Chapter 5

BENDING OF LIGHT IN ELLIS WORMHOLE GEOMETRY

Recently, Dey and Sen [1] derived the approximate light deflection angle $\alpha$ by an Ellis wormhole in terms of proper radial distance $\ell$ that covers the entire spacetime. On the other hand, Bodenner and Will calculated the expressions for light bending in Schwarzschild geometry using various coordinates and showed that they all reduce to a single formula when re-expressed in the coordinate independent language of "circumferential radius" $r_C$ identified with the standard radial coordinate $r_S$. We shall argue that the coordinate invariant language for two-way wormholes should be $\ell$ rather than $r_S$. Hence here we find the exact deflection $\alpha$ in Ellis wormhole geometry first in terms of $\ell$ and then in terms of $r_S$. We confirm the latter expression using three different methods. We argue that the practical measurement scheme does not necessarily single out either $\ell$ or $r_S$. Some errors existing in the literature are corrected.

5.1 Introduction

In an article, Dey and Sen [1] have calculated light bending in the weak field limit of Ellis geometry using the proper radial length $\ell$.\(^1\) Earlier, it was shown by Bodenner and Will [2]

\(^1\) The contents of this chapter are based on the paper: "Bending of light in Ellis wormhole geometry"; A. Bhattacharya and A. A. Potapov; Mod. Phys. Lett. A, 25, 2399 (2010).
that the expressions for light deflection in Schwarzschild gravity in terms of standard \( r_S \),
isotropic \( r_I \) and harmonic \( r_H \) radial coordinates differ. To resolve this apparent
difference, they converted the expressions in terms of some physically measurable distance and suggested
using the "circumferential radius" \( r_C = r_S \) in which the perimeter of a circle is \( 2\pi r_C \). Then
the differing expressions exactly coincided to second order with that obtained in terms of the
"standard" coordinate \( r_S \). The resulting single expression in terms of the coordinate \( r_S \) was
then designated to be the one in coordinate independent language. This made \( r_S \) look physically
preferable to other coordinates.

The purpose of this chapter to argue that the situation is geometrically very different in the
wormhole spacetime: The coordinate marker \( r_S \) does not completely cover the entire spacetime,
while the variable that does so is the proper length \( \ell \). We shall illustrate this generic fact by
considering the simplest Ellis wormhole geometry of Einstein gravity sourced by exotic scalar
field (having energy density \( \rho < 0 \), see Fig.5.1). Note that the choice of the standard radial
variable \( r_S \) restricts us to consider light rays passing only by one side of the wormhole rather
than passing through it. We shall derive here the exact expression for the bending of light first
in terms of \( \ell \), and next in terms of \( r_S \). The latter expression is confirmed by three different
methods of calculation, the last one probably not yet widely known. In the process, certain
errors existing in the literature are corrected. In the summary, we shall argue that the practical
way by which the radii of distant gravitating objects are inferred does not single out either \( \ell \)
or \( r_S \).

5.2 Direct integration method

To find the exact expression for \( \alpha \) in terms of \( \ell \), we shall consider general form of the metric:

\[
d\tau^2 = -A(r)dt^2 + B(r)dr^2 + C(r)d\Omega^2. \tag{5.2.1}
\]
for which the total deflection angle is obtained by direct integration [3]

\[ \alpha(r_0) = -\pi + 2 \int_{r_0}^{\infty} \frac{\sqrt{B(r)} dr}{\sqrt{C(r)} \sqrt{[C(r)/C(r_0)][A(r_0)/A(r)] - 1}} \tag{5.2.2} \]

where \( r_0 \) is the distance of closest approach of the light ray to the source. As an example, consider the Ellis wormhole metric\(^2\)

\[ d\tau^2 = -dt^2 + d\ell^2 + (\ell^2 + a^2) d\Omega^2 \tag{5.2.3} \]

where \( d\Omega^2 \equiv d\theta^2 + \sin^2 \theta d\phi^2, \ell \in (-\infty, +\infty) \) is the proper radial distance at fixed time \( t \) [4], \( \ell = 0 \) corresponds to throat radius, \( a \) is an arbitrary constant parameter (proportional to total scalar field energy). The metric (5.2.3) is shown [5] to fit a geodesically complete manifold on which \( \ell \in (-\infty, +\infty) \). Note also that there is a photon sphere defined by \( \frac{dC}{dt} = \frac{dA}{dt} \Rightarrow \ell^{\alpha} = 0 \), so the throat is also the photon sphere\(^3\). Formally comparing with the general metric (5.2.1), we get \( r \equiv \ell, r_0 = \ell_0, A = B = 1, C(\ell) = \ell^2 + a^2 \). Fortunately, this case is exactly integrable between \( \ell_0 \) and \( \infty \), so that we get

