Chapter 6

BRANS-DICKE WORMHOLE REVISITED

In this chapter, we shall derive a basic constraint to be satisfied by Brans class I solution to be a traversable wormhole. It will be argued that the wormhole could be traversible only \textit{in principle}, but not in practice. Using a recently proposed measure of total gravitational energy inside a static wormhole configuration, it is shown that the wormhole contains repulsive gravity required for the defocussing of orbits at the throat.

6.1 Introduction

Lorentzian wormholes are objects of research in the frontier of theoretical physics. Einstein’s gravitational field equations predict not only black holes, but also wormholes and neither of these has yet been ruled out by observations. Wormhole physics has lately gained a renewed impetus following the pioneering work by Morris, Thorne and Yurtsever [1] in 1988, although wormhole solutions were conceived as particle models by Einstein himself (Einstein-Rosen bridge [2]) in 1935. Wheeler’s geometrodynamics [3] tells us that wormholes are like topological handles connecting two distant regions of spacetime (like Klein bottle) or even two distinct universes.\footnote{The contents of this chapter are based on the paper: "Brans- Dicke wormhole revisited" ; A. Bhattacharya, I. Nigmatzyanov, R. Izmailov and K.K.Nandi ; Class. Quant. Grav. 26, 235017 (2009).} In
Ref[1], a general theoretical framework for these handles have been formulated and conceived as possible tunnels facilitating rapid interstellar travel. Over the years, however, newer possibilities have emerged. For instance, these objects offer an intriguing possibility as to whether they might act as effective gravitational lenses in astrophysical scenarios. This has been conjectured by Cramer et al. [4], who recommended an analysis of massive compact halo objects (MACHOs) search data for the detection of such lens effects. Such data also offer a distinct possibility which would allow us in future to distinguish between the strong field lensing effects caused by macroscopic wormholes and black holes [5-8]. Could it be that the early universe was populated by microsopic semiclassical wormholes? Though the answer is not definitively known, an analysis of the consequences can be found in Ref.[9]. Appearance of microscopic wormholes were speculated even in the popular articles on recent high energy LHC experiment. Kardashev, Novikov and Shatskiy [10] suggest that black holes could be past wormhole entrances. Undeniably, all these effects are still at a very speculative level, yet they are quite promising ones. The unique treatise by Visser [11] provides a detailed account of classical and quantum gravity wormholes.

An arena for the natural occurrence of classical Lorentzian wormholes is the Brans-Dicke (BD) theory which involves a scalar field. It might be recalled here that, in general, scalar fields have come to stay with us not only because of the Machian nature of BD theory, but also because the string theory, higher dimensional theories or \( f(R) \) gravity theories invariably predict their own modulo scalar fields. In the BD theory, a search for static wormholes has been initiated by Agnese and La Camera [12] who have shown that the BD scalar can play the role of exotic matter provided the coupling parameter \( \omega < -2 \). This work has been followed by investigations in other classes of Brans solutions [13] as well as in the Einstein conformally rescaled BD theory [14]. Several related works also exist in the literature, see for instance [15-23] – to mention only a few. Considering the importance of BD theory in the interpretation of various astrophysical phenomena, it is important that a thorough analysis of static spherically symmetric wormhole solutions be undertaken in this theory. This provides the motivation for the present paper.

The purpose of the present chapter is to report the following investigations: We derive the
basic constraint leading to the ranges for \( \omega \) for which traversable wormhole in the BD theory could be possible. One of the two ranges turn out to be different from that derived by Agnese and La Camera [12]. We discuss the violation of energy conditions on the basis of these ranges. A particularly interesting result is that the Horowitz-Ross naked black hole, defined later, has a wormhole analogue in the Brans class I solution. The impact of the ranges of \( \omega \) on traversibility is explicitly demonstrated. A recent definition of gravitational energy is also implemented in the BD wormhole to show that it contains repulsive gravity, as required.

The chapter is organized as follows: In Sec.6.2, we develop the usual geometry of Morris–Thorne wormholes for the Brans I solution. In Sec.6.3, we derive ranges of \( \omega \) and discuss the violation of energy conditions. In Sec.6.4, we discuss traversibility and in Sec.6.5, we compute the gravitational energy present in the wormhole. Sec.6.6 presents the conclusions and some relevant remarks.

