Loschmidt echo: effect of gapless phase

In the previous chapter, we have discussed the scaling of fidelity susceptibility (FS) and the Loschmidt echo (LE) along different paths of a three-spin interacting transverse Ising model consisting different types of isolated quantum critical points (QCPs). In this chapter, we want to extend the study of the LE for a two-dimensional Kitaev model residing on a honeycomb lattice which has a gapless phase \cite{27}. In this context, we consider a central spin model where the above mentioned model is chosen to be an environment and a central spin or qubit is coupled to all the spins of the environment. Although in chapter 2, we have not discussed the LE in the language of the central spin model, it is straightforward to make the connection. As observed in the previous chapter, here also the decay of the LE is highly influenced by the quantum criticality of the environmental spin model e.g., it shows a sharp dip close to the anisotropic quantum critical point (AQCP) of its phase diagram. The early time decay and the collapse and revival as a function of time at AQCP also exhibit interesting scaling behavior with the system size which is verified numerically. It has also been observed that the LE stays vanishingly small throughout the gapless phase of the model. The above study has also been extended to the one-dimensional Kitaev model i.e. when one of the interaction terms vanishes.
3.1 Model, Phase diagram and Anisotropic quantum critical point (AQCP)

As discussed in section 1.1.3, the Hamiltonian of the Kitaev model on a honeycomb lattice is given by

$$H = \sum_{j+l={\text{even}}} (J_1 \sigma^x_{j,l} \sigma^x_{j+1,l} + J_2 \sigma^y_{j-1,l} \sigma^y_{j,l} + J_3 \sigma^z_{j,l} \sigma^z_{j+1,l})$$

(3.1)

The exact solution of the above model has been demonstrated in section 1.1.3 of chapter 1. The eigenenergies of the $H_{\vec{k}}$ (see Eq. (1.46) of section 1.1.3) are given by

$$E_{\vec{k}}^\pm = \pm \sqrt{\alpha_k^2 + \beta_k^2}.$$  

(3.2)

This energy spectrum corresponds to two energy bands; it is noteworthy that for $|J_1 - J_2| \leq J_3 \leq (J_1 + J_2)$, the band gap $\Delta_k = E_{\vec{k}}^+ - E_{\vec{k}}^-$ vanishes for some particular $\vec{k}$ modes leading to the gapless phase of the Kitaev model. The phase diagram of the model is shown in an equilateral triangle satisfying the relation $J_1 + J_2 + J_3 = 4$ and $J_1, J_2, J_3 > 0$ (see Fig. (3.1)); one can easily show that the whole phase is divided into three gapped phases, separated by a gapless phase (inner equilateral triangle) which is bounded by gapless critical lines $J_1 = J_2 + J_3$, $J_2 = J_3 + J_1$ and $J_3 = J_1 + J_2$.

Figure 3.1: Phase diagram of the Kitaev model, satisfying $J_1 + J_2 + J_3 = 4$. The inner equilateral triangle corresponds to the gapless phase in which the coupling parameters satisfies the relations $J_1 \leq J_2 + J_3$, $J_2 \leq J_3 + J_1$ and $J_3 \leq J_1 + J_2$. Along the three paths I, II and III $J_3$ is varied, so as to study the LE. The path I, II and III are defined by the equations $J_1 = J_2$, $J_1 = J_2 + 1$ and $J_1 + J_3 = 4$ respectively.
On the critical line $J_3 = J_1 + J_2$ energy gap goes to zero for the four $\vec{k}$ modes given by $(k_x, k_y) = (\pm 2\pi/\sqrt{3}, 0)$ and $(\pm 2\pi/3, 0)$ which are the four corner points of Brillouin zone. One can now expand $\alpha_{\vec{k}}$ and $\beta_{\vec{k}}$ around the critical mode $\vec{k} = (2\pi/\sqrt{3}, 0)$ for $J_3 = J_1 + J_2$, in the form