\[ \alpha(\ell_0) = -\pi + 2\sqrt{\ell_0^2 + a^2} \text{EllipticK} \left( -\frac{a^2}{\ell_0^2} \right) \tag{5.2.4} \]

\[ = \frac{\pi a^2}{4\ell_0^2} - \frac{7\pi a^4}{64\ell_0^2} + O \left( \frac{a}{\ell_0} \right)^6 \tag{5.2.5} \]

where \( \ell_0 \) is the proper distance of closest approach. The expansion for \( \alpha(\ell_0) \) has been derived by Dey and Sen [1] using a different way\(^4\). The exact expression (5.2.4) nicely reveals the wormhole features (see Fig.5-1): Light bends towards the source in the sector \( \ell_0 \in (0, +\infty) \) (attractive gravity) and bends away from the source in the sector \( \ell_0 \in (0, -\infty) \) (repulsive gravity). The symmetric wormhole is massless on both sides, but it still interacts with test particles letting

\(^2\)The solution is simply the massless limit of the generic Ellis III wormhole [5].

\(^3\)The statement in Ref. [1] that there is no photon sphere is incorrect. The photon sphere at \( \ell^{\alpha} = 0 \) converts to \( r_S^{\alpha\alpha} = a \) via Eq.(5.2.6).

\(^4\)A minor correction: They obtained the expansion for \( \alpha \) which, in our notation, reads \( \alpha(\ell_0) = \frac{\pi a^2}{4\ell_0^2} - \frac{5\pi a^4}{32\ell_0^2} + O \left( \frac{a}{\ell_0} \right)^6 \). The second term is incorrect as its comparison with Eq.(5.2.5) shows.
We have chosen $a = 1$ for simplicity in Eq.(5.2.5). As we decrease the value of the closest approach distance $\ell_0$ from both right and left, we see that $\alpha(\ell_0)$ diverges to $\pm \infty$ as $\ell_0 \to \pm 0$.

Figure 5-1: We have chosen $a = 1$ for simplicity in Eq.(5.2.5). As we decrease the value of the closest approach distance $\ell_0$ from both right and left, we see that $\alpha(\ell_0)$ diverges to $\pm \infty$ as $\ell_0 \to \pm 0$.

them move along the geodesics of curved spacetime caused by a nontrivial scalar field$^5$.

We now make a change to the "circumferential" radial coordinate $r_C$ (or $r_S$) by

$$r_S = \sqrt{\ell^2 + a^2}$$

(5.2.6)

for which the circumference is $2\pi r_S$. Denoting the closest approach distance by $R_{S0} = \sqrt{\ell_0^2 + a^2} \Rightarrow \ell_0^2 = R_{S0}^2 - a^2$, we can convert Eq.(5.2.4) and expand it to obtain

$$\alpha(R_{S0}) = \frac{\pi a^2}{4R_{S0}^2} + \frac{9\pi a^4}{64R_{S0}^4} + O \left( \frac{a}{R_{S0}} \right)^6.$$  

(5.2.7)

To confirm that the expression for $\alpha(R_{S0})$ is indeed correct, we shall soon verify it by two other independent methods.

Let us calculate $\alpha$ directly by integration using metrics in different coordinates. First

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$^5$Ellis [5] concluded from the asymptotic expansion of his general Class III solution that the wormhole has Schwarzschild masses $m$ for $\ell \to +\infty$, and $-me^{-\alpha/\ell}$ for $\ell \to -\infty$. In the case under consideration, $m = 0$. The nontrivial scalar field is given by $\phi_\pm = \pm \left( \frac{\pi}{2} - \tan^{-1} \left( \frac{\ell}{\ell_0} \right) \right) \simeq \pm \frac{\pi}{2} \pm \frac{\alpha}{2}$, which yields "scalar charge" $\pm a$ leading to equal but opposite sign scalar field masses on either side. The scalar field causes spacetime to curve and we are studying light deflection due to only one (positive mass) side of the wormhole.
consider the metric (5.2.3) in standard coordinates so that it becomes

$$d\tau^2 = -dt^2 + \left(1 - \frac{a^2}{r_S^2}\right)^{-1} dr_S^2 + r_S^2 d\Omega^2, \quad r_S > a.$$  (5.2.8)