### 6.2 Brans I wormhole solution

The matter-free action in the Jordan frame is

\[
S = \frac{1}{2} \int d^4x \left( -g^{\frac{1}{2}} \right)^2 \left[ \phi R - \phi^{-1} \omega(\phi) g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} \right].
\]  

(6.2.1)

The field equations are

\[
\Box^2 \phi = 0
\]  

(6.2.2)

\[
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\omega \left[ \phi_{,\mu} \phi_{,\nu} - \frac{1}{2} g_{\mu\nu} \phi_{,\sigma} \phi^{,\sigma} \right] - \frac{1}{\phi} \left[ \phi_{,\mu} \phi_{,\nu} - g_{\mu\nu} \Box^2 \phi \right].
\]  

(6.2.3)

Where \( \omega \) is a constant dimensionless coupling parameter, \( \Box^2 \phi \equiv \left( \phi^{,\rho} \right)_{,\rho} \) and \( \phi \) is the Brans–Dicke scalar. The general solution in isotropic coordinates \((t, r, \theta, \phi)\) is given by

\[
d\xi^2 = -e^{2\alpha(\ell)} dt^2 + e^{2\beta(\ell)} dr^2 + e^{2\gamma(\ell)} r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right).
\]  

(6.2.4)
Brans derived four classes of solutions (I-IV), but the most widely used one is the class I solution [24]. It corresponds to a gauge $\beta - \nu = 0$ and is given by

$$e^{\alpha(r)} = e^{\alpha_0} \left[ \frac{1 - B/r}{1 + B/r} \right]^\frac{1}{\lambda},$$

(6.2.5)

$$e^{\beta(r)} = e^{\beta_0} \left[ 1 + B/r \right]^2 \left[ \frac{1 - B/r}{1 + B/r} \right]^\frac{\lambda - C - 1}{\lambda},$$

(6.2.6)

$$\varphi(r) = \varphi_0 \left[ \frac{1 - B/r}{1 + B/r} \right]^{\frac{\Omega}{\lambda}},$$

(6.2.7)

$$\lambda^2 = (C + 1)^2 - C \left( 1 - \frac{\omega C}{2} \right) > 0,$$

(6.2.8)

where $\alpha_0$, $\beta_0$, $B$, $C$, and $\varphi_0$ are constants. The constants $\alpha_0$ and $\beta_0$ are determined by asymptotic flatness condition as $\alpha_0 = \beta_0 = 0$. As it is, the above class of solutions represent a naked singularity at $r = B$ as the curvature invariants diverge there. When expressed as a wormhole, the throat has to be larger than the naked singularity radius so that a traveler might avoid encountering it.

In order to investigate the possibility if the above solution represents wormholes, it is convenient to cast the spacetime geometry in the Morris-Thorne canonical coordinates

$$d\tau^2 = -e^{2\Phi(R)} dt^2 + \left[ 1 - \frac{b(R)}{r} \right]^{-1} dR^2 + R^2 (d\theta^2 + \sin^2 \theta d\psi^2)$$

(6.2.9)

where $\Phi(R)$ and $b(R)$ are redshift and shape functions respectively. These functions are required to satisfy some constraints, enumerated in [1], in order that they represent a wormhole. It is, however, important to stress that the choice of coordinates is purely a matter of convenience and not a physical necessity. For instance, one could equally well work directly with isotropic coordinates or proper quantities using the expressions in Ref.[11], but the final conclusions would be the same.

Redefining the radial coordinate $r \to R$ in the metric (6.2.4)-(6.2.8) as

$$R = r e^{\beta_0} \left[ 1 + B/r \right]^2 \left[ \frac{1 - B/r}{1 + B/r} \right]^\frac{1}{\lambda}, \quad \Omega = 1 - \frac{C + 1}{\lambda},$$

(6.2.10)
we obtain the following functions for \( \Phi(R) \) and \( b(R) \):

\[
\Phi(R) = \alpha_0 + \frac{1}{\lambda} \left[ \ln \left( 1 - \frac{B}{r(R)} \right) - \ln \left( 1 + \frac{B}{r(R)} \right) \right], \tag{6.2.11}
\]

\[
b(R) = R \left[ 1 - \left\{ \frac{\lambda r^2(R) + B^2}{\lambda r^2(R) - B^2} \right\} \right]^2. \tag{6.2.12}
\]

The throat of the wormhole occurs at \( R = R_0 \) such that \( b(R_0) = R_0 \). This gives the minimum allowed \( r \)-coordinate radii \( r_0^\pm \) as

\[
r_0^\pm = \alpha^\pm B, \tag{6.2.13}
\]

\[
\alpha^\pm = (1 - \Omega) \pm \sqrt{\Omega(\Omega - 2)}. \tag{6.2.14}
\]

The values \( R_0^\pm \) can be obtained from Eq.(6.2.13) using this \( r_0^\pm \). Noting that \( R \to \infty \) as \( r \to \infty \), we find that \( b(R)/R \to 0 \) as \( R \to \infty \). Also, \( b(R)/R \leq 1 \) for all \( R \geq R_0^\pm \). The redshift function \( \Phi(R) \) has a singularity at \( r = r_S = B \). Therefore, in order that the wormhole be just geometrically traversible, the minimum allowed values \( r_0^\pm \) must exceed \( r_S = B \). We shall see below under what values of the coupling constant \( \omega \) it is possible.