$$\alpha_{\vec{k}} = 2 \left[ J_1 \sin \left( \frac{\sqrt{3}}{2} k_x - \frac{3}{2} k_y \right) + J_2 \sin \left( \frac{\sqrt{3}}{2} k_x + \frac{3}{2} k_y \right) \right],$$

$$\beta_{\vec{k}} = 2 \left[ J_3 + J_1 \cos \left( \frac{\sqrt{3}}{2} k_x - \frac{3}{2} k_y \right) + J_2 \cos \left( \frac{\sqrt{3}}{2} k_x + \frac{3}{2} k_y \right) \right],$$

$$\approx \sqrt{3}(J_2 - J_1)k_x + 3(J_1 + J_2)k_y,$$

$$\approx 2(J_3 - J_1 - J_2) + \frac{3}{4}(J_1 + J_2)k_x^2 + \frac{9}{4}(J_1 + J_2)k_y^2 + \frac{3\sqrt{3}}{2}(J_2 - J_1)k_x k_y,$$

$$= \frac{3}{4}(J_1 + J_2)k_x^2 + \frac{9}{4}(J_1 + J_2)k_y^2 + \frac{3\sqrt{3}}{2}(J_2 - J_1)k_x k_y,$$  \hspace{1cm} (3.3)

where $k_x$ and $k_y$ are the deviations from the above mentioned critical modes. We note that $\alpha_{\vec{k}}$ varies linearly and $\beta_{\vec{k}}$ varies quadratically in $k_x$ and $k_y$. The point $J_{xc} = 2J_1$ (where $J_1 = J_2$) denoted by A in (Fig. 3.1) needs to be checked carefully. This is an AQCP with energy dispersion $E_{\vec{k}} \sim k_x^2$ along $k_x$ ($k_y = 0$) and $E_{\vec{k}} \sim k_y$ along $k_y$ ($k_x = 0$). The corresponding dynamical exponents are given by $z_\perp = 1$ and $z_\parallel = 2$, respectively.

For $J_1 \neq J_2$, $J_{xc} = J_1 + J_2$ is also an AQCP that can be shown using a rotation to a new coordinate system

$$k_1 = \sqrt{3}(J_2 - J_1)k_x + 3(J_1 + J_2)k_y,$$

$$k_2 = 3(J_1 + J_2)k_x - \sqrt{3}(J_2 - J_1)k_y,$$  \hspace{1cm} (3.4)

in which $\alpha_{\vec{k}}$ and $\beta_{\vec{k}}$ take the form

$$\alpha_{\vec{k}} = k_1,$$

$$\beta_{\vec{k}} = c_1 k_1^2 + c_2 k_2^2 + c_3 k_1 k_2,$$  \hspace{1cm} (3.5)

where $c_1 = 9(J_1 + J_2)(4J_1^2 + 4J_2^2 + J_1 J_2)$, $c_2 = 27J_1J_2(J_1 + J_2)$, and $c_3 = 18\sqrt{3}J_1J_2(J_2 - J_1)$. Therefore, for a general AQCP ($J_1 \neq J_2$) the dispersion will vary linearly and quadratically along $\vec{k}_1$ and $\vec{k}_2$ directions, respectively, with two dynamical exponents $z_1 = 1$ and $z_2 = 2$. 

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3.2 Qubit coupled to $J_3$ term of the Kitaev Hamiltonian

In this section, we will provide a general calculation of the LE considering the Kitaev model on a honeycomb lattice as an environment ($E$) that is coupled to a central spin-$\frac{1}{2}$ ($S$). We shall denote the ground state and excited state of the central spin $S$ by $|g\rangle$ and $|e\rangle$ respectively. $S$ is coupled to $J_3$ term of $E$ Hamiltonian only when the central spin is in the excited state $|e\rangle$. Therefore the composite Hamiltonian takes the form

$$H_T (J_3, \delta) = \sum_{j+l=\text{even}} (J_1 \sigma^x_{j,j+1} \sigma^x_{j+1,j} + J_2 \sigma^y_{j-1,j} \sigma^y_{j,j+1} + J_3 \sigma^z_{j,j+1} + \delta |e\rangle \langle e| \sigma^z_j \sigma^z_{j+1}),$$

where $\delta$ is the coupling strength of $S$ to $E$. We shall work in the limit of $\delta \to 0$.

