5.1 Theoretical Modelling of Reynolds shear stress in $xz$- plane

The theoretical expressions of the stress fraction due to four quadrant events in Sections (2.6.2 and 4.2.10) are developed in this section. If the probability of $u'w'$ and the joint pdf of $u'$ and $w'$ are known, the contribution to the Reynolds shear stress due to four quadrant events can be easily calculated. So, in order to calculate the stress fraction due to the quadrant events we have to estimate the conditional probability distribution of the Reynolds shear stress in the respective quadrant. This theoretical development is basically the estimation of conditional pdf of the Reynolds shear stress. Antonia and Atkinson (1973) used a cumulant discards method to predict third and fourth order moments and the pdf of $u'w'$. Nakagawa and Nezu (1977) predicted the magnitude of the contributions of $u'w'$ due to the four quadrants events by considering conditional probability distribution functions of $u'w'$ derived using the cumulant discards method to the Gram-Charlier joint pdf $p(u', w')$ of $u'$ and $w'$. The derivation of the expression of stress fraction is given below.

For theoretical considerations, we normalize the velocity fluctuations $u'$ and $w'$ by dividing the standard deviation (r.m.s) in each direction so that $\hat{u} = u'/\sigma_u$, where $\sigma_u = \sqrt{u'^2}$ and
\( \hat{w} = w' / \sigma_w \), where \( \sigma_w = \sqrt{w'^2} \). Considering the joint probability function of \( \hat{u} \) and \( \hat{w} \) by \( p(\hat{u}, \hat{w}) \), its characteristic function by \( \chi(\alpha, \beta) \) with the moment of \( u'w' \) by \( m_{st} \) and corresponding cumulant \( q_{st} \), the characteristic function \( \chi(\alpha, \beta) \) can be defined as:

\[
\chi(\alpha, \beta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(\hat{u}\alpha + \hat{w}\beta)} p(\hat{u}, \hat{w}) \, d\hat{u} \, d\hat{w} \tag{5.1}
\]

where \( \alpha \) and \( \beta \) are its arguments. Here \( m_{st} \) and \( q_{st} \) correspond respectively to the coefficients in Taylor series expansions of \( \chi(\alpha, \beta) \) and \( \ln \chi(\alpha, \beta) \) are given by

\[
m_{st} = \frac{1}{i^{s+t}} \frac{\partial^{s+t}}{\partial \alpha^s \partial \beta^t} \chi(\alpha, \beta) | \alpha = \beta = 0 \tag{5.2}
\]

\[
q_{st} = \frac{1}{i^{s+t}} \frac{\partial^{s+t}}{\partial \alpha^s \partial \beta^t} \ln \chi(\alpha, \beta) | \alpha = \beta = 0 \tag{5.3}
\]

\[
\chi(\alpha, \beta) = \sum_{n=0}^{\infty} \frac{1}{n!} (\alpha \frac{\partial}{\partial \alpha'} + \beta \frac{\partial}{\partial \beta'})^n \chi(\alpha', \beta') | \alpha' = \beta' = 0 \tag{5.4}
\]

\[
\ln \chi(\alpha, \beta) = \sum_{n=1}^{\infty} \frac{1}{n!} (\alpha \frac{\partial}{\partial \alpha'} + \beta \frac{\partial}{\partial \beta'})^n \ln \chi(\alpha', \beta') | \alpha' = \beta' = 0 \tag{5.5}
\]

Using Eq. 5.2 in Eq. 5.4, one gets

\[
\chi(\alpha, \beta) = \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} m_{st} \frac{(i\alpha)^s (i\beta)^t}{s! t!} \tag{5.6}
\]

Using Eq. 5.3, Eq. 5.5 follows

\[
\ln \chi(\alpha, \beta) = \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} q_{st} \frac{(i\alpha)^s (i\beta)^t}{s! t!}, \quad \text{where} \quad q_{00} = 0; \tag{5.7}
\]

\[
\chi(\alpha, \beta) = \exp \left( \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} q_{st} \frac{(i\alpha)^s (i\beta)^t}{s! t!} \right)
\]
Comparing the like powers of Eq. 5.6 & Eq. 5.7, we get

\[
\begin{align*}
m_{10} &= q_{10} \\
m_{20} &= q_{20} + q_{10}^2 \\
m_{11} &= q_{11} + q_{10}q_{01} \\
m_{30} &= q_{30} + 3q_{20}q_{10} + q_{10}^3 \\
m_{21} &= q_{21} + q_{20}q_{10} + 2q_{11}q_{10} + q_{10}^2q_{01} \\
m_{40} &= q_{40} + 4q_{30}q_{10} + 3q_{20}^2 + 6q_{20}q_{10}^2 + q_{10}^4 \\
m_{31} &= q_{31} + 3q_{21}q_{10} + 3q_{20}q_{11} + 3q_{20}q_{10}q_{01} + 3q_{11}q_{10}^2 + q_{10}^3q_{01} \\
m_{22} &= q_{22} + 2q_{21}q_{01} + q_{20}q_{02} + 2q_{12}q_{10} + 2q_{11}^2 + 4q_{11}q_{10}q_{01} + q_{10}^2q_{01} + q_{10}^2q_{02} \\
\end{align*}
\]

Also, we have

\[
\begin{align*}
q_{10} &= m_{10} \\
q_{20} &= m_{20} - m_{10}^2 \\
q_{11} &= m_{11} - m_{10}m_{01} \\
q_{30} &= m_{30} - 3m_{20}m_{10} + 2m_{10}^3 \\
q_{21} &= m_{21} - m_{20}m_{01} - 2m_{11}m_{10} + 2m_{10}^2m_{01} \\
q_{40} &= m_{40} - 4m_{30}m_{10} - 3m_{20}^2 + 12m_{20}m_{10}^2 - 6m_{10}^4 \\
q_{31} &= m_{31} - m_{30}m_{01} - 3m_{21}m_{10} - 3m_{20}m_{11} + 6m_{20}m_{10}m_{01} + 6m_{11}m_{10}^2 - 6m_{10}m_{01} \\
q_{22} &= m_{22} - 2m_{21}m_{01} + 2m_{20}m_{01}^2 - m_{20}m_{02} - 2m_{12}m_{10} - 2m_{11}^2 + 8m_{11}m_{10}m_{01} - 6m_{10}^2m_{01} + m_{10}^2m_{02} \\
\end{align*}
\]
From the definition, we have

\[ m_{st} = \overline{u^s w^t} \]

\[ m_{10} = \overline{u} = \frac{\overline{u'}}{\sigma_u} = 0 \quad (\overline{u'} = 0) \]

\[ m_{01} = \overline{w} = \frac{\overline{w'}}{\sigma_w} = 0 \quad (\overline{w'} = 0) \]

\[ m_{20} = \overline{u'^2} = \frac{\overline{w'^2}}{\sigma_u^2} = \frac{\sigma_u^2}{\sigma_u} = 1 \]

\[ m_{02} = \overline{w'^2} = \frac{\overline{w'^2}}{\sigma_w^2} = \frac{\sigma_w^2}{\sigma_w} = 1 \]

\[ m_{11} = \overline{u'w'} = \frac{\overline{u'w'}}{\sigma_u \sigma_w} = -R < 0 \quad (\text{correlation coefficient}) \]