The integration (5.2.2) between $R_{S0}$ and $\infty$ yields

$$\alpha(R_{S0}) = -\pi + 2 \times \text{EllipticK} \left(\frac{a^2}{R_{S0}^2}\right)$$

$$\simeq \frac{\pi a^2}{4R_{S0}^2} + \frac{9\pi a^4}{64R_{S0}^4}$$  (5.2.9)

which is the same as in Eq.(5.2.7). Let us follow the same steps with harmonic coordinates defined by $r_S = r_H + a$. Denoting the closest approach distance by $R_{H0} = R_{S0} - a = \sqrt{\ell_0^2 + a^2} - a \Rightarrow \ell_0^2 = (R_{H0} + a)^2 - a^2$, we can convert Eq.(5.2.5) to obtain

$$\alpha(R_{H0}) \simeq \frac{\pi a^2}{4R_{H0}^2} - \frac{\pi a^3}{2R_{H0}^4} + \frac{57\pi a^4}{64R_{H0}^4}.  \quad (5.2.10)$$

The metric in harmonic coordinates is

$$d\tau^2 = -dt^2 + \left[1 - \frac{a^2}{(r_H + a)^2}\right]^{-1} dr_H^2 + (r_H + a)^2 d\Omega^2.  \quad (5.2.11)$$

With the new values of $A, B$ and $C$, carrying out the integration (5.2.2) between $R_{H0}$ and $\infty$, we obtain the same expansion as in Eq.(5.2.10).

The expressions (5.2.5), (5.2.7) and (5.2.10) are all formally different. According to [2], the expression should be (5.2.9) because it is stated in the coordinate independent language $r_C = r_S = \sqrt{\ell^2 + a^2} = r_H + a$, based on the demand that the angular part in the metric be $r_C^2 d\Omega^2$. Certainly it makes sense for the Schwarzschild geometry because it has a center (By Birkhoff’s theorem, the entire Schwarzschild mass can be thought of as concentrated at the center $r_C = 0$). The use of $r_C$ as in (5.2.9) does not however expose the other side of the wormhole because $r_C \in [a, \infty)$ by definition. To accommodate that side, we can alter the above demand by saying that the radial part be the spatially invariant $d\ell^2$, which makes sense because a wormhole has no center but a hole at $\ell = 0$ connecting two regions charted by $\ell \in (-\infty, 0] \cup [0, +\infty)$. Hence, in this case, the coordinate invariant expression should be
Eq.(5.2.4) instead of Eq.(5.2.9). However it is always possible to change over to any coordinate marker but we shall choose $r_S$ only to demonstrate the validity of other two methods as they have been originally worked out in that marker. There is no new physics involved.

We emphasize that there is nothing wrong in doing calculations with $r_C$ but for interpretation of the full wormhole geometry, one always needs to convert the final expression in terms of $\ell$. For notational clarity, hereinafter we shall drop $S$ from $r_S$ and $R_{S0}$, and but retain the same meanings.

### 5.3 Perturbation method (Bodenner & Will)

This method calculates the bending in the next higher order beyond the first term. In the standard metric (5.2.8), the equation for light trajectory on the equatorial plane ($\theta = \pi/2$) is given by

$$\frac{d^2u}{d\varphi^2} + u = 2a^2u^3 - \frac{a^2u}{b^2}$$

where $u = 1/r$ and $b$ is the impact parameter. Assume small perturbations to the zeroth order solution $u = u_0$ ($a = 0$) as follows:

$$u = u_0 + u_1 + u_2.$$  \hfill (5.3.2)

Then the path equation is linearized to

$$\frac{d^2u_0}{d\varphi^2} + u_0 = 0$$  \hfill (5.3.3)

$$\frac{d^2u_1}{d\varphi^2} + u_1 = 2a^2u_0^3 - \frac{a^2u_0}{b^2}$$  \hfill (5.3.4)

$$\frac{d^2u_2}{d\varphi^2} + u_2 = 6a^2u_0^2u_1 - \frac{a^2u_1}{b^2}.$$  \hfill (5.3.5)
The zeroth order solution is a straight line \( u_0 = \frac{\cos \varphi}{R} \), where \( R \) is the perpendicular distance from the abscissa. The full solution up to order \( a^4 \) is

\[
u = \frac{\cos \varphi}{R} - \frac{a^2}{16b^2R^3}[(4R^2 - 7b^2)\cos \varphi + b^2\cos 3\varphi + 4(2R^2 - 3b^2)\varphi \sin \varphi]
+ \frac{a^4}{256b^4R^5}[\{4b^2R^2(24\varphi^2 - 37) + 32R^4(1 - \varphi^2) + 12b^4(13 - 6\varphi^2)\} \cos \varphi
+ (28b^2R^2 - 42b^4) \cos 3\varphi + b^4 \cos 5\varphi + 312b^4 \varphi \sin \varphi - 296b^2R^2 \varphi \sin \varphi
+ 64R^4 \varphi \sin \varphi - 36b^4 \varphi \sin 3\varphi + 24b^2R^2 \varphi \sin 3\varphi].
\]