### 6.3 Violation of energy conditions

The expressions for the stress components in the orthonormal rest frame of the observer are [1]

\[
\rho = \frac{b'}{R^2}, \tag{6.3.1}
\]

\[
p_{R} = 2 \left( 1 - \frac{b}{R} \right) \frac{\Phi'}{R} - \frac{b}{R^3} \tag{6.3.2}
\]

\[
p_{\theta} = p_{\psi} = \left( 1 - \frac{b}{R} \right) \left[ \Phi'' + \Phi'^2 + \frac{\Phi'}{R} - \frac{b' R - b}{2 R^2} \left( \Phi' + \frac{1}{R} \right) \right], \tag{6.3.3}
\]

where primes denote differentiation with respect to \( R \). Thus the energy density of the wormhole material is given by

\[
\rho(R) = (R^{-2})(db/dR) \tag{6.3.4}
\]

so that

\[
\rho = - \frac{4 B^2 r^4 Z^2 [(C + 1)^2 - \lambda^2]}{\lambda^2 (r^2 - B^2)^4}, \tag{6.3.5}
\]
where
\[
Z \equiv \left( \frac{r - B}{r + B} \right)^{(C+1)/\lambda}.
\] (6.3.6)

To have a wormhole spacetime, we shall allow the source scalar field to violate one or more energy conditions, particularly the Weak Energy Condition (WEC) \( \rho > 0 \) and/or the Null Energy Condition (NEC) \( \rho + p_R \geq 0 \) where \( \rho \) is the matter energy density and \( p_R \) is the radial pressure. (Transverse pressures \( p_\theta, p_\phi \) are not considered as they refer strictly to ordinary matter.) The violation of NEC is a minimal requirement to have defocussing of light trajectories (repulsive gravity) passing across the wormhole throat. Since it is a minimal requirement, a stronger violation (namely, of WEC) allowing defocussing is not ruled out. The necessity of NEC violation in wormholes is provided by the Topological Censorship Theorem [25] and by dynamical circumstances [26].

At \( r = r_S = B \), the density \( \rho \) diverges because it is a surface where all curvature invariants diverge (naked singularity). However, this surface is inaccessible to a traveller as the minimum radius he/she reaches during travel is \( r_0^\pm > r_S \). We see from Eq.(6.3.5) that the WEC is violated at all \( r \) including the throat under the condition
\[
(C + 1)^2 > \lambda^2.
\] (6.3.7)

Next we want to derive the specific dependence of the constraints on the coupling parameter \( \omega \). For this purpose, henceforth we shall consider the weak field value of \( C \), viz.,
\[
C = -\frac{1}{\omega + 2}.
\] (6.3.8)
Putting this in Eq.(6.2.8), we get
\[
\lambda = \pm \sqrt{\frac{2\omega + 3}{2\omega + 4}}.
\] (6.3.9)

(i) Let us consider the + sign before \( \lambda \). In the limiting case, \( C(\omega) \rightarrow 0 \), \( \lambda(\omega) \rightarrow 1 \) as \( \omega \rightarrow \pm \infty \), one simply recovers the Schwarzschild exterior metric in standard coordinates from Eqs.(6.2.9),(6.2.11) and (6.2.12), so that \( b(R) = 2M \) and \( \rho = 0 \) (Schwarzschild exterior vacuum). It is clear that \( \lambda^2 > 0 \) for all \( \omega \) except in the range \(-2 < \omega < -\frac{3}{2}\), and hence we exclude it from consideration. We find that the wormhole condition, viz., \((C + 1)^2 > \lambda^2\) is satisfied only in the
range $\omega < -2$. This implies $\gamma = C + 1 = \frac{\omega + 1}{\omega + 2} > 1$, which is exactly the result obtained by Agnese and La Camera [12]. For $\omega < -2$, it can also be checked from Eq.(6.2.13) that, out of the two throat radii $r_0^\pm$, only $r_0^+ > B$, as is required to avoid the singular surface. For $\omega > -2$, which we have already excluded, the throat radii become negative.

Figure 6-1: The simultaneous inequalities $r_0^+ > B$ and $f = (C + 1)^2 - \lambda^2 > 0$ determine the range $\omega < -2$, when $\lambda$ is positive. In all the figure we have set units in which $B = 1$. The functions $r_0^+$ and $f$ are expressed in terms of $\omega$ using Eqs.(6.2.10), (6.2.13) and (6.3.8), (6.3.9).