(5.10)

Using Eq. 5.10, Eq. 5.9 follows

\[ q_{00} = 0 \]

\[ q_{10} = 0 \]

\[ q_{20} = 1 \]

\[ q_{11} = -R \]

\[ q_{30} = m_{30} \]

\[ q_{21} = m_{21} \]

\[ q_{40} = m_{40} - 3 \]

\[ q_{31} = m_{31} + 3R \]

\[ q_{22} = m_{22} - 2R^2 - 1 \]

(5.11)

\[ q_{st} \text{ for } s < t \text{ can be obtained by merely exchanging } s \text{ and } t \text{ in the terms of } q_{st} \text{ for } s \geq t \]

In turbulent phenomena the cumulants of extremely high order can usually be neglected, and even in the theory of isotropic turbulence the fourth-order cumulant terms are sometimes discarded (Rotta, 1972). So, taking into account the cumulants of less than fourth order and
from Eq. 5.7 using Eq. 5.11, we have

\[
\ln \chi(\alpha, \beta) = -\frac{1}{2}(\alpha^2 - 2R\alpha \beta + \beta^2) + \sum_{s+t=3}^{4} \frac{q_{st}}{s!t!} \alpha^s \beta^t
\]  

(5.12)

Through an inverse transformation of (Eq. 5.1) in which the terms of \( \chi(\alpha, \beta) \) of less than fourth order are taken into account, \( p(\hat{u}, \hat{w}) \) can be written as

\[
p(\hat{u}, \hat{w}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i(\hat{u}\alpha + \hat{w}\beta)} \chi(\alpha, \beta) \, d\alpha \, d\beta
\]

(5.13)

Since \( p(\hat{u}, \hat{w}) \) be the joint density function of \( \hat{u} \) and \( \hat{w} \) which have zero means and unit variances. Under the assumptions that all the moments of \( \hat{u} \) and \( \hat{w} \) exist and some other general conditions (Chambers. 1967), the joint density function (Eq. 5.13) \( p(\hat{u}, \hat{w}) \) can be expanded as a series of derivatives of the standard bivariate normal density function \( \phi(\hat{u}, \hat{w}) \) (Mardia, 1970):

\[
p(\hat{u}, \hat{w}) = \phi(\hat{u}, \hat{w}) + \sum_{s+t=3}^{4} \frac{q_{st}}{s!t!} \frac{\partial^{s+t} \phi(\hat{u}, \hat{w})}{\partial \hat{u}^s \partial \hat{w}^t} \]

(5.14)

where

\[
\phi(\hat{u}, \hat{w}) = \frac{1}{2\pi(1-R)^{0.5}} e^{-\frac{\hat{u}^2 + 2R\hat{u}\hat{w} + \hat{w}^2}{2(1-R^2)}}
\]

(5.15)

and \( H_{st}(\hat{u}, \hat{w}) \) is Hermite polynomial of order \( (s + t) \) in two variables.

Equations (5.14) represents a special form of joint probability density distribution of the Gram-Charlier type in bivariate case.

The probability distribution of \( \hat{u} \):

\[
p(\hat{u}) = \phi(\hat{u}) + \sum_{s=3}^{4} (-1)^s \frac{q_{s0}}{s!} \frac{\partial^s \phi(\hat{u})}{\partial \hat{u}^s}, \quad \phi(\hat{u}) = \frac{1}{(2\pi)^{0.5}} e^{-\frac{\hat{u}^2}{2}}
\]

(5.16)
Similarly the probability distribution of \( \hat{w} \):

\[
p(\hat{w}) = \phi(\hat{w}) + \sum_{s=3}^{4} (-1)^{s} \frac{\partial^{s} \phi(\hat{w})}{\partial \hat{w}^{s}} = e^{-\hat{w}^{2}/2}
\]

By a change of variables (Eq. 5.14) may be reduced to

\[
p_{\tau n}(\tau n) = \int_{-\infty}^{\infty} \frac{R}{|\hat{u}|} p(\hat{u}, -R\tau n/\hat{u}) \, d\hat{u}
\]

When all cumulants \( q_{st} \) in Eq. 5.14, Eq. 5.16 and Eq. 5.17 with \( s + t \geq 3 \) are equal to zero, the Gram-Charlier distribution becomes the same as the Gaussian one, \( q_{st} \) gives a measure of the skewness of the distribution. Let \( \tau n = \frac{u'w'}{\sigma_u \sigma_w} \) be the normalized Reynolds stress, its probability distribution by \( p_{\tau n}(\tau n) \):

\[
\tau n = \frac{u'w'}{\sigma_u \sigma_w} = \frac{u' w'_{st}}{\sigma_u \sigma_w} = -\frac{\hat{w}}{R} (\because \frac{\hat{w}^{2}}{\sigma_u \sigma_w} = -R)
\]

set \( \Psi = \hat{u} \)

\[
\frac{\partial(\tau n, \Psi)}{\partial(\hat{u}, \hat{w})} = \begin{vmatrix}
\frac{\partial \tau n}{\partial \hat{u}} & \frac{\partial \tau n}{\partial \hat{w}} \\
\frac{\partial \Psi}{\partial \hat{u}} & \frac{\partial \Psi}{\partial \hat{w}}
\end{vmatrix} = \frac{\hat{u}}{R}
\]

By a change of variables (Eq. 5.14) may be reduced to

\[
p_{\tau n}(\tau n) = \int_{-\infty}^{\infty} \frac{R}{|\hat{u}|} p(\hat{u}, -R\tau n/\hat{u}) \, d\hat{u}
\]

\[
p_{\tau n}(\tau n) = \frac{R}{\pi(1-R^{2})^{0.5}} e^{-\frac{R^{2}}{1-R^{2}}} \int_{0}^{\infty} e^{-\frac{s^{2} + \frac{R^{2}(\tau n/\hat{u})^{2}}{1-R^{2}}} \left[ 1 + \sum_{s+t=3}^{4} \frac{q_{st}}{s! t!} \right] \left[ H_{st}(\hat{u}, -R\tau n/\hat{u}) + H_{st}(-\hat{u}, R\tau n/\hat{u}) \right]} \, d\hat{u}
\]

Let \( p_{Q1}(\tau n), p_{Q2}(\tau n), p_{Q3}(\tau n) \) and \( p_{Q4}(\tau n) \) be the probability distributions of \( Q1, Q2, Q3 \) and \( Q4 \), respectively. Therefore

\[
p_{\tau n}(\tau n) = p_{Q1}(\tau n) + p_{Q2}(\tau n) + p_{Q3}(\tau n) + p_{Q4}(\tau n)
\]

