ and travel in opposite directions resulting to an immediate increase in $S(t)$ for $t > 0$. This is not the case when $L_1 \neq L_A$ and will soon be discussed separately. As both of them reaches the boundary at $t = L/4$ and gets reflected, $S$ starts decreasing and eventually shows a dip when $t = L/2$ after which both the QPs belonging to a pair exchange their systems and once again the EE increases after $t = L/2$. At $T = 2L/v_{\text{max}} (= L)$, both the QPs arrive at the starting point and $S$ shows a dip once again after which the pattern repeats. This is also shown in Fig. 4.4 with $L' = 0.0$. If we now perform local $h$-quenching at a general site $L'$ of the spin chain, we observe that the evolution of the EE in double quenches follows the single quench case but accompanied by deviations at certain times which can once again be explained using the QP picture. We observe that the double quench case follows the single quench case ($\delta = 0.0$) till $t = t'/2$ after which there is a sudden deviation or increase from the single quench case. This is because one of the QPs ($QP_R^1$) produced at $L'$ (which is not present in single quench case) enters the system B at this time whereas the other QP of the same pair
Figure 4.3: The EE as a function of time after a local $J$-quench in the middle of the chain ($L_1 = L/2$) along with $h$-quenching at $L' = L/4$ for different interaction strengths $\delta$. Here, we consider $L_A = L_1 = L/2$ as the subsystem with total system size $L = 300$. The time scales $t'/2$, $t'$ and $t''$ can be seen in this figure which agrees well with our explanations. In this case, $t' = 75$ and $t'' = 225$. Another time scale observed is around $t = 187$ which is when $QP^1_R$ enters system A resulting to an increase in the EE as $QP^1_L$ is in system B during that time. Note that the effect of $QP^1$ is very small compared to $QP^2$.

remains in system A. This results to an extra increase in $S$. On the other hand, at $t = t'$, the $QP^2_L$ reaches back to $L_1 = L_A$ after reflection at $L'$ where we find a sharp decrease in $S$ as both $QP^2_R$ and $QP^2_L$ are now in system B. The EE keeps on decreasing (after a slight increase) till $QP^2_R$ enters system A at $t = 2(L - L_1)/v_{\text{max}}$ as a result of getting reflected from the right boundary, after which we observe a sharp increase in the EE. One more time scale observed corresponds to $t'' = 2(L - L')/v_{\text{max}}$ which was also observed in the LE. At this time, the $QP^2_R$ and $QP^2_L$ return back to $L_A$ and exchange their systems, as also discussed before with reference to the LE.

The main difference between the analysis of the LE and the EE is that in the case of the LE, we are interested in time scales at which the produced QPs come back to its origin. In the case of the EE, we are interested in the timescales in which one of the QPs belonging to a pair crosses one system and goes to the other system. If this crossover results to both the QPs to be in the same system, then the EE decreases, otherwise it increases. Here, since we considered the geometry where $L_1 = L_2 = L_A$, many of the time scales are hidden. In the next section, we apply the same ideas to the case when $L_1 \neq L_2$ but $L_1 = L_A$ along with a special discussion for the most general case when $L_1 \neq L_2 \neq L_A$, and verify the QP picture proposed.
4.2. Loschmidt echo and entanglement entropy for critical chain

4.2.2 LE and EE for \(L_1 \neq L_2\)

We are now interested in the local quenching of asymmetric spin chain (\(L_1 \neq L_2\)). Let us first study the evolution of the LE. For \(J\)-quenching alone [135], the LE shows three time scales given by \(t_3\), \(t_4\) and the time period \(T\) discussed in section 4.2.1 (see \(L' = 0\) plot of Fig. 4.5). The nature of these plots are discussed in details in Ref. [135] using conformal field theory. When the transverse field term at \(L'\) is changed from \(h\) to \(h + \delta\) together with \(J\)-quenching, we expect to see the following additional time scales in parallel with the discussion in the previous section: \(t_1 = (2L')/v_{\text{max}}\), \(t' = 2(L_1 - L')/v_{\text{max}}\), \(t'' = 2(L - L')/v_{\text{max}}\). Fig. 4.5 shows the LE as a function of time after \(J\)-quenching at \(L_1 = L/3\) and \(h\)-quenching at different sites \(L'\). All the above mentioned time scales can be clearly seen in this figure except \(t_1\). We shall try to argue for this latter.