To analyze the energy conditions at and away from the throat, let us compute the radial pressure, which, using Eqs.(6.2.11), (6.2.12) in (6.3.2), turn out to be

$$p_R = -\frac{4Br^3Z^2}{\lambda^2(r^2 - B^2)^4}[\lambda C(r^2 + B^2) - Br(C^2 - 1 + \lambda^2)].$$  

Adding Eqs.(6.3.5) and (6.3.10), we get

$$\rho + p_R = -\frac{4Br^3Z^2}{\lambda^2(r^2 - B^2)^4}[\lambda C(r^2 + B^2) + 2Br(C + 1 - \lambda^2)].$$  

Let us concentrate on Eq.(6.3.11) and the inequality (6.3.7). We find from Fig.6-1 that $r_0^+ > B$ and $f = (C + 1)^2 - \lambda^2 > 0$ hold simultaneously when $\omega < -2$. Also Fig.6-2 shows that $\rho < 0, \rho + p_R < 0$ for a typical value, say, $\omega = -5$ and for all values of $r \geq r_0^+$. These imply that both WEC and NEC are violated all the way up from the throat. Moreover, $\rho \to 0, \rho + p_R \to 0$ as $r \to \infty$ (which is the same as $R \to \infty$), in accordance with the asymptotic flatness of the
Figure 6-2: **WEC and NEC violation for a range of** \( r \in [r_0^+, 20] \) **and typical value,** \( \omega = -5 \) [case (i)]. See Eqs. (6.3.5) and (6.3.11).

BD solution under consideration.

Figure 6-3: **The simultaneous inequalities** \( r_0^+ > B \) **and** \( f = (C + 1)^2 - \lambda^2 > 0 \) **determine the range** \(-\frac{3}{2} < \omega < -\frac{4}{3}\), **when** \( \lambda \) **is negative.**

(ii) Next let us consider the – sign before \( \lambda \). In the limiting case, \( C(\omega) \to 0, \lambda(\omega) \to -1 \) as \( \omega \to \pm\infty \). This has the effect that we can no longer recover the Schwarzschild solution because \( g_{00} \) and \( g_{11} \) diverge at \( r = B \). However, we shall concentrate only on the finite values of \( \omega \) allowing wormhole spacetime. It turns out that within a small range, \(-\frac{3}{2} < \omega < -\frac{4}{3}\), wormholes are possible. From Fig.6-3, it follows that the conditions \( r_0^+ > B \) and \( f = (C + 1)^2 - \lambda^2 > 0 \) are simultaneously satisfied in that range. Also Fig.6-4 shows that \( \rho < 0, \rho + p_R < 0 \) for a typical
value, say $\omega = -1.4$ and for all $r \geq r^+_0$. In other words, all the features of case (i) exist in this case too.

The above analysis indicates that $(C + 1)^2 > \lambda^2$ is a basic constraint to be satisfied by the BD constants, supporting the discussion in Refs.[13,14,27,28]. This concludes our analysis about the allowed values for $\omega$, that is, depending on the sign of $\lambda$, wormholes are possible when $\omega < -2$ as in (i) and $-\frac{3}{2} < \omega < -\frac{4}{3}$ as in (ii).

### 6.4 Traversability

Does the BD solution under consideration represent a humanly traversible wormhole? In general, there are several constraints for human traversibility as discussed in Ref.[1]. One condition for traversibility is that the spacetime be asymptotically flat, which is indeed the case with the solution (6.2.5)-(6.2.8). However, the most important condition is that the tidal forces be finite and tolerable everywhere, especially at the throat.

The differential of the radial tidal acceleration $\Delta a^r$ in the static orthonormal frame $(\hat{e}_t, \hat{e}_\rho, \hat{e}_\varphi, \hat{e}_\theta)$ is given by

$$\Delta a^r = -R_{R'R}^{R} \xi^R,$$

(6.4.1)

where $\xi^R$ is the radial component of the separation vector and the curvature tensor component.
This component is invariant under a Lorentz boost \([1,29]\). For the metric given by Eqs.\((6.2.11)\) and \((6.2.12)\), we find in the freely falling orthonormal frame \((\vec{e}_0, \vec{e}_1, \vec{e}_2, \vec{e}_3)\) with velocity \(v\) the following expression

\[
|\mathbf{R}_{\bar{R}\bar{R}Rt} - \mathbf{R}_{\bar{R}\bar{R}t}| = \left| \begin{array}{c}
(1 - b/R) \left\{ -\Phi'' + \frac{b'R - b}{2R(R - b)} \Phi' - (\Phi')^2 \right\} \right|.
\]

\((6.4.2)\)

A point must be clarified here. In Ref.\([11]\), the components of the Riemann tensor are given in mixed \((1,3)\) form whereas we are using here the purely covariant \((0,4)\) form, following the notation in Refs.\([1,29]\). This does not make any difference; we could as well use the mixed notation. Actually, the expression in Eq.\((6.4.3)\) as well as all other components are exactly the same as the ones given in Ref.\([11]\). This happens because both are calculated in the orthonormal frame in which raising and lowering of indices is done only by the locally flat Minkowski metric \(\eta_{\mu\nu} = (-1, +1, +1, +1)\).