From Eq. 5.14 and Eq. 5.19, \( p_{Qi}(\tau n)(i = 1, 2, 3, 4) \) can be derived by using conditional calcu-
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For example, \( pQ_2(\tau_n) \) becomes

\[
p_{Q_2}(\tau_n) = \frac{ReR_e}{2\pi(1-R^2)^{0.5}} \int_0^\infty e^{-\frac{\xi^2+(\xi/\eta)^2}{2}} \left\{ 1 - \frac{1}{(1-R^2)^2} \left\{ B_1\eta^3 - B_1^*\eta^3 - B_2\eta^2(\xi/\eta) + \right\} B_2^*\eta(\xi/\eta)^2 - B_3\eta + B_3^*\eta(\xi/\eta) \right\} \frac{d\eta}{\eta}
\]

(5.21)

which is an infinite series of terms of the form

\[
I_{nm} = \int_0^\infty e^{-\frac{\xi^2+(\xi/\eta)^2}{2}} \eta^n (\xi/\eta)^m \frac{d\eta}{\eta}
\]

(5.22)

where \( \xi = \frac{Rn}{(1-R^2)} \) and \( \eta = \frac{\xi}{\sqrt{1-R^2}} \)

\[
B1 = \frac{1}{6}q_{30} + \frac{1}{2}Rq_{21} + \frac{1}{2}R^2q_{12} + \frac{1}{6}R^3q_{03}
\]

\[
B2 = \frac{1}{2}Rq_{30} + (R^2 + \frac{1}{2})q_{21} + (R + \frac{1}{2}R^3)q_{12} + \frac{1}{2}R^2q_{03}
\]

(5.23)

\[
B3 = \frac{1}{2}q_{30} + \frac{3}{2}Rq_{21} + (R^2 + \frac{1}{2})q_{12} + \frac{1}{2}Rq_{03}
\]

and \( B_1^* \) is \( B_i \) with \( q_{st} \) replaced by \( q_{ts} \).

From the definition of laplace transform, George F. Simmons (Second edition, Tata McGraw-Hill, Chap.9, pp.382) can be given as indicated by

\[
F(s) = \int_0^\infty e^{-st} f(t) dt
\]

(5.24)

From Eq. 5.22 using Eq. 5.24 we have

\[
I_{nm} = \int_0^\infty e^{-\frac{\xi^2+(\xi/\eta)^2}{2}} \eta^n (\xi/\eta)^m \frac{d\eta}{\eta} = \xi^m |\xi|^{\frac{(n-m)}{2}} K_{\frac{n-m}{2}} (|\xi|)
\]

(5.25)

\[
K_{\nu+1}(\xi) = 2\nu \xi^{-1} K_\nu(\xi) + K_{\nu-1}(\xi), \quad K_{-\nu}(\xi) = K_\nu(\xi)
\]

(5.26)

where \( K_\nu \) is the \( \nu^{th} \)-order modified Bessel function of second kind

From Eq. 5.21 using Eq. 5.23, Eq. 5.25 and Eq. 5.26 we have

\[
p_{Q_2}(\tau_n) = p_G(\tau_n) + \psi^-(\tau_n) \quad (\tau_n > 0).
\]

(5.27)
Similarly we have

\[ p_{Q_1}(\tau_n) = p_G(\tau_n) + \psi^+(\tau_n) \quad (\tau_n < 0). \] (5.28)

\[ p_{Q_2}(\tau_n) = p_G(\tau_n) - \psi^+(\tau_n) \quad (\tau_n < 0). \] (5.29)

\[ p_{Q_3}(\tau_n) = p_G(\tau_n) - \psi^-(\tau_n) \quad (\tau_n > 0). \] (5.30)

where

\[ p_G(\tau_n) = \frac{R}{2\pi} e^{R \xi} \frac{K_0(|\xi|)}{\sqrt{1 - R^2}} \] (5.31)

\[ \psi^+(\tau_n) = \frac{R}{2\pi} e^{R \xi} K_\frac{1}{2}(\xi) \frac{\sqrt{|\xi|}}{(1-R)^2} \{(1 + R)(\frac{S^+}{3} + D^+)|\xi| - \left(\frac{2 - R}{3}S^+ + D^+\right)\} \] (5.32)

\[ \psi^-(\tau_n) = \frac{R}{2\pi} e^{R \xi} K_\frac{1}{2}(\xi) \frac{\sqrt{|\xi|}}{(1-R)^2} \{(1 - R)(\frac{S^-}{3} + D^-)|\xi| - \left(\frac{2 + R}{3}S^- + D^-\right)\} \] (5.33)

and

\[ S^\pm = \frac{S_w \pm S_u}{2} = \frac{q_{03} \pm q_{30}}{2}, \quad D^\pm = \frac{D_w \pm D_u}{2} = \frac{q_{21} \pm q_{12}}{2} \] (5.34)

\( S_u \) and \( S_w \) are the skewness factors of \( u \) and \( w \) respectively, as \( S_u = \bar{u}^3 \) and \( S_w = \bar{w}^3 \). \( D_u \) and \( D_w \) correspond to turbulent diffusion in the \( x \) and \( z \) directions respectively, as \( D_u = \bar{u} \bar{w}^2 \) and \( D_w = \bar{w} \bar{u}^2 \), and are also called as diffusion factors.

Apply the conditional probability \( p_{Q_i}(\tau_n)(i = 1, 2, 3, 4) \) and (Eq. 5.20), \( p_{\tau_n}(\tau_n) \) becomes \( 2p_G(\tau_n) \). i.e

\[ p_{\tau_n}(\tau_n) = 2p_G(\tau_n) \] (5.35)

Then the contribution to the Reynolds stress \( S_{i,H} \) corresponding to each event can represented by

\[
\begin{align*}
S_{i,H} &= \int_{H}^{\infty} \tau_n p_{Q_i}(\tau_n) d\tau_n > 0 \quad (i = 2, 4) \\
S_{i,H} &= \int_{-\infty}^{H} \tau_n p_{Q_i}(\tau_n) d\tau_n < 0 \quad (i = 1, 3)
\end{align*}
\] (5.36)
5.1.1 Joint probability density function $p(\hat{u}, \hat{w})$

The joint probability density function $p(\hat{u}, \hat{w})$ by Eqs (5.14) and (5.15) using Gram-Charlier type is shown in Fig. 5.1 at the bed level (zero level) at the location D over the scour hole generated by the cylinder diameter $D_c = 3.2$ cm. This probability density function has been plotted for corresponding time interval given in figure (scour evolution over time). It is clearly observed from the figures that initially the joint probability density function shows normal distribution, while as time increases it deviates from the normality, but gradually it recovers to some extent after a certain time.