We consider that the $S$ is initially in a generalized state $|\phi(0)\rangle_S = c_g|g\rangle + c_e|e\rangle$ (with the coefficients satisfying the condition $|c_g|^2 + |c_e|^2 = 1$), and the $E$ is initially in the ground state $|\varphi(J_3, 0)\rangle$. Following the similar steps of calculation as in section 1.2.2 we obtain the expression of the LE given as

$$\mathcal{L} (J_3, t) = |\langle \varphi(J_3, 0) \rangle \exp(-iH(J_3 + \delta)t)|\varphi(J_3, 0)\rangle|^2.$$  \hspace{1cm} (3.6)

Here, we have exploited the fact that the $|\varphi(J_3, 0)\rangle$ is an eigenstate of the Hamiltonian $H(J_3)$.

As defined in Eq. 1.53 of section 1.1.3, the complete ground state of $H(J_3)$ is given by

$$|\varphi(J_3, 0)\rangle = \prod_k \left[ \frac{1}{2} (a^\dagger_k - e^{i\gamma_k} b^\dagger_k) (a^\dagger_k + i b^\dagger_k) \right]|\Phi\rangle.$$  \hspace{1cm} (3.7)

where $\vec{k}$ runs over half of the Brillouin zone of the hexagonal lattice. Following mathematical steps identical to those in the Appendix A, it can be shown that Eq. (3.7) leads to the expression for the LE given by

$$\mathcal{L}(J_3, t) = \prod_{\vec{k}} \mathcal{L}_{\vec{k}} = \prod_{\vec{k}} [1 - \sin^2(2\gamma_{\vec{k}}) \sin^2(\epsilon_{\vec{k}}(J_3 + \delta)t)],$$  \hspace{1cm} (3.8)

where, $\tan \theta_{\vec{k}}(J_3 + \delta) = \alpha_{\vec{k}}/\beta_{\vec{k}}^\prime$ and $\gamma_{\vec{k}} = [\theta_{\vec{k}}(J_3) - \theta_{\vec{k}}(J_3 + \delta)]/2$. Here $\beta_{\vec{k}}^\prime$ corresponds to the value with $J_3 + \delta$ instead of $J_3$. The expression for the LE closely resembles that of the case when the transverse Ising chain is chosen to be the environment [53]. For numerical analysis of Eq. (3.8), we shall use $k_x$ and $k_y$ in terms of two independent variables $v_1$ and $v_2$, with $0 \leq v_1, v_2 \leq 1$. The $k_x$ and $k_y$ are given by [26, 222]

$$k_x = \frac{2\pi}{\sqrt{3}}(v_1 + v_2 - 1), \hspace{0.5cm} k_y = \frac{2\pi}{3}(v_2 - v_1).$$  \hspace{1cm} (3.9)
which span the rhombus uniformly. Avoiding the corner points of the Brillouin zone (where the LE results in a zero value), we vary \( v_1 \) and \( v_2 \) from \( 1/(2N) \) to \( 1 - 1/(2N) \) in steps of \( 1/N \), where \( N \) is the system size \([28]\) and consider only the half of the Brillouin zone using the condition \((v_1, v_2) \geq 1\).