Let us now move on to the calculation of the EE for the same situation (\(L_1 \neq L_2\), but \(L_1 = L_A\)). Fig. 4.6 shows the time evolution of the EE after single and double quenching at time \(t = 0\). One can observe the difference in the LE and the EE between the times \(t_3\) and \(t_4\) for \(\delta = 0\) case (see Fig. 4.5 and Fig. 4.6). In this time range the LE remains constant. On the other hand the EE decreases slowly and then starts increasing at \(t = t_4\).
Figure 4.5: The plot shows the LE as a function of time for double quenches with $L_1 = 100, L_A = 100$ and $L = 300$ and different $L'$. The first peak of the LE occurs at time $t' = 2(L_1 - L')/v_{\max}$ ($t' = 25$ and $50$ for $L' = L/4$ and $L/6$ respectively). The other time scales are $t_3 = 100$, $t_4 = 200$ and $t''$. We note that $t'' = 225$ and $250$ for $L' = L/4$ and $L/6$ respectively.

as discussed in Ref. [135], the increase being due to arrival of $QP^2_R$ in system A. Moving to the double quenches, the discussion is almost same as for the case $L_1 = L_2 = L/2$ in the previous section. The EE more or less follows the $\delta = 0$ case and we see special time scales at $t'/2$, $t'$ and $t''$. See caption of Fig. 4.6 for more details.

We now consider the most general situation with $L_1 \neq L_2 \neq L_A$ and calculate the time evolution of the EE. Here, $J$-quenching is again performed at $L_1$, but the subsystem is assumed to be of length $L_A = L/2$, different from $L_1$. The EE as a function of time after the double quenches is shown in Fig. 4.7. Let us define $l = (L_A - L_1)$ as the distance between the right end of the subsystem A and the site of $J$-quenching. As in other cases, we discuss explicitly the case with $L' < L_1 < L_A$ below, and try to present some other examples through the figures. For $\delta = 0$, the EE remains at a constant value for small times and starts increasing at $t = l/v_{\max}$ when $QP^2_R$ hits at the boundary of the two subsystems and enters the subsystem $B$ [130]. Note the contrast between $L_1 = L_A$ where $S(t)$ increases immediately and $L_1 \neq L_A$ where $S(t)$ is constant initially. The EE shows a sharp decrease at $t = (2L_1 + l)/v_{\max}$ when $QP^2_L$ enters system B after getting reflected from the left boundary. Similar to the previous case (i.e., for $L_1 \neq L_2$ and $L_1 = L_A$), there is a 'decay region' between the time range $[(2L_1 + l)/v_{\max}, (L_2 + L_B)/v_{\max}]$ where EE decays very slowly, after which there is an increase as both the QPs are now in different subsystems. In this case, since the site for $J$-quenching does not coincide with the subsystem size, there are many additional time scales as discussed below, the appearance of which puts the picture of travelling QP on stronger footing. Let us now come back to the double quenches. Following the double quenches, the EE more or less
Figure 4.6: Time evolution of the EE for the same case as in Fig. 4.5 but different $L'$. For $L' = L/4$, the deviation from single J-quench case appears at $t'/2 = 12.5$ whereas at $t' = 25$, $QP_L^2$ enters system B after getting reflected at $L'$ where its other partner $QP_R^2$ is already present. The decrease in the EE continues till $QP_R^2$ enters system A at $t = L_2 = 200$. We also see a dip at $t = t'' = 225$ where $QP_L^2$ and $QP_R^2$ exchanges their systems. Similarly, one can argue for the evolution of the EE when $L' = 3L/4$. The deviation from the single quench case begins at $t = 62.5$. In this case, $QP_L^2$ enters system B at $t = 100$ which causes a sharp decrease at this time. On the other hand, $QP_R^2$ enters system A at $t = 125$ resulting to an increase in the EE as its other counterpart is still in B. The natural increase at $t = 200$ which is there for only J-quenching case can also be observed. This might be due to the fact that any reflection at $L'$ is not perfect and there is a possibility of getting a transmitted component of the QP wave, also discussed in sections 4.3 and 4.5. The time scales $t'' = 225$ and $T = 300$ are also present.

