Chapter 2

Hartman effect and dissipative Quantum Tunneling in a Parabolic barrier

2.1 Introduction

The issue of tunneling time is labeled with controversies from the beginning of it’s emergence. The controversy starts from the very fact that most of the definitions of tunneling time seems to predict superluminal tunneling velocities. One prediction, in the form of what is known as the Hartman effect [18], states that the tunneling time becomes independent of barrier length for thick enough barriers, ultimately resulting in unbounded tunneling velocities. Experiments done with single photons, classical light waves, and microwaves all show this apparent superluminality. Shortly after the discovery of quantum mechanical tunneling, Condon in 1930 posed the question of the speed of the tunneling process [20]. MacColl [21] carried out an approximate wave-packet analysis of the time-dependent Schrödinger equation that suggested that tunneling takes no appreciable time. The problem lay dormant for 30 years until technological advances made possible thin film devices based on quantum tunneling. Hartman revisited the question in an effort to understand the frequency limitations for tunneling devices involving metal insulator metal thin film sandwiches. Using the method of stationary phase previously applied to scattering problems by Eisenbud [22], Bohm [23], and Wigner [24], Hartman obtained an analytical expression for the time delay
(the group delay or phase time) in barrier tunneling that suggested a finite but short traversal time that saturates with distance. The origin of these paradoxical effects has been a mystery for decades. In the recent years Hartman’s work on tunneling time has drawn much attention of the community [18]. At the time of it’s publication, little attention was paid to Hartmans theoretical work on tunneling time of wavepackets in the sixties. Hartman analyzed the temporal aspects of tunneling by writing down solutions of the time-dependent Schrödinger equation in terms of a superposition integral over stationary state solutions weighted by a Gaussian momentum distribution function. Without explicitly evaluating the integrals he could infer certain properties of the transmitted wave packet by examining the magnitude and phase of the integrand. His main striking result was that under certain circumstances (opaque barrier) the tunneling time is independent of barrier length and the traversal time can be less than the time that would be required to travel a distance equal to the barrier length in vacuum. Many physicists hesitated to deal with Hartmans results since a very fast tunneling, or a zero tunneling time holds a serious consequence of violating Einstein’s postulate of Special Theory of Relativity. Hartman effect was first re-examined by Fletcher [25] within stationary phase method for quasi-monochromatic non-relativistic particles tunneling through the potential barriers. Furthermore, on recent times, experiments with photonic band gap structures [26, 27] showed apparent superluminality. These observations as well as the theoretical predictions lead towards the phenomena of superluminal barrier traversal. Regarding the explanation of this apparent superluminality, it is important to mention some publications over the past two decades or so [28, 29, 30, 31, 32, 33, 34, 35]. Some suggestions have been made [30, 31, 32, 33, 34] to explain faster than light phenomena by the concept of energy storage and release in the barrier region. The argument is that the group delay, which is directly related to the dwell time by an additive self interference term [35], is actually the lifetime of stored energy (or stored particles) leaking through both ends of the barrier. The saturation of group delay with barrier length has been a mysterious phenomena for decades. It’s existence has been verified for tunneling of all kinds of waves, starting from matter waves, electromagnetic waves to even sound waves. Experimentally it has been observed using electromagnetic waves and sound waves and there is no question about its existence. If the interpretation of the group delay in tunneling
is taken to be some kind of transit time then the Hartman effect would naturally lead to superluminal and unbounded group velocities. On the other hand, if the group delay in tunneling is not considered to be a transit time but a lifetime then the Hartman effect is easy enough to explain. Simply speaking, the origin of the Hartman effect is the saturation of stored energy with barrier length. Since the group delay is proportional to stored energy, it saturates as the stored energy saturates.

The phase time or group delay is defined as the energy differentiation of the phase change due to transmission. \( \tau_{GT} = \frac{\hbar d\phi_T}{dE} \) where \( \phi_T \) is the phase change due to transmission. The total group delay is defined as

\[
\tau_G = |T|^2 \tau_{GT} + |R|^2 \tau_{GR} \tag{2.1}
\]

where \( T \) and \( R \) are the transmission and reflection coefficients respectively.

In case of symmetric barriers \( \tau_G = \tau_{GT} = \tau_{GR} \).