We now put the values of \(C(\omega)\) and \(\lambda(\omega)\) from Eqs \((6.3.8), (6.3.39)\) in the right hand side of Eq.\((6.4.3)\) and write \(|\mathbf{R}_{\bar{1}\bar{0}\bar{1}\bar{0}}| = g(\omega, r, B)\). Let us vary \(r\) away from the throat for fixed values of \(\omega\) within the ranges derived in the previous section. To be suitable for comfortable travel by a human of length 2 mtrs, the tidal acceleration should be roughly of the order of one Earth gravity \(g_\oplus \left( = \frac{GM_\oplus}{r_\oplus^2} \right) \approx 980 \text{ cm/sec}^2\). In relativistic units, it should be \(\frac{g_\oplus}{c^2}\). Since \(\xi^R\) has the dimension \(L\) of length, looking at Eq.\((6.4.1)\), we get that the dimension of \(\mathbf{R}_{\bar{R}\bar{R}\bar{R}t}\) must be \(\frac{g_\oplus}{c^2} \sim L^{-2}\). Therefore, the right hand side Eq.\((6.4.3)\) should be less than \(\lesssim \frac{g_\oplus}{c^2x_{\text{2mtrs}}} \sim 10^{-20}\text{cm}^{-2}\) \([1]\), which requires that the magnitude of the curvature components should be very close to zero in the orthonormal frame of the traveller. Such a condition is easily provided by \(\Phi = 0\), which is not the case here. To have an idea of the magnitudes involved, we focus on a typical value in case (i), say, \(\omega = -5\) and observe the following:

The plot of \(g(r)\) vs \(r\) shown in Fig.6-5 in units \(B = 1\) reveals a remarkable feature of Brans class I wormhole. At the throat location, \(r_0^+ = 1.958\), we see that \(g(r_0^+)\) is of the order of \(10^{-3}\text{cm}^{-2}\), which increases up to \(g(r) = 10^{-2}\text{cm}^{-2}\) at a location \(r = 2.313\) away from the throat! Thereafter, \(g(r)\) decays rapidly. That is, an infalling observer meets the maximum
radial tidal force not at the throat but above it. This phenomenon is very much analogous to the idea of naked black holes first discussed by Horowitz and Ross [29]. They defined the naked black hole as a spacetime in which an infalling observer meets the maximum tidal force not at the horizon but above it. In a freely falling frame, the curvature components could be larger than those at the horizon. Since the region of large tidal forces is visible to distant observers, Horowitz and Ross called such objects “naked black holes.” In our case, the role of horizon is played by the throat of the analogue wormhole. However, no such phenomenon occurs for the range in case (ii) – there is a steady decrease in curvature right from the throat up, as can be seen from Fig.6-6.

As we read off from Fig.6.5, the maximum tidal force unfortunately is still $10^{18}$ times more than that on Earth’s gravity field. Hence the wormhole is not humanly traversible. Since human travel is out of question, we might try to calculate the speed $v$ of an inanimate test particle passing through the wormhole. The lateral tidal forces in the Lorentz boosted frame of the particle are [29]

\[
\begin{align*}
\mathbf{R}_{\tilde{\varphi}^2} &= \mathbf{R}_{\tilde{\varphi}^2} + \left( \frac{v^2}{1 - v^2} \right) \left( \mathbf{R}_{\tilde{\varphi}^2} + \mathbf{R}_{\tilde{\varphi}^2} \right) \\
\mathbf{R}_{\tilde{\varphi}^3} &= \mathbf{R}_{\tilde{\varphi}^3} + \left( \frac{v^2}{1 - v^2} \right) \left( \mathbf{R}_{\tilde{\varphi}^3} + \mathbf{R}_{\tilde{\varphi}^3} \right)
\end{align*}
\]  