Figure 5.1: Joint probability density function $p(\hat{u}, \hat{w})$ of fluctuating velocity components ($u'$, $w'$) at the location D shown in Fig. 4.16
Figure 5.1: (Continued)
The contours of $p(\hat{u}, \hat{w})$ computed from the Eq. 5.14 are drawn at location $A$ (bed level surface) at three different heights ($z/h = 0.018, 0.154, 0.269$) in Fig. 5.2; and at location $D$ (close to the cylinder) within the scour mark generated by $D_c = 3.2\text{cm}$ at heights ($z/h = -0.08, 0.018, 0.269$) in Fig. 5.3. It is clearly observed from the Fig. 5.2 that as the height increases from the smooth bed surface (near-wall) to the outer layer, there is a little change of pattern from the triangular shape to the circular, whereas Fig. 5.3 shows a drastic change of pattern from the bottom to the outer layer. At the bottom surface $z/h = -0.080$ (below the bed level surface), the contours of $p(\hat{u}, \hat{w})$ look like ellipse inhabiting the ejection and sweeping events implying the downward flux of momentum towards the bed. As the vertical distance increases from the bottom to the outer layer ($z/h \geq 0.018$) the ellipse-typed contour occupying the second and fourth quadrants changes to circular contour with equally favored...
all four quadrants. The symmetry about the origin of \((\hat{u}, \hat{w})\)-plane corresponds to the wake-interference roughness regime (Papanicolaou et al., 2001). In the outer layer above the level surface \((z/h \geq 0.018)\), the contour pattern is almost identical.

Figure 5.2: Contour plots of joint probability density function \(p(\hat{u}, \hat{w})\) at three different heights at the location \(A\) on the smooth surface for the case \(D_c=3.2\) cm.
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Figure 5.3: Contour plots of joint probability density function \( p(\hat{u}, \hat{w}) \) at three different heights at the location \( D \) (close to the cylinder) within the scour region generated by the cylinder diameter \( D_c = 3.2\) cm.

The contours of \( p(\hat{u}, \hat{w}) \) are plotted for all three cylinder diameters \( (D_c = 3.2, 4.2, 6.0\) cm) at the level surface \( (z/h = 0.018) \) of location \( A \) (smooth surface) in Fig. 5.4; and at the same level \( (z/h = 0.018) \) in location \( D \) (close to the cylinder) within the scour hole in Fig. 5.5. Fig. 5.4 shows that the contours of \( p(\hat{u}, \hat{w}) \) are almost triangular in shape and independent of cylinder diameters, that means, there are no effects of scour holes generated by different cylinder \( D_c \).

The contours occupy all the four quadrants almost uniformly (Fig. 5.4). In the Fig. 5.5, as the
cylinder diameter \((D_c)\) increases, there is a clear shift of contours from the triangular shape to ellipse for \(D_c = 6.0\text{cm}\) occupying the most favored quadrants - ejection and sweep implying the downward flux of momentum (Strom and Papanicolaou, 2007). Eventually, increase of cylinder diameter leads to increase in width of scour holes and hence enhancement of turbulence.

Figure 5.4: Contour plots of \(p(\hat{u}, \hat{w})\) for three different cylinder diameters \((D_c=3.2, 4.2\) and \(6.0\text{cm}\)) at the location \(A\) (smooth surface) at the level surface \(z/h = 0.018\).
Figure 5.5: Contour plots of $p(\hat{u}, \hat{w})$ at three different cylinder diameters ($D_c = 3.2$, 4.2 and 6.0cm) at the location $D$ (scour region, close to the cylinder) at the bed level surface $z/h = 0.018$.

The theoretical curves of $p(\hat{u}, \hat{w})$ given by the Eq. 5.14 using Gram-Charlier type of bivariate case are shown in Fig. 5.6 at the near-bed points of all eight locations $A$ to $H$ across the equilibrium scour mark generated by the cylinder diameter $D_c = 3.2cm$. It is clearly observed that at location $A$ on the smooth level surface, the density function shows the bivariate Gaussian distribution. As the horizontal distance increases from $A$ to $H$ along the scour hole, how the distribution of $p(\hat{u}, \hat{w})$ deviates from the Gaussian curve. At location $B$ at the edge of the sour
hole, the distribution inhabits within the inward and outward interaction quadrants implying the upward flux of momentum, whereas at locations $C$ and $D$ (close to the cylinder) within the scour region, it inhabits within the ejection and sweep quadrants implying the downward flux of momentum. Again at the location $E$ and $F$ on the cylinder and the location $H$ on the deposited materials, the distribution represents the upward flux of momentum and at the location $G$ (on the ridge) it inhabits the downward flux of momentum. The prominent deviation of the distribution takes place at locations $C$ and $D$ within the scour mark, and at $E$ on the edge of the cylinder. Thus it is observed that the orientation of the theoretical curves of joint probability density function changes cyclically with the horizontal distance depending on the local velocity structures. The $p(\hat{u}, \hat{w})$ have its maximum value about 0.25 at the location $E$ on the positive side of $\hat{u}$ and minimum peak is 0.05 at $C$ and $G$. 

Figure 5.6: Joint probability density function $p(\hat{u}, \hat{w})$ at eight different locations A to H along the flow for $D_c = 3.2\text{cm}$ at the bed surface.
The variation of contours of $p(\hat{u}, \hat{w})$ from Eq. (5.14) at the bottom-most points of all eight longitudinal locations $A$ to $H$ across the scour hole generated by a fixed cylinder diameter ($D_c = 3.2$ cm) are plotted in Fig. 5.7. It is interesting to see how the bottom stress fraction is changing along the horizontal direction from the location $A$ to $H$. As the distance increases from $A$ to $H$, the contour changes drastically with different shapes from triangular to ellipses and back, which means $p(\hat{u}, \hat{w})$ changes with location from the smooth surface to the deposited materials through scour hole. At location $B$, i.e. at the edge of the scour hole, the contours inhabit within the outward and inward interaction quadrants implying the upward flux of momentum, whereas at location $D$ within the scour region, it is reverse, i.e. the contours inhabit within the ejection and sweep quadrants implying the downward flux of momentum, and again at location $E$ (at the edge of the cylinder), it changes to upward flux of momentum and it sustains up to the lo-
cation $H$ except the location $G$, where the contours are almost circular. Thus, it seems that the orientation of the contours changes cyclically depending on the local bottom shear stress.
Figure 5.7: Contour plots of $p(\hat{u}, \hat{w})$ at eight different locations A to H along the flow for $D_c = 3.2\text{cm}$ at the bed surface.
5.1.2 Probability density functions $p(\hat{u})$ and $p(\hat{w})$

The probability density function of $p(\hat{u})$ is presented using the measured and computed data for $D_c = 3.2cm$ in Fig. 5.8(a) on the smooth surface at $A$ and in Fig. 5.8(b) within the scour hole at $D$ at different heights. It is observed from the Fig. 5.8(a) that the theoretical curves of the probability density of $\hat{u}$ computed by the Eq. 5.16 are fairly good agreement with experimental data for $D_c = 3.2cm$ at $A$ for three different heights $(z/h = 0.018, 0.158, 0.269)$. The computed values of $p(\hat{u})$ at $D$ are shown with the observed data in Fig. 5.8(b), it shows a good agreement at the bed level height $z/h = 0.018$ except at heights $z/h = -0.080, 0.269$. Figures 5.9 (a,b) show the $p(\hat{u})$ of stream-wise velocity $\hat{u}$ at $A$ and $D$ at a fixed height $z/h = 0.018$ for different cylinder of diameters $D_c = 3.2, 4.2, 6.0cm$. It is found that the probability density of $\hat{u}$ computed from Eq. 5.16 is fairly good agreement with experimental data at $A$ for three different cylinder of diameters $(D_c = 3.2, 4.2, 6.0cm)$ shown in Fig. 5.9(a), whereas the $p(\hat{u})$ at $D$ shows a good agreement with the experimental data for the cylinder of diameters $D_c = 3.2, 6.0cm$ except with some deviation for the diameter $D_c = 4.2cm$ in Fig. 5.9(b).