Regardless of transmission or reflection, the dwell time is a measure of the time spent by a particle in the barrier region \( A < x < B \). It is given by the expression

\[
\tau_D = \int_A^B \frac{|\psi(x)|^2}{j_{in}} \, dx \tag{2.2}
\]

where \( \psi(x) \) is the wave function corresponding to energy \( E \) and \( j_{in} = \frac{\hbar k}{m} \) is the flux of the incident particles. This equation gives us the time that the incident flux has to be turned on, to provide the accumulated particle storage in the barrier. Even though the group delay is defined in terms of the energy derivative of a phase shift due to the transmission or reflection and the dwell time by an integral over a probability density, the two quantities are not unrelated. In fact, they are practically equal for energies above the barrier height and differ by a self-interference delay for below-barrier energies. The relation between group delay \( (\tau_G) \), dwell time \( (\tau_D) \) and self-interference delay \( (\tau_I) \) is given by [34]

\[
\tau_G = \tau_D + \tau_I \tag{2.3}
\]

Before going further, we will first discuss this relation explicitly. The group delay measures the delay between appearance of a wave packet at the beginning and end of the barrier. By the method of stationary phase, it is given by the energy derivative of the transmission phase shift
\[ \tau_{GT} = \hbar \frac{d\phi_T}{dE} \quad (2.4) \]

Here \( \phi_T = \phi_t + kL \) is the phase shift due to transmission, where \( L \) is the length of the barrier and \( k \) is the wave number. Similarly we can also define the group delay for the reflected wave as

\[ \tau_{GR} = \hbar \frac{d\phi_R}{dE} \quad (2.5) \]

where \( \phi_R \) is the reflection phase shift. The time independent Schrödinger equation for \( \psi(x) \) can be written as

\[ -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V\psi = E\psi \quad (2.6) \]

It’s complex conjugate equation is written as

\[ -\frac{\hbar^2}{2m} \frac{\partial^2 \psi^*}{\partial x^2} + V\psi^* = E\psi^* \quad (2.7) \]

Taking the derivative of equation (2.6) we get

\[ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \left( \frac{\partial \psi}{\partial E} \right) + V \frac{\partial \psi}{\partial E} = E \frac{\partial \psi}{\partial E} + \psi \quad (2.8) \]

Now multiplying equation (2.7) by \( \frac{\partial \psi}{\partial E} \) and equation (2.8) by \( \psi^* \) and then subtracting, the above equation gives

\[ \frac{\hbar^2}{2m} \frac{\partial}{\partial x} \left( \frac{\partial \psi \psi^*}{\partial E} - \psi^* \frac{\partial^2 \psi}{\partial E \partial x} \right) = |\psi|^2 \quad (2.9) \]

Integrating equation (2.9) over the limit 0 to \( L \), we get

\[ \left( \frac{\partial \psi \psi^*}{\partial E} - \psi^* \frac{\partial^2 \psi}{\partial E \partial x} \right)_{L} - \left( \frac{\partial \psi \psi^*}{\partial E} - \psi^* \frac{\partial^2 \psi}{\partial E \partial x} \right)_{0} = \frac{2m}{\hbar^2} \int_{0}^{L} |\psi|^2 dx \quad (2.10) \]

In front of the barrier \((x \leq 0)\) the is incident and reflection component of the wave function can be written as

\[ \psi_I = e^{ikx} + Re^{-ikx} \quad (2.11) \]

Behind the barrier there is only a transmission component

\[ \psi_{III} = Te^{ikx} \quad (2.12) \]
Using these wave functions in equation (2.10), we get [35]

\[
-2ik \left[ |T|^2 \frac{d|T|}{dk} + |R|^2 \frac{d|R|}{dk} + i \left( |T|^2 \frac{d\phi}{dk} + |R|^2 \frac{d\phi}{dk} + \frac{Im(R)}{k} \right) \right] \frac{\partial k}{\partial E} = \frac{2m}{\hbar^2} \int_0^L |\psi|^2 dx
\]  
(2.13)

Now since \(|R|^2 + |T|^2 = 1\), we get

\[
\tau_G = \frac{\int_0^L |\psi|^2 dx}{f_m} - \frac{Im(R)}{k} \hbar \frac{\partial k}{\partial E}
\]  
(2.14)