(6.4.4)
Since, by spherical symmetry, \( R_{\theta \theta t} = R_{\varphi \varphi t} \) and \( R_{\varphi R \theta R} = R_{\varphi \varphi R^2 R} \), we get, plugging in the relevant expressions [1]

\[
\left| R_{\bar{\varphi} \bar{\theta} \bar{R} \bar{R}} \right| = \left| R_{\bar{\theta} \bar{\theta} \bar{R} \bar{R}} \right| = \left| \frac{1}{2R^2(1-v^2)} \left[ v^2 \left( b' - \frac{b}{R} \right) + 2(R - b)\Phi' \right] \right| \\
= \left( \frac{2Br^3Z^2}{1-v^2} \right) \times \frac{[2Br(C+1) + \lambda(r^2 + B^2)(Cv^2 + v^2 - 1) - 2Br\lambda^2v^2]}{\lambda^2(r^2 - B^2)^4} \quad (6.4.5)
\]

Assuming the lateral tidal force to be of the same order of magnitude as that of the maximum radial tidal force, we get

\[
\left| R_{\bar{\varphi} \bar{\theta} \bar{R} \bar{R}} \right| = \left| R_{\bar{\theta} \bar{\theta} \bar{R} \bar{R}} \right| \lesssim 10^{-2} cm^{-2}, \quad (6.4.6)
\]

which leads to a particle velocity \( v \sim 0.66c \). This means that a freely falling test particle accelerates from near zero velocity at one mouth to as high as \( 0.66c \) at the radius \( r = 2.313 \) and then decelerates to velocity \( 0.18c \) at the throat \( r_0^+ = 1.958 \) before emerging into the other mouth. Of course, the above assumption is not mandatory; the velocities will be less and less as we go on decreasing the value on the right hand side of (6.4.5). Overall, our conclusion is that the wormhole is traversible only “in principle” since the tidal forces are finite and decay to zero at infinity, but not traversible in practice by humans.
6.5 Total gravitational energy

The total gravitational energy in localized sources having static spherical symmetry and satisfying energy conditions is negative (attractive gravity). A natural query is how the gravitational energy behaves under circumstances where energy conditions are violated. To answer this, the known expression for the gravitational energy has to be suitably adapted to account for situations like the ones occurring in wormhole spacetimes.

All wormhole solutions require exotic material for their construction. However, to our knowledge the gravitational energy content in the interior of exotic matter distribution has not yet been studied. An intuitive approach in this direction is provided by the formulation of gravitational energy by Lynden-Bell, Katz and Bičák [30]. The energy formulation there is intended for isolating and calculating the total attractive gravitational energy $E_G$ of stationary gravity fields. In our view, their formulation did not require any compelling restriction on the energy conditions of the source matter.

The total gravitational energy $E_G$ appropriate for ordinary matter in a normal star is given in [30] as

$$E_G = Mc^2 - E_M = \frac{1}{2} \int_0^r [1 - (g_{rr})^{1/2}] \rho r^2 dr$$

(6.5.1)

where the total mass-energy within the standard coordinate radius $r$ is provided by Einstein’s equations as

$$Mc^2 = \frac{1}{2} \int_0^r \rho r^2 dr$$

(6.5.2)

and the sum of other forms of energy like rest energy, kinetic energy, internal energy etc is defined by

$$E_M = \frac{1}{2} \int_0^r (g_{rr})^{1/2} \rho r^2 dr.$$  

(6.5.3)

The factor $\frac{1}{2}$ comes from $\frac{4\pi}{8\pi}$. Note that $E_G < 0$ for ordinary matter configuration that has attractive gravity.

We shall adapt $E_G$ to wormhole geometry and distinguish it by $\bar{E}_G$. By construction, the wormhole geometry has a hole instead of a center and so we shall change the lower limit of integration in Eq.(6.5.2) to the minimum allowed radius or throat $R_0$ defined by $b(R_0) = R_0$. 
The radius $R$ has the significance that it is the embedding space radial coordinate; it decreases from $+\infty$ to $R = R_0$ in the lower side and again increases to $+\infty$ in the upper side. This requires us to change the integrals (6.5.2) and (6.5.3) to

$$Mc^2 = \frac{1}{2} \int_{R_0}^{R} \rho R^2 dR + \frac{R_0}{2} \quad (6.5.4)$$

$$E_M = \frac{1}{2} \int_{R_0}^{R} (g_{RR})^{\frac{1}{2}} R^2 dR, \quad (6.5.5)$$

the entire spacetime geometry being assumed to be free of singularities. The constant $\frac{R_0}{2}$ in Eq.(6.5.4) comes from the integration of Einstein’s equation $\frac{\partial M}{\partial R} = \frac{1}{2} \rho R^2$ and we shall choose it so as to offset the inner boundary term $\frac{b(R_0)}{2}$ coming from the integration. When $\rho = 0$, we should fix $R_0 = 0$ in order to recover $M = 0$. In geometries with a regular center, one has $R_0 = 0$, the above then reproduces Eqs.(6.5.2) and (6.5.3) respectively.