follows the single quench case. The first deviation resulting to an increase in the EE (similar to the one discussed before) occurs at $t = (L_A - L')/v_{\text{max}}$, when $QP_R^1$ enters the subsystem B. By definition, the time scale $t'$ is the time taken by the $QP_L^2$ to get reflected at $L'$ and come back to system A, which in this particular case is given by $t' = (2(L_1 - L') + l)/v_{\text{max}}$. On the other hand, the time scale $t''$, defined as the time taken by $QP_L^2$ (or $QP_R^2$) to undergo double reflections at $L'$ and one of the boundaries, gets divided into two scales. This is because the distance travelled by $QP_L^2$ is smaller than $QP_R^2$, which was not the case in our previous discussions where $L_1 = L_A$. Let us define these two timescales by $t''_1 = (2(L_1 - L') + l + 2L_B)/2$ (for $QP_L^2$) and $t''_2 = t''_1 + 2l/v_{\text{max}}$ (for $QP_R^2$). All these time scales are clearly shown in Fig. 4.7. We would like to point out here that the basic physics related to tracking of QPs remain same when we change the position of $L'$, but the formula for these time scales may have to be changed as can be seen in the $L' = 3L/4$ case discussed in Fig. 4.7. Also, the above discussion will be correct if $(L_A - L')/2 < L_1 - L'$, i.e., $QP_R^1$ reaches $L_A$ before $QP_L^2$. In the opposite case also, one needs to simply apply the same ideas to get the right picture of dynamics. It
4.3 Entanglement entropy for a ferromagnetic chain

In this section, we briefly discuss the evolution of entanglement entropy when the total spin chain is in the ferromagnetic phase. Here, we consider local quenching of asymmetric spin chains \((L_1 \neq L_2)\) with \(L_1 = L_A\). We concentrate upon two different cases to calculate the EE after single or double quenches: one where the total spin chain is deep inside the ferromagnetic phase (see Fig. 4.8) and the other where the spin chain is close to the critical point (see Fig. 4.9). Let us first consider the spin chain with \(h = 0.5\) at all sites. Fig. 4.8 shows time evolution of the EE after single quench at \(L_1\) (\(J\) quench) and also after the double quenches, namely, \(h\) quench at \(L'\) and \(J\) quench at \(L_1\). For the single quench \((L' = 0)\), the EE detects \(t_3\) and \(t_4\) successfully. Note the difference between the

is also to be mentioned that we do observe some extra time scales, some of which can be explained and are discussed in the caption of Fig. 4.7.

Figure 4.7: Time evolution of the EE after a local \(J\)-quenching at \(L_1 = L/3\) and \(h\)-quenching at different sites \(L'\) with \(\delta = 1.0\). Here, the subsystem is of length \(L_A = L/2\) which does not coincide with cut resulting to few more relevant time scales. For \(L' = L/5 = 60\), the first deviation (or increase) from the single quench case appears at \(t = 45\). The EE decreases at \(t = t' = 65\) when \(QP_{L'}^2\) and \(QP_{R}^2\) are in the same subsystem B. The next increase in the EE would be at \(t = 175\) when \(QP_{R}^2\) enters subsystem A. The split of time scale \(t''\), as discussed in the text, can also be seen with dips at \(t_1'' = 215\) and \(t_2'' = 265\). The case with \(L' = 3L/4\) is all the more interesting. The deviation occurs at \(t = (L' - L_A)/2 = 37.5\). The sharp decrease in this case occurs at \(t = 100\) which is the time taken by \(QP_{R}^2\) to get reflected at \(L'\) and enter system A so that both \(QP_{R}^2\) and \(QP_{L}^2\) are in system A. But at \(t = 125\), \(QP_{L}^2\) enters system B after getting reflected from the left boundary resulting to an increase in the EE. \(t_1'' = 200\) (due to \(QP_{L}^2\)) and \(t_2'' = 250\) (due to \(QP_{R}^2\)). We note extra dips around \(t = 187\) which seems to be due to \(QP_{L}^1\) entering system B after reflection from the left boundary.
critical and ferromagnetic region for times up to \( t_3 \). In the critical case, the EE increases at \( t = 0 \) followed by a decrease which starts around \( t_3/2 \) when the \( QP_L^2 \) gets reflected from the boundary, though the decrease is sharper at \( t_3 \). On the other hand, in the ferromagnetic region, we see a sudden increase in the EE followed by an almost constant EE region up to \( t_3 \) (no decrease at \( t_3/2 \)) after which it decreases suddenly. This hints to the fact that in the ferromagnetic region, QPs are more point like particles, and hence its location can be known precisely. But in the critical case, these QPs are extended wavepackets as also mentioned in Ref. [151], and hence the reflection at the boundary is felt also at \( L_1 \). Similar to the critical case (see Sec. 4.2.2), the EE decays between times \( t_3 \) to \( t_4 \). In this time range the fastest moving quasi-particles do not contribute in the EE. Let us now discuss the time evolution of EE after double quenches. One can observe clearly the time scales \( t'/2 \) and \( t' \) from Fig. 4.8. Interestingly, the EE starts decreasing after \( t' \) and it continues up to \( t_4 \). The sharp increase in the EE at \( t_4 \) is due to the fact that at \( t_4 \), \( QP_R^2 \) enters system A whereas \( QP_L^2 \) is still in system B. It is to be mentioned that in the ferromagnetic case, \( t'' \) is not clearly visible, which once again can be explained due to the point like nature of QPs in the ferromagnetic region. For \( t < t'' \), \( QP_L^2 \) is in system B and \( QP_R^2 \) is in system A. They exchange their systems at \( t'' \), thus contributing to the entropy equally for \( t < t'' \) and \( t > t'' \). On the other hand, the critical case distinguishes between QPs approaching \( L_1 \) and moving away from \( L_1 \) due to the finite extent of QP.