Figure 3.2: LE as a function of parameter \( J_3 \) \((J_3 \text{ is varied along path I})\) shows a sharp dip at point A \((J_3 = 2 - \delta)\) and after a small revival in the gapless phase it again decays at point B, \( J_3 = 0 \) (see Fig. 3.1) with \( N_x = N_y = 200, \delta = 0.01 \) and \( t = 10 \). Inset (a) shows the variation in the LE when the parameter \( J_3 \) is varied along the path \( \Pi \) \((J_1 = J_2 + 1)\) in the phase diagram for \( N_x = N_y = 200, \delta = 0.01 \) and \( t = 10 \) clearly showing a sharp dip at point \( P \) \((J_3 = 2 - \delta)\) and again rise at point \( Q \) \((J_3 = 1 - \delta)\). Inset (b) marks the dip in LE when \( J_3 \) is varied along the path \( \Pi' \) \((J_1 + J_3 = 4)\), for this case \( N = 400, \delta = 0.01 \) and \( t = 10 \) so that \( E \) realize the change in the behavior at \( J_3 = 2 - \delta \). Details of these three cases is provided in the sections 3.2.1, 3.2.1 and 3.2.1 respectively.

The LE is calculated numerically as a function of \( J_3 \) using Eq. (3.8) and it shows dip at all critical points. To illustrate this, we choose three paths along which the interaction \( J_3 \) is varied. In the first case \( J_3 \) is varied along the path \( J_1 = J_2 \) (path ‘I’ in Fig. 3.1) so that the model enters from the gapped phase to the gapless phase (extending in the region \( J_3 \in [0, 2] \)) crossing the AQCP (point ‘A’ in Fig. 3.1) at \( J_3 = 2 - \delta \). The LE shows a sharp dip at point A and there is a revival with a small magnitude which again decays at the end point B, \( J_3 = 0 \) (see Fig. 3.2). Now surprising result shows up when the path is so chosen (path ‘II’ in Fig. 3.1), given by the equation \( J_1 = J_2 + 1 \) that the system enters the gapless phase through an AQCP with \( J_1 \neq J_2 \), (denoted by ‘P’ in Fig. 3.1). The LE shows a sharp dip at the point P and stays close to its minimum value (with a small revival as observed in path I) throughout the gapless phase and again shows a rise when the system exits the gapless phase through the point Q. In contrary, for the case when \( J_3 \) is changed along the line \( J_1 + J_3 = 4, J_2 = 0 \) (path ‘III’ in Fig. 3.1), one observes only a single drop in the LE near \( J_3 = 2 - \delta \) (see Fig. 3.2), inset (b)); this is associated
with the critical point of the one-dimensional Kitaev model. In the next section we will study the scaling of the short time behavior of the LE close to these critical points and the collapse and revival of the LE with time when the $E$ is right at these critical point.

### 3.2.1 Path I: anisotropic quantum critical point ($J_1 = J_2$)

As discussed in section 3.1, $J_{3c} = 2J_1$ is an AQCP with critical exponents $\nu_\perp = z_\perp = 1$ along $\hat{j}$ direction and $\nu_\parallel = 1/2, z_\parallel = 2$ along $\hat{i}$ direction. At this point energy gap vanishes for the three critical modes given by $(2\pi/\sqrt{3}, 0)$ and $(0, \pm 2\pi/3)$ in half of the Brillouin zone. Now we will study the short time behavior of the LE (in Eq. (3.8)) close to the AQCP. We define a cutoff frequency $K_c = (K_{x,c}, K_{y,c})$ such that modes up to this cut-off only are considered to calculate the decay of LE at short time close to the AQCP.

Then the LE is given by

$$L_c(J_3, t) = \prod_{k_x,k_y>0} K_c \cdot L_k.$$

(3.10)

We define the quantity $S(J_3, t)$, such that $S(J_3, t) = \ln L_c = -\sum_{k_x,k_y>0} |\ln L_k|$. Expanding around one of the critical mode upto the cut-off, we get $\sin^2 \varepsilon_k(J_3 + \delta)t \approx 4(J_3 + \delta - 2J_1)^2 t^2$ and $\sin^2 (2\gamma_k) \approx 9J_1^2 k_0^2 \delta^2/(J_3 - 2J_1)^2 (J_3 + \delta - 2J_1)^2$ (see Eq. (3.3))

therefore we obtain,

$$S(J_3, t) \approx -\frac{36E(K_c) J_1^2 \delta^2 t^2}{(J_3 - 2J_1)^2},$$

(3.11)

where $E(K_c) = 4\pi^2 N_c (N_c + 1) (2N_c + 1)/54N_y^2$ and $N_c$ is an integer nearest to $3N_yK_c/2\pi$.