![Figure 5.8: Plots of $p(\hat{u})$ against $\hat{u} = u'/\sigma_u$ for $D_c$= 3.2cm. (a) At the location $A$ (smooth surface): ◦ (....), $z/h = 0.018$; + (- -), $z/h = 0.158$; ▽ (-), $z/h = 0.269$: and (b) at the location $D$ (scour region): ◦ (....), $z/h = -0.080$; + (- -), $z/h = 0.018$; ▽ (-), $z/h = 0.269$. Here dot, dash and solid lines represent the theoretical value.](image)
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Figure 5.9: Plots of $p(\hat{u})$ against $\hat{u} = u'/\sigma_u$ at the bed level $z/h = 0.018$. (a) At the location $A$ (smooth surface): $\circ (.....), D_c = 3.2cm; + (- -), D_c = 4.2cm; \triangleright (-), D_c = 6.0cm$, and (b) At location $D$ (scoured region) $\circ (.....), D_c = 3.2cm; + (- -), D_c = 4.2cm; \triangleright (-), D_c = 6.0cm$. Here dot, dash and solid lines represent the theoretical value.

The data plotted in Figs. 5.10(a, b) are the probability density $p(\hat{w})$ of vertical velocity component over the scour hole generated by $D_c = 3.2cm$ at $A$ and $D$. It is found that the probability density of $\hat{w}$ calculated from Eq. 5.17 is fairly good agreement with experimental value at $A$ for all three different heights ($z/h = 0.018, 0.158, 0.269$) shown in Fig. 5.10(a).

The probability density of vertical velocity component at location $D$ within the scour hole is plotted with experimental data for all three different heights, and it shows a good agreement at the heights $z/h = 0.018, 0.269$ above the bed level except some deviation at the near wall $z/h = -0.080$ (Fig. 5.10b). Figures 5.11(a,b) shows the computed and observed values of probability density of vertical velocity component at $A$ on the smooth surface and at $D$ within the scour hole at a fixed height $z/h = 0.018$ (bed level) for different cylinder diameters $D_c = 3.2, 4.2, 6.0cm$. It is observed that the probability density of $\hat{w}$ calculated from Eq. 5.17 is fairly good agreement with experimental value at both the locations $A$ and $D$ for all three cylinder diameters $D_c = 3.2, 4.2, 6.0cm$ except with some shift for the cylinder diameter $D_c = 6.0cm$ at location $D$ within scour region.
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Figure 5.10: Plots of $p(\hat{w})$ against $\hat{w} = w'/\sigma_w$ for $D_c = 3.2$cm. (a) At the location $A$ (smooth surface): $\circ (\ldots), z/h = 0.018; + (- -), z/h = 0.158; \triangleright (-), z/h = 0.269$; and (b) at the location $D$ (scour region): $\circ (\ldots), z/h = -0.080; + (- -), z/h = 0.018; \triangleright (-), z/h = 0.269$. Here dot, dash and solid lines represent the theoretical value.

Figure 5.11: Plots of $p(\hat{w})$ against $\hat{w} = w'/\sigma_w$ at the bed level $z/h = 0.018$. (a) At the location $A$ (smooth surface): $\circ (\ldots), D_c = 3.2$cm; $+ (- -), D_c = 4.2$cm; $\triangleright (-), D_c = 6.0$cm, and (b) at the location $D$: $\circ (\ldots), D_c = 3.2$cm; $+ (- -), D_c = 4.2$cm; $\triangleright (-), D_c = 6.0$cm. Here dot, dash and solid lines represent the theoretical value.

In the overall trend, it is observed from the figures that at the location $A$ on the smooth surface, both $p(\tilde{u})$ and $p(\hat{w})$ indicate symmetrical distributions about the zero axis for all three different heights, which agree well with the experimental data. But within scour region at the location $D$, except few cases both $p(\tilde{u})$ and $p(\hat{w})$ have a good agreement with the experimental data. This discrepancy may be due to the strong turbulence in the near bed region within the
scour holes. Despite the fact that the qualitative behavior of the distributions are fairly well and explained by the Gram-Charlier distribution.

### 5.1.3 Probability density function $p_{\tau_n}(\tau_n)$ of normalized Reynolds shear stress

The measured and computed values of $p_{\tau_n}(\tau_n)$ of normalized Reynolds shear stress $\tau_n$ are plotted in Fig. 5.12(a,b) for $D_c = 3.2 cm$ at the locations $A$ (smooth level surface) and $D$ (close to the cylinder). It is noted from the figures that the probability density of $(\tau_n)$ calculated from Eq. 5.35 is fairly well with experimental data at both locations $A$ and $D$ for all three heights. At location $A$ on the smooth level surface, the distribution profiles of $p_{\tau_n}(\tau_n)$ are almost identical for all heights (Fig. 5.12a), whereas at $D$ within the scour region, peak of the distribution profile decreases with the increase of vertical height (Fig. 5.12b). In fact, the maximum peak occurs at the near-bed $(z/h = -0.080)$ within the scour region. It may be noted that $(\tau_n) = 0$ have singularity at origin.

![Figure 5.12](image)

Figure 5.12: Plots of $p_{\tau_n}(\tau_n)$ against $\tau_n$ for $D_c = 3.2 cm$. (a) At $A$ on the smooth surface: $\circ (- -)$, $z/h = 0.018$; $+ (- -)$, $z/h = 0.158$; $\triangleright (-)$, $z/h = 0.269$, and (b) at $D$ (close to the cylinder) within the scour region: $\circ (- -)$, $z/h = -0.080$; $+ (- -)$, $z/h = 0.018$; $\triangleright (-)$, $z/h = 0.269$. Here dash-dot, dash and solid lines represent the theoretical value.

Figures 5.13(a,b) present the plots of computed and measured values of $p_{\tau_n}(\tau_n)$ against normalized Reynolds shear stress $(\tau_n)$ for the locations $A$ and $D$ at a level surface $z/h = 0.018$. 