For wormholes, however, $R_0 \neq 0$. The difference between the above integrals, viz.,

$$\bar{E}_G = Mc^2 - E_M = \frac{1}{2} \int_{R_0}^{R} [1 - (g_{RR})^{\frac{1}{2}}] \rho R^2 dR + \frac{R_0}{2} \quad (6.5.6)$$

is what we define here as the total gravitational energy of wormholes within the region of integration [31]. Clearly, it is a straightforward adaptation of Eq.(6.5.1) to wormhole geometry. However, one immediately notices that due to the presence of the nonzero last term, the sign of $\rho$ does not necessarily determine the sign of $\bar{E}_G$, as would be the case otherwise.

Looking at the MTY form (6.2.9), the Eq.(6.5.6) is rephrased on one side of the wormhole as

$$\bar{E}_G^+ = \frac{1}{2} \int_{r_0^+}^{r_1} [1 - (g_{RR})^{\frac{1}{2}}] \rho R^2 \frac{dR}{dr} dr + \frac{R_0^+}{2} \quad (6.5.7)$$

and on the other side as

$$\bar{E}_G^- = -\frac{1}{2} \int_{r_0^-}^{r_1} [1 - (g_{RR})^{\frac{1}{2}}] \rho R^2 \frac{dR}{dr} dr - \frac{R_0^-}{2} \quad (6.5.8)$$

where $r_1$ is an arbitrarily chosen fixed radius, $R_0^+$ is the throat radius in $R$–coordinate to be computed from Eq.(6.2.10) and

$$g_{RR} = \left[1 - \frac{b(R)}{R}\right]^{-1}. \quad (6.5.9)$$
Signature protection in the metric (6.2.9) requires that $1 - (g_{RR})^{1/2} < 0$, which, together with the information that $\rho < 0$ should yield a positive value of the integral in (38). All the factors in the integrand are now known, but it turns out that an analytic integration is not possible. We therefore do the numerical integration with $B = 1$, $r = r_0^+$, $r_1 = 20$ (say), and using the Eqs.(6.2.10), (6.5.7)-(6.5.9), we find for typical values of $\omega$ from Sec.6.3 that

$$\omega = -5, \ r_0^+ = 1.96, \ R_0^+ = 5.82, \ \tilde{E}_G^- = 3.74 = -\tilde{E}_G^- \quad (6.5.10)$$

$$\omega = -1.4, \ r_0^+ = 2.92, \ R_0^+ = 8.26, \ \tilde{E}_G^- = 5.50 = -\tilde{E}_G^- \quad (6.5.11)$$

which suggest that the total gravitational energy in the spherical shell is positive indicating repulsive gravity. Likewise, $\tilde{E}_G^- < 0$ for either value of $\omega$, implying attractive gravity.

### 6.6 Conclusions

Static spherically symmetric wormholes in the Brans-Dicke theory have been reasonably well discussed in the literature, but a thorough analysis of many features is still lacking. In the present work, our aim was to fill that gap obtaining several key results.

Our analysis fundamentally supports the basic constraint on the BD constants, viz., $(C + 1)^2 > \lambda^2$, obtained earlier by Nandi et al [13,14], re-discussed and refined by Bloomfield [27,28]. The discussion in Sec.6.3 shows that much depends on the sign of $\lambda$. We have found a new range on the coupling parameter, viz., $-3/2 < \omega < -4/3$ for negative $\lambda$, and $\omega < -2$ for positive $\lambda$. The latter range was obtained earlier by Agnese and La Camera [12] using a different gauge. One might ask what would happen if we allow a thorough mix-up, that is, if we interchange signs of $\lambda$ keeping the ranges for $\omega$ uninterchanged, or even getting out of the suggested ranges? We will then see that the throat radii become either negative or complex or fall below the singular radius $B$. All these have to be disregarded as being unphysical. Next, we have shown that both WEC and NEC are violated for the ranges of $\omega$ specified above.

The next result is about traversability. A remarkable aspect, heretofore unnoticed, of the Brans class I solution is that it is the wormhole analogue of the Horowitz-Ross naked black hole: An infalling observer meets the maximum radial tidal force not at the throat but above it. It is then quantitatively shown that Brans-Dicke wormhole can not be humanly traversible because
of the occurrence of tidal force $10^{18}$ times stronger than that on Earth’s surface. However, it is traversible “in principle”: An inanimate test particle can traverse through it with a velocity profile as derived for the case $\omega = -5$ in Sec.6.4.

We have shown in Sec.6.5 that the gravitational energy content in an arbitrary spherical shell around the wormhole throat is positive implying repulsive gravity. The other side has attractive gravity. Hence the wormhole is like a Janus faced object that sucks in matter at one mouth and spews out at the other. All the above results might have important implications in astrophysics.