This is nothing but equation (2.3) stating the linear relationship between phase time and dwell time, where the self-interference term is given by [35]

\[
\tau_i = -\frac{Im(R)}{k} \hbar \frac{\partial k}{\partial E}
\]  
(2.15)

When the reflectivity is high, the incident pulse spends much of its time dwelling in front of the barrier as it interferes with itself during tunneling process. This excess dwelling is interpreted as the self-interference delay. This term can be successfully disentangled from the dwell time [35]. Now if the surrounding of the barrier is dispersionless, then the self-interference term vanishes, resulting in the equality of group delay and dwell time [33]. In that case the dwell time will give the lifetime of energy storage in the barrier region. In a relatively recent work [36], Jakiel et.al. have shown that Hartman effect is valid for all known expressions of the mean tunneling time, in various non-relativistic approaches, with finite width barriers without absorption or dissipation. It is to be noted that in reality, tunneling experiments are hard to perform with a setup which is isolated from the environment. In a recent work [37] (which will be discussed in the next chapter), we have formulated the expression of dwell time in presence of dissipation by the formalism of weak measurement. We have shown that inclusion of dissipative interaction precludes the zero time tunneling. But in that work we have not discussed the behavior of dwell time with increasing barrier thickness. In this work, it is our aim to calculate the dwell time theoretically for tunneling through a dissipative inverted parabolic barrier to find whether “Hartman effect” exists in presence of dissipative interaction with the environmental bath modes; i.e. whether the dwell time saturates with increasing barrier thickness or not. The inverted harmonic potential or parabolic barrier, which provides an infinite potential barrier can be used as a toy model for tunneling, where exact Gaussian
wave packets may be found as solutions. The inverted harmonic oscillator problem attracts a great deal of attention not only for being one of the exactly solvable potential in quantum mechanics [38, 39, 40, 41, 42, 43, 44, 45] but also because of having wide range of applicability in many branches of physics. It receives quite a formidable number of applications in many branches of physics from high energy to solid state theory. To incorporate dissipation, we will use the formalism of Yu [46] and Yu et.al. [47], where evolution of wave function and tunneling in dissipative system has been discussed. In Section 2.2 we will briefly discuss the background formalism, in Section 2.3 we will derive the dwell time using the discussed formalism and finally in Section 2.3 we will summarize with some concluding remarks.

2.2 Background formalism

Yu et.al. [47] have shown that for the simplest example of dissipative system, a harmonic oscillator coupled to a heat bath in a special case of ohmic dissipation, a self contained treatment can be used to get wave function evolution, where path integral technology need not be used. Later in another paper [46], Yu used this particular formalism in the quantum tunneling case for a inverted harmonic oscillator to obtain the tunneling probability and current density. Here we will use this formalism to get the dwell time for such systems. Using the notations of Ref. [46, 47] and quoting formulae from there, we first identify the total Hamiltonian of the system of inverted harmonic potential with a coupled infinite harmonic oscillator heat bath as

$$H = \frac{p^2}{2M} - \frac{1}{2}M(\omega_0^2 + \Delta \omega^2)q^2 + q \sum_j c_j x_j + \sum_j \left( \frac{p_j^2}{2m_j} + \frac{1}{2}m_j \omega_j^2 x_j^2 \right)$$

(2.16)

From this Hamiltonian, the quantum Langevin equation can be derived as

$$\ddot{q} + \eta \dot{q} - \omega_0^2 q = f(t)$$

(2.17)

where $\eta$ is the damping constant and $f(t)$ is the Brownian motion driving force given by

$$f(t) = -\sum_j \frac{c_j}{M} \left( x_{j0} \cos \omega_j t + \dot{x}_{j0} \sin \omega_j t \frac{\sin \omega_j t}{\omega_j} \right)$$

(2.18)
This equation (2.17) can be solved to give

\[ q(t) = a_1(t)q_0 + a_2(t)\dot{q}_0 + \sum_j [b_{j1}x_{j0} + b_{j2}\dot{x}_{j0}] \]  

(2.19)

where \( q_0, \dot{q}_0, x_{j0}, \dot{x}_{j0} \) are the initial position and velocity of the system and the bath respectively, and

\[ a_1 = e^{-\left(\frac{\eta}{2}\right)t}\left(\cosh \omega t + \frac{\eta}{2\omega} \sinh \omega t\right), \]

\[ a_2 = e^{-\left(\frac{\eta}{2}\right)t}\frac{\sinh \omega t}{\omega} \]  