Figure 4.8: The plot shows the EE as a function of time for single and double quenches when the whole chain is non-critical (\( h = 0.5 \)) with \( L_1 = 100 \), \( L_A = 100 \) and \( L = 300 \) and different \( L' \). For \( L' = L/4 \), the deviation in the EE from single quenching case starts at \( t = t'/2 \) where \( t' = 50 \), with \( v_{\text{max}} = 2h = 1 \). For \( t > 50 \), both \( QP_L^2 \) and \( QP_R^2 \) are in system B leading to decrease in the EE which continues up to \( t_4 = 400 \). Similarly, for \( L' = L/5 \) one can find \( t' = 80 \) and the figure shows the expected behavior. The absence of \( t'' \) is explained in the text.
Figure 4.9: The EE as a function of time for the same situation as in Fig 4.8 but fixing \( h \) at 0.99. In this case the value of \( v_{\text{max}} \) is 1.98. This gives \( t' = 25.25 \) for \( L' = L/4 \). We do see timescales \( t'/2 \) and \( t' \) along with \( t_3 (\sim 101) \) and \( t_4 (\sim 202) \). We also observe a peak near \( t'' (\sim 227) \).

We now move to explain the quenching results when the final system remains close to the quantum critical point. The evolution of the EE as a function of time following single and double quenches is shown in Fig. 4.9 for \( h = 0.99 \). In this case also, we observe a sudden increase/deviation from the single quench case at \( t'/2 \) followed by a sharp decrease at \( t' \). We also observe an increase immediately after \( t' \) which is different from the \( h = 0.5 \) case and similar to the critical case. This may be because the QP which is now more like an extended object with extended wavefunction is only partially reflected at \( L' \) as compared to localized QP deep inside the ferromagnetic phase having less wavelike properties. Rest of the discussion is the same in this case and discussed in details in the caption of Fig. 4.9.

4.4 Comparison between \( QP^1 \) and \( QP^2 \)

In this section we try to argue why the effect of \( QP^2 \) is stronger than \( QP^1 \). As discussed before, the initial state, which is no longer the ground state of the final Hamiltonian, is a source of QPs. Also, since the perturbation studied in this work is local and very small, the QPs are produced at the site of perturbation only, i.e., at \( L' \) and \( L_4 \). These QPs have energies given by the eigenvalues of the final Hamiltonian. Let \( q \) identifies QP \( \eta_q \) having eigenenergy \( \Lambda_q \). QPs of energy \( \Lambda_q \) are produced at the perturbation site with probability \( f_q \). Numerically, one can obtain \( f_q \) by calculating the expectation value