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for different diameters $D_c = 3.2, 4.2, 6.0\, \text{cm}$ of cylinder. It is found that the probability density of $(\tau_n)$ calculated from Eq. 5.35 shows fairly good agreement with observed values for both $A$ and $D$. At $A$ on the smooth level surface, the profiles of $p_{\tau_n}(\tau_n)$ of normalized $\tau_n$ are almost identical at near-bed level $z/h = 0.018$ for all cylinder diameters ($D_c$) (Fig. 5.13a), while at location $D$ within the scour mark, peak of the distribution increases with increase of $D_c$ at the same height $z/h = 0.018$ (Fig. 5.13b).

![Figure 5.13: Plots of $p_{\tau_n}(\tau_n)$ against $\tau_n$ at the level surface $z/h = 0.018$.](image)

Figure 5.13: Plots of $p_{\tau_n}(\tau_n)$ against $\tau_n$ at the level surface $z/h = 0.018$. (a) At the location $A$: o (- -), $D_c = 3.2\, \text{cm}$; + (- -), $D_c = 4.2\, \text{cm}$; ⊳ (-), $D_c = 6.0\, \text{cm}$, and (b) at the location $D$: o (- -), $D_c = 3.2\, \text{cm}$; + (- -), $D_c = 4.2\, \text{cm}$; ⊳ (-), $D_c = 6.0\, \text{cm}$. Here dash-dot, dash and solid lines represent the theoretical value.

5.1.4 Conditional statistics of Reynolds stress

Contributions of stress fraction $|S_{i,H}|$ to the Reynolds shear stress are estimated from the velocity data collected in the presence of cylinder diameter $D_c = 3.2\, \text{cm}$. The stress fraction $|S_{i,H}|$ are plotted against the threshold parameter $H$ for each of the four quadrants at the location $A$ on the level surface at three different heights ($z/h = 0.018, 0.158, 0.269$) in Fig. 5.14(a); at location $D$ within the deepest part of the equilibrium scour hole at the levels ($z/h = -0.080, 0.018, 0.269$) in Fig. 5.14(b); and at location $H$ above the deposited materials (on the ridge) for different heights ($z/h = 0.040, 0.158, 0.269$) in Fig. 5.14(c). It is observed that the quadrant-wise estimates to the Reynolds shear stress from the velocity measurements:
outward interaction \( (Q_1) \), ejection \( (Q_2) \), inward interaction \( (Q_3) \), sweep \( (Q_4) \) are fairly in good agreement with theoretical values calculated from Eq. 5.36.

At the location \( A \) on the smooth level surface, sweep event \( (Q_4) \) is much greater than the ejection \( (Q_2) \) near the boundary at \( z/h = 0.018 \) (Grass, 1971; Nakagawa and Nezu, 1977; Rau-pach, 1981), whereas away from the boundary (outer region) at heights \( z/h = 0.158, 0.269 \), ejection dominates over sweep event (Lopez and Garcia, 1999; Hurther and Lemmin, 2000). It may be noted that in the quadrant \( Q_2 \), the strength of ejection increases with increase in vertical distance from the near-bed surface, and the events in the other quadrants are little bit irregular.

On the other hand, both the interactions- outward and inward are almost identical for all heights (Fig. 5.14a). At the near-bed level \( z/h = 0.018 \) for \( H = 8 \), the contributions from the quadrants are \( S_{1,8} = -0.340, S_{2,8} = 0.557, S_{3,8} = -0.345, S_{4,8} = 0.874 \), the sum of them is 0.750, which is 75\% of average shear stress \( \tau_{xz} \); and at the level \( z/h = 0.158 \) for \( H = 8 \), the contributions from the quadrants are \( S_{1,8} = -0.169, S_{2,8} = 0.654, S_{3,8} = -0.140, S_{4,8} = 0.262 \), the sum is 0.607, which is 61\% of average shear stress \( \tau_{xz} \).

At location \( D \) (close to the cylinder) in the scour region due to \( D_c = 3.2 \, cm \), sweep event \( (Q_4) \) is greater than the ejection \( (Q_2) \) from the bottom to the level surface \( (z/h < 0.018) \), but above the level \( (z/h > 0.018) \), \( Q_2 > Q_4 \), i.e., ejection event is greater than sweep in the outer region of the main flow (Fig. 5.14b). At location \( D \) within scour region, the contribution to \( |S_{i,H}| \) for a fixed threshold parameter \( H \) from all four quadrants increases with increase of vertical distance, whereas at location \( A \) on the level surface, the similar phenomenon occurs only for ejection event \( (Q_2) \), but magnitudes of stress fraction in other three quadrants first decrease and then increase with \( z/h \). For example, at the level \( z/h = -0.08 \) for \( H = 8 \), the contributions from
the quadrants are $S_{1,8} = -0.000, S_{2,8} = 0.087, S_{3,8} = -0.070, S_{4,8} = 0.272$, the sum of which is 0.289, i.e., 29% of average shear stress $\tau_{xz}$. At the level $z/h = 0.018$ for $H = 8$, the contributions from the quadrants are $S_{1,8} = -0.699, S_{2,8} = 1.234, S_{3,8} = -1.193, S_{4,8} = 1.576$, the sum of which is 0.918, i.e., 92% of average shear stress $\tau_{xz}$. At the level $z/h = 0.269$ for $H = 8$, the contributions from the quadrants are $S_{1,8} = -1.843, S_{2,8} = 2.801, S_{3,8} = -1.670, S_{4,8} = 1.621$, the sum of which is 0.909, i.e., 91% of average shear stress $\tau_{xz}$.

At the interface of sweep and ejection events, kolk-boils phenomenon occurs (Ojha and Mazumder, 2008), which is observed to appear at the lee side of the waveform structures in the river environments. Upward tilting stream-wise vortex in the river flows is known as kolk; and the circular or oval shape structures known as boils move-up to the water surface and merges with the surroundings (Nezu and Nakagawa, 1993). The geometry of the scour mark starting from the edge looks like a lee side of dunes. The flow separates at the ridge of the scour hole resulting the formation of vortices, hence the occurrence of kolk-boils phenomenon at the scour marks (Venditti and Bauer, 2005).

At location $H$ on the deposited material behind the cylinder, ejection event ($Q_2$) is much greater than the sweep event ($Q_4$) away from the boundary for heights $z/h = 0.158, 0.269$, while near the boundary at $z/h = 0.041$, both ejection and sweep events are almost negligible. The outward interaction ($Q_1$) is much greater than the inward interaction ($Q_3$) near the boundary at $z/h = 0.041$, while away from the boundary at heights $z/h = 0.158, 0.269$, both outward and inward interactions are almost identical with negligible value at higher threshold parameter $H$. For example, at the level $z/h = 0.075$ for threshold parameter $H = 8$, the contributions from the quadrants are $S_{1,8} = -0.019, S_{2,8} = 0.0755, S_{3,8} = -0.008, S_{4,8} = 0.111$, the sum of which is 0.16, i.e., 16% of average shear stress $\tau_{xz}$. At the level $z/h = 0.269$ for $H = 8$, the contributions
from the quadrants are $S_{1,8} = -0.154, S_{2,8} = 0.722, S_{3,8} = -0.123, S_{4,8} = 0.201$, the sum of which is 0.65, i.e., 65% of average shear stress $\tau_{xz}$. Moreover, sum of contributions of all quadrant events increases with increase in height $z/h > 0.018$ for threshold parameter $H > 0$.