(2.20)

\[ b_{j1} = -\frac{c_j}{M} \int_0^t a_2(t') \cos \omega (t - t') dt', \]

\[ b_{j2} = -\frac{c_j}{M} \int_0^t a_2(t') \sin \omega (t - t') dt' \]  

(2.21)

with \( \omega = \left(\omega_0^2 + \frac{\eta^2}{4}\right)^{1/2} \). The solution can be used to show that the wavefunction of the system plus bath can be written as [47]

\[ \psi(q, \{\xi_j\}, t) = \psi\left(q - \sum_j \xi_j, t\right) \Pi_j \chi_j(\xi_j, t) \]  

(2.22)

Now let us assume that the initial wavefunction is a Gaussian wave packet centered at the right of the peak of the potential by \( z_0 \) has the form

\[ \psi_0(q, t = 0) = (2\pi\sigma^2)^{-1/4}e^{-\left(q - z_0\right)^2/4\sigma^2 + ikq} \]  

(2.23)

Following [47], the Green’s function can be derived as

\[ G(q, q_0; t, 0) = \left(\frac{M}{2\pi\hbar a_2}\right)^{1/2} \times \exp \left[\frac{iM}{2\hbar a_2} (a_1 q_0^2 + a_2 e^{\eta t} q^2 - 2q_0q)\right] \]  

(2.24)

Then the wavefunction \( \psi(q, t) \) can be calculated as

\[ \psi(q, t) = \left(2\pi\sigma^2\right)^{-1/4}(a_1 + i\omega_0 a_2 r^2) \times e^{-\left(q - z_0\right)^2/4\sigma^2 + i(c_2 q^2 + c_1 q + c_0)} \]  

(2.25)

This is a Gaussian distribution with a width of

\[ \sigma_0^2 = \sigma^2(a_1^2 + r^4\omega_0^2 a_2^2) \]  

(2.26)

where \( r = \sigma_0/\sigma \) and \( q_c = a_1 z_0 + a_2 \hbar k/M \). The coefficients in the exponential term are

\[ c_2 = \frac{Me^{\eta t}}{4\hbar} \frac{d}{dt}(\ln \sigma_0^2) \]  

(2.27)
2.3 Dwell time in dissipative medium and Hartman effect

\[ c_1 = \frac{Me^{nt}}{4\hbar} \left( -2q_c \frac{d}{dt} (\ln \sigma^2_\theta) + 4\dot{q}_c \right) \]  (2.28)

\[ c_0 = \frac{k\alpha_2}{4} e^{nt} \left( q_c \frac{d}{dt} (\ln \sigma^2_\theta) - 2\dot{q}_c \right) + \frac{q_c z_0}{4\sigma^2_\theta} \omega^2_0 a_2 r^2 \]  (2.29)

The wave function expressed in equation (2.25) gives us the expression for the current density

\[ J = \frac{\hbar}{2Me^{nt}} \left( \psi \frac{\partial \psi^*}{\partial q} - \psi^* \frac{\partial \psi}{\partial q} \right) = \frac{\hbar}{M e^{nt}} |\psi|^2 (2c_2 q + c_1) \]  (2.30)

Putting the value of \( \psi, c_1, c_2 \) we get

\[ J = \left( 2(q - q_c) \frac{d}{dt} (\ln \sigma^2_\theta) + 4\dot{q}_c \right) \times \frac{(a_1^2 + r^4 \omega^2_0 a_2^2)}{\sqrt{2\pi} \sigma^2} e^{-\frac{(q-q_c)^2}{2\sigma^2}} \]  (2.31)

We will utilize equation (2.31) to get the dwell time in the next section.