\[
f_q = \langle \psi_i | \eta_q^\dagger \eta_q | \psi_i \rangle
\]
which is proportional to the number of QPs $\eta_q$ present in the initial state $|\psi_i\rangle$. This expression can be written in terms of $\Phi_q$, $\Psi_q$ of the final Hamiltonian and the matrix $G^i$ (see Appendix C.2) with respect to the initial Hamiltonian. A comparison of $f_q$ for the $h$-quench alone ($h$ changed from 1 to 2 at $L'$), $J$-quench alone (0 to 1 at $L_1$) and both quenches together is shown in Fig. 4.10. Clearly, QP creation probability is an order of magnitude higher in case of $J$-quench alone when compared to $h$-quench. This hints to the fact that the $J$-quench is the main source of QP production and hence our numerical results are dominated by the dynamics of $QP^2$.

![Figure 4.10: Variation of $f_q$ with $q$ for a critical chain. The dashed line or the lowest curve corresponds to the single $h$-quench case (no $J$-quench) which clearly is an order of magnitude smaller than the single $J$-quench (no $h$ quench), the dotted line. The case of double quench is shown by solid line which more or less overlaps with the case when only $J$-quench is performed. Similar behavior is also observed for a ferromagnetic chain.]

4.5 Conclusions and Discussions

In this chapter, we studied the effect of two simultaneous local quenches in an otherwise uniform transverse Ising chain of length $L$ with open boundary conditions. Initially, the system is prepared in the ground state of the transverse Ising chain having a uniform transverse field $h$ and the interaction strength set to unity at all sites except $J_{L_1}$ and $J_L$ where it is zero. The first quench corresponds to sudden increase of $J_{L_1}$ from zero to 1, and the second quench involves the sudden change of the transverse field from $h$ to $h + \delta$ at site $L'$. We argued that the sites of the two local quenches are source of QP production as there is a local increase in energy due to the quenches. These QPs are wavepackets of low lying excitations of the final Hamiltonian. As discussed in Refs. [150, 151], the QPs are localized in the ferromagnetic region and behave more like classical particles,
whereas they are extended objects/wavepackets as the critical point is approached.

We numerically studied the evolution of the LE and entanglement entropy after the double quenches and explained the evolution using the QP picture. The envelope of the curve is dictated by the fastest moving QPs. We showed taking examples that most of the timescales can be explained using the propagation of QPs. The most interesting phenomena that is observed numerically is the partial or full reflection of the QPs at $L'$, the site of $h-$quench. Only if we include such a phenomena that we can explain certain numerically observed time scales. As mentioned in details, the most relevant time scales are $t'$, $t_3$, $t_4$ and $t''$ (for $L_1 = L_A$) which are all due to the $QP^2$ pair, or the pair produced at the $J$-quenching site. The presence of the other set of QPs, namely, $QP^1$ is clearly seen in the evolution of the EE. We have shown that the probability of the QPs produced due to $h$-quench is roughly an order of magnitude smaller than the $J$-quench which could be the reason for stronger effect of $QP^2$ in the evolution of the LE and the EE.

The double quenches deep inside the ferromagnetic phase can be very nicely described by the point like the QPs where all the time scales are sharply observed. We have contrasted this ferromagnetic case with the double quenches in the critical phase and proposed the reasons for their differences. It seems that the reflection of $QP^2$ at $L'$ in a critical chain is only partial having a transmitted component also. This can be attributed to extended wavepacket nature of QPs at the critical point. One can then explain the decrease of the EE at $t = t_3$, slight increase of the EE for $t > t'$ after the sharp decrease at $t'$, and the dip at $t = T$.

Our main aim in this chapter is to study the dynamical evolution of the LE and the EE after double quenches and see if one can explain the behavior, atleast qualitatively, using propagation of QPs. We have demonstrated here that this indeed is possible. Though we can not propose a general formula for all the timescales involved as it depends on which QP arrives at the subsystem first, which in turn depends on the location of $L'$, $L_1$ and $L_A$, but the basic idea gives us the right picture. We have checked this for other cases also which are not presented in this chapter. The QP picture does explain many features, if not all, of the dynamical evolution of the LE and the EE that occur in double quenches studied here. We have provided some arguments for the immediate increase of the EE after $t'$ for a critical chain which is related to the extended nature of the QPs at the critical point.