It is interesting to note that for a fixed $D_c = 3.2cm$ and threshold parameter $H = 0$, contributions to the $|S_{i,0}|$ $i = 1, 2, 3, 4$ from all four quadrant events at the near-bed region ($z/h = 0.018$) are larger on the level surface at $A$ than the near-bed level ($z/h = -0.08$) within the scour region at $D$. On the other hand, contributions from all four quadrants to the stress fraction at the outer layer $z/h > 0.018$ on scour hole at $D$ are larger than the outer layer on the level surface at $A$ (Fig. 5.14a,b). The results were consistent with the results of Krogstad et al. (1992), who indicated that in the outer layer over the rough surface, contributions from second ($Q_2$) and fourth ($Q_4$) quadrants to the Reynolds shear stress are greater than on the smooth surface, considering the reduction of mean flow velocity within the scour hole as analogous with the velocity over the rough surface. The overall trend of these plots shows that the contributions of ejections and sweeps to the shear stress at all concerned locations ($A$, $D$ and $H$) away from the wall are much higher than that of inward and outward interactions. It is observed from the Fig. 5.14b that the typical characteristic of all quadrant events at the levels $z/h = 0.018$, 0.269 has much longer tail with the threshold parameter $H$ compared with the all quadrants events at near-bed level ($z/h = -0.08$). It also seems that the contributions of ejections and sweeps to the shear stress at $D$ away from the wall are much higher than that of other locations ($A$ and $H$), which means that the sweep and ejection events provide extraction of energy from the circulation to generate more turbulence at $D$. Furthermore, the tails of all quadrant events in $D$ appear longer with the parameter $H$ than in the case of the locations $A$ and $H$.

The contributions of $|S_{i,H}|$ to the Reynolds shear stress at the level surface $z/h = 0.018$ for
different cylinder diameters ($D_c = 3.2, 4.2, 6.0 \text{cm}$) are plotted against the threshold parameter $H$ for each of the four quadrants in Fig. 5.15(a) for location $A$, and in Fig. 5.15(b) for location $D$ in the scour region. At both the locations $A$ and $D$ theoretical and experimental values are fairly well except for $D_c = 6.0 \text{cm}$ at location $D$. At $A$ on the smooth surface level in the near-wall region ($z/h = 0.018$), sweep event of the $Q_4$ dominates over the ejection in $Q_2$ for all the cylinder diameters ($D_c = 3.2, 4.2, 6.0 \text{cm}$), whereas $Q_3 > Q_1$ for $D_c = 6.0 \text{cm}$ and $Q_3 \approx Q_1$ for the cylinder diameters $D_c = 3.2, 4.2 \text{cm}$. On the other hand, at $D$ within the scour hole at surface level ($z/h = 0.018$), the sweep event is greater than the ejection event for the diameters $D_c = 3.2, 4.2 \text{cm}$, and it is reversed for the larger diameter $D_c = 6.0 \text{cm}$. On the other hand, inward interaction ($Q_3$) is greater than outward interaction ($Q_1$) for the threshold parameter $H$ and for all $D_c$. At location $D$ within scour region, it is interesting to note that the both sweep and inward interactions to $|S_{i,H}|$ for fixed $H$ decrease with increase of cylinder diameter $D_c$ at the level $z/h = 0.018$, whereas at $A$ on the smooth surface level, all the events to stress fraction $|S_{i,H}|$ decrease for $D_c = 3.2, 4.2 \text{cm}$ except for $D_c = 6.0$. Moreover, it is interesting to note that at the location $D$ (close to the cylinder) at the surface level $z/h = 0.018$, sum of contributions of all quadrant events decreases with increase of cylinder diameter $D_c$ for the threshold parameter $H > 0$, whereas the mean shear stress $\tau_{xz}$ at that level increases with cylinder diameter $D_c$.

From the observations it is revealed that at the near-bed (surface level) $z/h = 0.018$ and threshold parameter $H = 0$, contributions of all four quadrant events to $|S_{i,0}| i = 1, 2, 3, 4$ for $D_c = 3.2 \text{cm}$ are larger within the scour region for $D_c = 3.2 \text{cm}$ at $D$ than that on the locations $A$ and $H$; whereas contributions of all quadrant events for $D_c = 4.2, 6.0 \text{cm}$ at the same level $z/h = 0.018$ on scour hole at $D$ are less than that on the smooth surface at $A$ (Fig. 5.15a, b). The wider scour holes generated by larger cylinder diameters $D_c = 4.2, 6.0 \text{cm}$ may act as

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diffusive agent. It is observed that the interaction components (outward and inward) were each smaller in magnitude than either ejection or sweep contributions at all levels, which implies that the downward flux of momentum is much more significant than the outward flux of momentum (Figs. 5.14-5.15).

Figure 5.14: Plots of stress fractions $S_{i,H}$ against hole size $H$ for $D_c = 3.2$cm. (a) At the location $A$: ◦ ($\cdot -$), $z/h = 0.018$; + ($\cdot$), $z/h = 0.158$; ✢ ($\cdot$), $z/h = 0.269$, (b) at $D$ close to the cylinder: ◦ ($\cdot$), $z/h = -0.080$; + ($\cdot$), $z/h = 0.018$; ✢ ($\cdot$), $z/h = 0.269$ and (c) at $H$: ◦ ($\cdot$), $z/h = 0.041$; + ($\cdot$), $z/h = 0.158$; ✢ ($\cdot$), $z/h = 0.269$. Here dash-dot, dash and solid lines represent the theoretical value.
5.2 Verification of theoretical model to the Reynolds shear stresses in \( xy \), \( xz \) and \( yz \)-planes

In order to apply the cumulant discard method, we define the following variables: \( u' \), \( v' \) and \( w' \) are the zero mean fluctuating streamwise, lateral and vertical velocity components. Let us assume variables \( \hat{u} \), \( \hat{v} \) and \( \hat{w} \) are equal to \( u'/\sqrt{u'^{2}}, v'/\sqrt{v'^{2}} \) and \( w'/\sqrt{w'^{2}} \). We shall quantify the contributions from the different planes of the relative shear stress for the covariance terms \( \tau_1 = \frac{u'v'}{u'^{2}}, \tau_2 = \frac{u'w'}{u'^{2}} \) and \( \tau_3 = \frac{v'w'}{v'^{2}} \). Three characteristic functions \( \chi_1(\alpha, \beta), \chi_2(\alpha, \gamma) \) and \( \chi_3(\beta, \gamma) \), expressed as the Fourier transform of the joint probability density functions \( p_1(\hat{u}, \hat{v}) \), \( p_1(\hat{u}, \hat{w}) \) and \( p_1(\hat{v}, \hat{w}) \), respectively, can be expressed as function of the moment and cumulant generating functions in which \( m_{1, st} = \bar{u}^{s}\bar{v}^{t}, m_{2, st} = \bar{u}^{s}\bar{w}^{t} \) and \( m_{3, st} = \bar{v}^{s}\bar{w}^{t} \) denote the moments of \((s + t)^{th}\) order and \( q_{1, st}, q_{2, st} \) and \( q_{3, st} \) correspond to the cumulants of \((s + t)^{th}\) order. Nakagawa and Nezu (1977) expressed the conditionally sampled probability density over
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the four quadrant of covariance events $\tau_1$. The following equations are given:

\[
\begin{align*}
 p_{j,2}(\tau_j) &= p_{jG}(\tau_j) + \psi_j^-(\tau_j) \quad (\tau_j > 0) \\
 p_{j,4}(\tau_j) &= p_{jG}(\tau_j) - \psi_j^- (\tau_j) \quad (\tau_j > 0) \\
 p_{j,1}(\tau_j) &= p_{jG}(\tau_j) + \psi_j^+(\tau_j) \quad (\tau_j < 0) \\
 p_{j,3}(\tau_j) &= p_{jG}(\tau_j) - \psi_j^+(\tau_j) \quad (\tau_j < 0)
\end{align*}
\]

(5.37)

where the index $i$ in $p_{j,i}$ denotes the quadrant index in the $j^{th}$ plane with planes 1, 2 and 3 corresponding to the $xy$, $xz$ and $yz$-planes respectively. The probability density $p_{jG}(\tau_j)$ is directly developed from the corresponding bivariate normal distribution where

\[
\begin{align*}
 p_{jG}(\tau_j) &= \frac{R_j}{2\pi} e^{R_j\xi_j} \frac{K_0(|\xi_j|)}{\sqrt{1-R_j^2}} \\
 \psi_j^+(\tau_j) &= \frac{R_j}{2\pi} e^{R_j\xi_j} K_\frac{1}{2}(\xi_j)|\xi_j| \left\{ (1+R_j)(\frac{S_j^++D_j^+}{S_j^+})|\xi_j| - \left(\frac{2-R_j}{S_j^+}ight)S_j^+ + D_j^+ \right\} \\
 \psi_j^- (\tau_j) &= \frac{R_j}{2\pi} e^{R_j\xi_j} K_\frac{1}{2}(\xi_j)|\xi_j| \left\{ (1-R_j)(\frac{S_j^-+D_j^-}{S_j^-})\xi_j - \left(\frac{2+R_j}{S_j^-}\right)S_j^- + D_j^- \right\} \\
 \xi_j &= \frac{R_j\tau_j}{(1-R_j^2)}; \quad S_j^\pm = \frac{q_{j,03}+q_{j,30}}{2}; \quad D_j^\pm = \frac{q_{j,21}+q_{j,12}}{2}
\end{align*}
\]

(5.38)

where $R_j$ and $K_0(\xi)$ are the corresponding correlation coefficient and zeroth-order modified Bessel function of second kind. The nonconditionally sampled probability function of shear stress is:

\[
p_j(\tau_j) = p_{j,1}(\tau_j) + p_{j,2}(\tau_j) + p_{j,3}(\tau_j) + p_{j,4}(\tau_j)
\]

(5.39)

The conditional sampled probability function of shear stress is

\[
p_j(\tau_j) = 2p_{jG}(\tau_j)
\]

(5.40)

Then the contribution to the Reynolds stress $S_{i,H}$ corresponding to each quadrant and plane represented by

\[
\begin{align*}
 S_{i,H\mid j} &= \int_{H}^{\infty} \tau_j p_{j,i}(\tau_j) \, d\tau_j \quad (i = 2, 4) \\
 S_{i,H\mid j} &= \int_{-\infty}^{-H} \tau_j p_{j,i}(\tau_j) \, d\tau_j \quad (i = 1, 3) \quad j = 1, 2 \text{ and } 3
\end{align*}
\]

(5.41)
Note that: \( j = 1, j = 2 \) and \( j = 3 \) correspond to \( xy, xz \) and \( yz \)-planes

### 5.2.1 Probability density functions \( p_j(\tau_j)(j = 1, 2, 3) \) of normalized Reynolds shear stresses

The measured and computed values of \( p_j(\tau_j)(j = 1, 2, 3) \) of normalized Reynolds shear stresses \( \tau_j(j = 1, 2, 3) \) are plotted in Fig. 5.16(a,b,c) for \( D_c = 4.2cm \) at the locations \( A \) (smooth level surface), \( D \) (close to the cylinder) and \( H \). It is noted from the figures that the probability density of \( (\tau_j) \) calculated from Eq. 5.40 is fairly well with experimental data at all the locations \( A, D \) and \( H \) for height \( z/h = 0.192 \). The following relations of distribution profiles \( p_j(\tau_j)(j = 1, 2, 3) \) are as follows: at location \( A \) on the smooth level surface;

\[
p(\tau_2) > p(\tau_1) > p(\tau_3) \quad \text{(Fig. 5.16a)}
\]

at location \( D \); \( p(\tau_1) \approx p(\tau_3) > p(\tau_2) \) (Fig. 5.16b) and \( H \) satisfies \( p(\tau_2) > p(\tau_1) > p(\tau_3) \) (Fig. 5.16c). It may be noted that

\[
(\tau_j)(j = 1, 2, 3) = 0 \text{ have singularity at origin.}
\]
Figure 5.16: Plots of $p_j(\tau_j); j = 1, 2, 3$ against $\tau_j; j = 1, 2, 3$ at height $z/h = 0.192$ for $D_c = 4.2$ cm. (a) At the location $A$: ◦ (· −), $xy$-plane; + (- -), $xz$-plane; ▽ (-), $yz$-plane, (b) at $D$ close to the cylinder: ◦ (· −), $xy$-plane; + (- -), $xz$-plane; ▽ (-), $yz$-plane and (c) at $H$: ◦ (· −), $xy$-plane; + (- -), $xz$-plane; ▽ (-), $yz$-plane. Here dash-dot, dash and solid lines represent the theoretical value.

5.2.2 Conditional statistics of Reynolds shear stresses

The measured and computed values of stress fractions of different planes shown in Fig. 5.17 (a, b, c). It shows that theoretical models well fitted in three different planes.
Figure 5.17: Plots of stress fractions $S_{i,H}$ against hole size $H$ at height $z/h = 0.192$ for $D_c = 4.2$ cm. (a) At the location $A$: ○ (- -), $xy$-plane; + (- -), $xz$-plane; ≥ (-), $yz$-plane. (b) At $D$ close to the cylinder: ○ (- -), $xy$-plane; + (- -), $xz$-plane; ≥ (-), $yz$-plane and (c) At $H$: ○ (- -), $xy$-plane; + (- -), $xz$-plane; ≥ (-), $yz$-plane. Here dash-dot, dash and solid lines represent the theoretical value.