Finally, we wish to make some remarks clarifying certain issues:

(1) We should make it clear that the Brans-Dicke class I solution (6.2.5)-(6.2.8) represents a naked singularity at $r = B$ since curvature invariants diverge there. Nonetheless, precisely this class of solution has also been PPN expanded to interpret the solar system tests. Exactly these tests follow also from the PPN expansion of the Brans-Dicke \textit{field equations per se}. Alternatively, the class of solutions can be interpreted to describe a special category of black holes, asymptotically flat only if $\omega < -\frac{3}{2}$, and with a horizon of infinite surface area giving rise to the so called cold black holes [32]. It is also possible to interpret the class of solutions as pure wormhole solutions as long as the throat radius $r_0^+ > B$, so that effectively a traveler skirts this forbidden singularity radius. Campanelli & Lousto [33] and Bronnikov [32] investigated the general properties including the structure and stability of black hole and wormhole solutions in the BD theory. We have argued above that wormhole solutions arise even when the condition $\omega < -\frac{3}{2}$ is violated [Sec.6.3, case (ii)]. It would be interesting to examine how the present work relates to those just mentioned.

(2) We recall that the BD theory is Machian such that the scalar field $\varphi$ is more like the zeroth component of gravitational potential than any material field. In addition, it couples non-minimally in the Hilbert-Einstein action. On the other hand, in the conformally rescaled Einstein frame, the redefined scalar couples minimally and provides unambiguous source stresses. So the procedure should be to first derive the metric and corresponding stresses in the Einstein frame, then via certain operations get to those in the Jordan frame. For instance, the Bronnikov-Ellis solution [19] in the minimally coupled theory can be easily transferred into Brans class I solution by redefining the constants together with inverse Dicke transformations,
as already shown in Ref.[22]. Under these operations, the Einstein frame stress components readily convert into those in the BD Jordan frame. In this sense, one talks about the stress components of the BD scalar field, just as was done in Ref.[12]. In fact, the definition of energy density and pressure is frame dependent: these quantities have different behaviours in the so-called Einstein’s frame, with gravity minimally coupled to the scalar field, or in the Jordan’s frame, with gravity non-minimally coupled to the scalar field. Black hole solutions can occur only when \( \omega < -3/2 \) and this implies a negative energy in all space-time when the solution is transposed to the Einstein’s frame. In this frame the energy density of the scalar field is positive when \( \omega > -3/2 \). The behaviour is different when the Jordan’s frame is used. This may explain why negative energy is obtained for the case \( \omega = -1.4 \), as shown in Fig.6.4 [34].

(3) A word of caution: During the discussion in Sec.6.3, we had taken the limit \( \omega \to \pm \infty \) to arrive at the Schwarzschild solution, which is correct. But there is a prevailing belief that general relativity is always recovered from BD theory in limit \( \omega \to \pm \infty \). This is now known not to be correct, see Refs.[35-38].

### 6.7 Figure captions

**Fig.6-1.** The simultaneous inequalities \( r_0^+ > B \) and \( f = (C + 1)^2 - \lambda^2 > 0 \) determine the range \( \omega < -2 \), when \( \lambda \) is positive. In all the figure we have set units in which \( B = 1 \). The functions \( r_0^+ \) and \( f \) are expressed in terms of \( \omega \) using Eqs.(6.2.10), (6.2.13) and (6.3.8), (6.3.9).

**Fig.6-2.** WEC and NEC violation for a range of \( r \in [r_0^+,20] \) and typical value, \( \omega = -5 \) [case (i)]. See Eqs.(6.3.5) and (6.3.11).

**Fig.6-3.** The simultaneous inequalities \( r_0^+ > B \) and \( f = (C + 1)^2 - \lambda^2 > 0 \) determine the range \( -\frac{3}{2} < \omega < -\frac{4}{3} \), when \( \lambda \) is negative.

**Fig.6-4.** WEC and NEC violation for a range of \( r \in [r_0^+,20] \) and typical value, \( \omega = -1.4 \) [case (ii)]. See Eqs.(6.3.5) and (6.3.11).

**Fig.6-5.** Plot of \( g(r) \) vs \( r \) for a range of \( r \in [r_0^+,10] \) and typical value, \( \omega = -5 \) [case (i)]. See Eq.(6.4.3).

**Fig.6-6.** Plot of \( g(r) \) vs \( r \) for a range of \( r \in [r_0^+,10] \) and typical value, \( \omega = -1.4 \) [case (ii)] See Eq.(6.4.3).
6.8 References


[34] We thank an anonymous referee for the explanation.