2.3 Dwell time in dissipative medium and Hartman effect

Dwell time is defined as the average number of particles within the barrier region divided by the average number entering (or leaving) the barrier per unit time. It corresponds to the average time spent by a particle within the barrier irrespective of whether it is finally reflected or transmitted. If we follow Winful’s explanation [31, 32, 33, 34, 35], when the surroundings of the barrier is dispersionless, it represents the lifetime of energy storage in the barrier region. The dwell time in a neighborhood of \( q \) is defined as the ratio between the particle number in the interval \([q, q + dq]\) and the incoming current

\[ d\tau^D = \frac{|\psi(q)|^2}{J_{in}} \]  (2.32)

Obviously, Eq. (2.32) describes a balance equation: in the stationary case the injected current equals the decay rate of the probability in \([q, q + dq]\). The dwell time \( \tau^D \) of a finite region within the context of a stationary state scattering problem is obtained via a spatial integration of Eq. (2.32). So the dwell time \( \tau^D \) is given by

\[ \tau^D = \frac{M}{\hbar k} \int_{q_0}^{q} |\psi(q)|^2 dq = \frac{M}{\hbar k} \int_{q_0}^{q} \rho(q) dq \]  (2.33)

where \( J_{in} = \frac{\hbar k}{M} \) is the incident flux. We are considering constant incident flux for simplicity. Now we are dealing with non-stationary quantum states interacting
with the environment. So the wave function and consequently the probability
density function will also be dependent on time. Differentiating with respect to
time:
\[ \dot{\tau}_D = \frac{M}{\hbar k} \int_{q_0}^{q} \frac{\partial p(q,t)}{\partial t} dq \] (2.34)

Using the continuity equation:
\[ \dot{\tau}_D = -\frac{M}{\hbar k} \int_{q_0}^{q} \frac{\partial J(q,t)}{\partial q} dq = \frac{M}{\hbar k} [J(q_0) - J(q)] \] (2.35)

Now integrating equation (2.35) with respect to time:
\[ \tau_D = \frac{M}{\hbar k} \int_{0}^{T} [J(q_0) - J(q)] dt \] (2.36)

where measurements are made at \( t = 0 \) and \( t = T \). For long time measurement,
we can set \( T = \infty \). As we have discussed that the initial wavefunction is centered
at the right of the peak of the potential, it is travelling from right to left. So let
us now set \( q_0 = 2z_0 \) and \( q = 0 \). Then using equation (2.31) we get
\[ J(q_0) - J(q) = 4z_0 \frac{d}{dt} (\ln \sigma^2_\theta) \frac{(a_1^2 + r^4 \omega_0^2 a_2^2)}{\sqrt{2\pi \sigma^2}} e^{-\frac{x^2}{2\sigma^2}} \] (2.37)

Considering the spread of the wavefunction to be time dependent and replacing
\( \sigma \) be \( \sigma_\theta \) in equation (2.37), then putting in equation (2.36), we get
\[ \tau_D = \frac{4Mz_0}{\hbar k} \int_{0}^{\infty} \frac{d}{dt} \left( \frac{(a_1^2 + r^4 \omega_0^2 a_2^2)}{\sqrt{2\pi \sigma^2}} e^{-\frac{x^2}{2\sigma^2}} \right) d(x^2) \] (2.38)

where \( x^2 = a_1^2 + r^4 \omega_0^2 a_2^2 \). Putting \( \frac{z_0}{\sqrt{2\sigma}} = \zeta \):
\[ \tau_D = \frac{4M\zeta}{\hbar k \sqrt{\pi}} \int_{0}^{\infty} \frac{d}{dx^2} \left( e^{-\frac{x^2}{2\sigma^2}} \right) d(x^2) \] (2.39)

Again we substitute \( y = \zeta/x \). Now as \( t \to 0, x \to 1 \) and \( y \to \zeta \); as
\( t \to \infty, x \to \infty \) and \( y \to 0 \). So equation (2.39) becomes
\[ \tau_D = \frac{8M\zeta}{\hbar k \sqrt{\pi}} \int_{0}^{\zeta} \frac{e^{-\frac{y^2}{2\sigma^2}}}{y^2} dy \] (2.40)

The remaining problem is that the integration do not converge at \( y = 0 \). So instead
of taking the limit of \( x \) to be \( \infty \), we take a finitely long time \( t = T_{long} \), for which
\( x \to \zeta \) and \( y \to 1 \). So now the dwell time is found to be
\[ \tau_D = \frac{8M\zeta}{\hbar k \sqrt{\pi}} \left[ \frac{1}{\zeta} e^{\frac{-\zeta^2}{2\sigma^2}} \right] \] (2.41)
2.3. Dwell time in dissipative medium and Hartman effect