We therefore find an exponential decay of the LE in the early time limit given by

$$L_c(J_3, t) \approx \exp (-\Gamma t^2)$$

(3.12)

where $\Gamma = 36E(K_c) J_1^2 \delta^2/(J_3 - 2J_1)^2$. The anisotropic nature of the quantum critical point is reflected in the fact that $\Gamma$ scales as $1/N_y^2$ and is independent of $N_x$. Further using the expression of $L_c(J_3, t)$, one can easily observe that it is invariant under the transformation $N_y \rightarrow N_y\alpha, \delta \rightarrow \delta/\alpha$ and $t \rightarrow t\alpha$, where $\alpha$ is some integer.

Now we fix $J_1 = J_2 = 1$ and $J_3 = 2 - \delta$ (point ‘A’ in Fig. (3.1)) and observe collapse and revival of LE with time (presented in the Fig. 3.3). The time period of collapse and revival is proportional to $N_y$, and is unaffected by the changes in $N_x$; this confirms the scaling result of the decay rate $\Gamma$ for the short time limit near the AQCP discussed above.

### 3.2.2 Path II: anisotropic quantum critical point ($J_1 \neq J_2$)

It has been shown that the point P in Fig. 3.1 ($J_1 \neq J_2, J_{3,c} = J_1 + J_2$) is an AQCP which can be seen by choosing directions $\hat{k}_1 = \sqrt{3}(J_2 - J_1) \hat{i} + 3(J_1 + J_2) \hat{j}$ (see section 3.1) and
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Figure 3.3: The collapse and revival of LE with $t$ at the AQCP (point ‘A’ in Fig. 3.1) for $J_1 = J_2 = 1$, $\delta = 0.01$ and $J_3 = 2 - \delta$, keeping $N_x (= 100)$, fixed and varying $N_y$ verifies the scaling relations satisfied by $N_y$, $\delta$ and $t$ as discussed in the text. The inset shows that the quasi-period of the collapse and revival is independent of $N_x$.

The critical exponents associated with this critical point are given by $\nu_1 = z_1 = 1$, and $\nu_2 = 1/2$, $z_2 = 2$ along $\hat{k}_1$ and $\hat{k}_2$ directions, respectively. To calculate the early time scaling in a similar spirit as in the previous section, we expand Eq. (3.8) near one of the critical modes up to the cut-off $K_c$ to obtain

$$
\sin^2 (\varepsilon_k (J_3 + \delta)) t \approx 4 (J_3 + \delta - J_1 - J_2)^2 t^2
$$

In the short time limit, the LE becomes

$$
\mathcal{L}_c (J_3, t) \approx \exp \left( -\Gamma t^2 \right)
$$

where, $\Gamma = \delta^2 \mathcal{E} (K_c) / (J_3 - J_1 - J_2)^2$, $\mathcal{E} (K_c) = 8\pi^2 J_2^2 N_c (N_c + 1) (2N_c + 1) / 3N^2$ and $N_c$ is an integer nearest to $NK_c / 4\pi J_2$.

In fact, comparing with the previous section 3.2.1, one can see that in this case $k_1$ (instead of $k_y$) appears in the expression of the LE in the short time limit. Further, from Eq. (3.13) and the expression of $\gamma$ one observes that $L_c (J_3, t)$ is invariant under the transformation $N_x = N_y = N \rightarrow N\alpha, \delta \rightarrow \delta / \alpha$ and $t \rightarrow t\alpha$, with $\alpha$ being some integer which is also observed in the collapse and revival behavior (see Fig. 3.4).