Here ‘$Erf$’ stands for error function. Generally, the width of the potential ($z_0$) is greater than the width of the wavefunction ($\sigma$). Also we will study the behavior of dwell time with increasing width of the barrier. Therefore the exponentially decaying term $e^{-\zeta^2/\zeta}$ would be less significant. So neglecting this term from equation (2.41) we get

$$\tau^D = \frac{8M\zeta}{\hbar k \sqrt{\pi}} \left[ \frac{1}{e} + \sqrt{\pi}(Erf(1) - Erf(\zeta)) \right]$$

(2.42)

Expressing the dwell time in terms of potential width $w = 2z_0$, we get

$$\tau^D = \frac{2M}{\hbar k \sqrt{\pi}} \left( \frac{w}{\sigma} \right)^2 \left[ \frac{1}{e} + \sqrt{\pi}(Erf(1) - Erf(w/2\sigma)) \right]$$

(2.43)

\[ F(\frac{w}{\sigma}) \]

Figure 2.1: $F(\frac{w}{\sigma})$ with the scaled potential width $\frac{w}{\sigma}$

The figure shows that in presence of dissipative interaction, the dwell time is not saturating with increasing barrier width, but it increases as the barrier width increase. Now let us compare this result with the classical case of a particle traveling in presence of a frictional force proportional to velocity ($v$) having the equation of motion

$$\dot{v} + \gamma v = 0$$

(2.44)

The solution of this equation will be of the form

$$v = v_0 e^{-\gamma t}$$

(2.45)

Consider that the classical particle takes $\tau^{CL}$ time to travel the distance $w_{cl}$ in presence of the usual velocity dependent frictional force. Then integrating equation (2.45), we get

$$\tau^{CL} = \frac{1}{\gamma} \ln \left( \frac{1}{1 - \frac{w_{cl}}{v_0}} \right)$$

(2.46)
A condition must be imposed that $\gamma \leq \frac{1}{\tau_0}$, where $\tau_0 = \frac{w_{cl}}{v_0}$ is the time taken by the particle to cross the distance $w_{cl}$ with constant velocity $v_0$ in absence of dissipation. Because otherwise the particle will lose all of its kinetic energy before crossing the barrier. Now considering $\gamma \ll \frac{1}{\tau_0}$ and expanding the logarithmic term in equation (2.46) and neglecting 3rd and higher order terms, we get

$$\tau_{CL} = \alpha w_{cl}^2 + \beta w_{cl}$$

(2.47)

where $\alpha = \gamma/v_0^2$ and $\beta = 1/v_0$. From FIG.1 and FIG.2 we see that the increment of dwell time with barrier width follow almost the same nature of that of a classical particle in presence of velocity dependent frictional force. So here we find that inclusion of dissipative interaction makes the behavior of dwell time quasi-classical. Also such dependence of dwell time on the barrier width precludes the afore mentioned “Hartman effect”.

2.4 Summary

It is evident from the above analysis that due to interaction with the environmental bath modes the tunneling time does not saturate with increment of barrier width. Unlike Hartman effect, in this case the dwell time depends on barrier thickness. As we have mentioned earlier, explanation of the saturation of tunneling time can be given by the concept of energy storage and release in the barrier region [34]. Group delay, which is equal to dwell time in absence of self interference, is proportional to the stored energy and it saturates (in absence of dissipation), as the stored energy saturates. In presence of dissipative interaction, tunneling entity loses energy to the interacting bath modes and in the process the saturation of energy is prevented. As the barrier length increases, the

Figure 2.2: $\tau^{CL}$ with $w_{cl}$ taking $\alpha = .01$ and $\beta = .1$
interaction with the bath modes also increase, resulting more and more energy loss. So the dwell time increases with increase of barrier length instead of getting saturated. The continuous interaction with the environmental bath modes makes the behavior of dwell time quasi-classical. In fact the quasi-classical behavior of dwell time due to the non-conservative interaction with environmental bath modes is analogous to the phenomena of decoherence. Since tunneling is a purely quantum phenomena, the cause of it’s occurrence is deeply rooted with quantum interference, anything such as dissipation (which causes the system to decohere) will destroy the process, which in turn precludes the Hartman effect.