3.2.3 Path III: one-dimensional quantum critical point ($J_2 = 0$)

As mentioned already, along the line $J_1 + J_3 = 4, (J_2 = 0)$, the two dimensional spin model reduces to an equivalent one-dimensional spin chain with energy gap vanishing at $J_1 = J_3$ for $k_c = \pi$ and the corresponding dynamical exponent being $z = 1$. We shall now expand $\sin (\varepsilon_k (J_3, \delta) t)$ and $\sin (2\gamma_k)$ around the critical mode $k_c$ to analyze the short time decay of LE, resulting into $\sin^2 (\varepsilon_k (J_3 + \delta) t) \approx 4 (J_3 + \delta - J_1)^2 t^2$ and

$$
\hat{k}_2, \text{ perpendicular to } \hat{k}_1 [28].
$$
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Figure 3.4: Variation of the LE with $t$ at the AQCP P ($J_1 = 3/2, J_2 = 1/2$ and $J_3 = 2 - \delta$) shows collapse and revival with different $N_x = N_y = N$ and $\delta = 0.001$.

$$\sin^2(2\gamma_k) \approx J_1^2 k^2 \delta^2 / (J_3 - J_1)^2 (J_3 + \delta - J_1)^2.$$ The LE hence takes the form

$$\mathcal{L}_c(J_3, t) \approx \exp \left(-\Gamma t^2\right)$$

(3.14)

where, $\Gamma = 4J_1^2 \delta^2 \mathcal{E}(K_c) / (J_3 - J_1)^2$ and $\mathcal{E}(K_c) = 4\pi^2 N_c (N_c + 1) (2N_c + 1) / 6N^2$, $N_c$ is an integer nearest to $NK_c / 2\pi$. From Eq. (3.14), we find that the LE shows a similar scaling relation as is expected for a one-dimensional chain with $z = 1$ [53]; this is also confirmed by studying the collapse and revival of the LE (see Fig. (3.5)).
3.3 Conclusion

In this chapter we study a variant of the central spin model in which a central spin (qubit) is globally coupled to an environment which is chosen to be a two-dimensional Kitaev model on a honeycomb lattice through the interaction term $J_3$. Using the exact solvability of the Kitaev model, we have derived an exact expression of the LE when the interaction $J_3$ is varied in a way such that the system enters the gapless phase crossing the AQCP of the phase diagram. However, the behavior of the LE as a function of $J_3$ depends upon the path along which $J_3$ is varied. In the case when the AQCP, Q (with $J_1 \neq J_2$ see Fig. 3.1) is crossed, one observes a complete revival of the echo when the system exits the gapless phase to re-enter the gapped phase; this is in contrast to the case $J_1 = J_2$. For the case of $J_2 = 0$ there is only one sharp dip at the critical point $J_3 = 2 - \delta$ which is associated with the QCP of the one-dimensional Kitaev model.

The early time scaling behavior for both the paths I and II close to the AQCP bear the signature of the fact that the gapless phase is entered crossing an AQCP with different exponents along different spatial directions. This is also confirmed by studying the collapse and the revival of the LE as a function of time. However, one does not observe a perfect collapse and revival (except for the equivalent one dimensional case); this may be because of the proximity to a gapless phase. The quasi-period of collapse and revival in all cases scale with the system size as $N^z$. The case with $J_2 = 0$ reflects the fact that the system is essentially one-dimensional in this limit.

It should be noted that in a recent work, Pollmann et al [128], have studied the problem of the LE in a transverse Ising spin chain in the presence of a longitudinal field; more precisely they calculated the magnitude of the overlap between the final state reached following a slow quench across the QCP and its time evolved counterpart at time $t$ (generated following the time evolution with the final Hamiltonian). They observe a cusp-like minimum in the echo as a function of time in the limit when the spin chain is integrable. However, this behavior is smeared in the non-integrable case (with non-zero longitudinal field) thus providing a probe for integrable versus non-integrable behavior. In the present chapter, we however deal with a situation in which the spin chain is suddenly quenched across the QCP, and observe the collapse and revival only at the QCP. Since the coupling a qubit with a spin chain is equivalent to a sudden change of a parameter of the Hamiltonian.