We now assume that \( u \to 0 \) at \( \varphi = \pi/2 + \delta \), where \( \delta \) is so small that \( \sin \delta \approx \delta \), \( \cos \delta \approx 1 \). Further, neglecting \( \delta^2 \), we obtain from Eq.(5.3.6), \( \delta = P/Q \), where

\[
P = 2a^2\pi \{16b^2(5a^2R^2 + 2R^4) - b^4(87a^2 + 48R^2) - 16a^2R^4\}
\]

\[
Q = 8a^4R^4(4 + \pi^2) - 8b^2a^2R^2 \{a^2(11 + 3\pi^2) + 8R^2\}
+ b^4 \{a^4(61 + 18\pi^2) + 32a^2R^2 - 256R^4\}.
\]

Expanding \( \delta \) in powers of \( (a/R) \), we obtain

\[
\delta = \left(\frac{3\pi}{8R^2} - \frac{\pi}{4b^2}\right)a^2 + \left(\frac{3\pi}{16b^4} + \frac{93\pi}{128R^4} - \frac{3\pi}{4b^2R^2}\right)a^4 + O(a^6).
\]

The value of \( b \) is obtained from the minimum of \( R \), which is just the closest approach distance \( R_0 \) for which \( du/d\varphi = 0 \). This yields

\[
b = R_0
\]

[See Eq.(14) of Ref.[2] for details]. We now have to find the minimum value of \( R \), which is the closest approach distance \( R_0 \). The minimum of \( R \) is the maximum of \( u \), which can be shown by differentiation to occur at \( \varphi = 0 \). Putting \( \varphi = 0 \) in (5.3.6), setting \( u_{\text{max}} = 1/R_0 \), and inverting, we get

\[
\frac{1}{R} = \frac{1}{R_0} - \frac{a^2}{8R_0^3} + O(a^4).
\]
Inserting the values of \( b \) and \( 1/R \) from Eqs.(5.3.10) and (5.3.11) into Eq.(5.3.9), we get

\[
\alpha (R_0) = 2\delta = \frac{\pi a^2}{4R_0^2} + \frac{9\pi a^4}{64R_0^4} + O\left(\frac{a}{R_0}\right)^6.
\]  

(5.3.12)

Which is exactly the same expression as in Eq.(5.2.5).

A little thought will indicate that we could as well do all the above steps in terms of \( \ell \) ending up with the more appropriate coordinate invariant form, Eq. (5.2.5). Alternatively, just use \( R_0 = \sqrt{\ell_0^2 + a^2} \) in Eq. (5.3.12) to retrieve Eq. (5.2.5)

5.4 Invariant angle method (Rindler & Ishak)

The method of Rindler and Ishak [6] is based on the invariant formula for the cosine of the angle \( \psi \) between two coordinate directions \( d \) and \( \delta \) such that

\[
\cos \psi = \frac{g_{ij} d^i \delta^j}{\left(g_{ij} d^i d^j\right)^{1/2} \left(g_{ij} \delta^i \delta^j\right)^{1/2}},
\]  

(5.4.1)

Where \( g_{ij} \) is the metric tensor, and in the equatorial plane \( d = (dr, d\phi) = (A,1) d\phi, (d\phi < 0) \), \( \delta = (\delta r, 0) = (1,0) \delta r \), where \( \frac{dr}{d\phi} = A(r, \phi) \) (See Ref.[7] for the detailed diagram). Eq. (5.4.1) applied to the metric (5.2.8) then yields

\[
\cos \psi = \frac{|A|}{\left[A^2 + B(r) r^2\right]^{1/2}},
\]  

(5.4.2)

where \( B(r) = \left[1 - \frac{a^2}{r^2}\right]^{-1} \). In a more convenient form, the final expression for \( \psi \) to be used is

\[
\tan \psi = \frac{B^{1/2} r}{|A|}
\]  

(5.4.3)

The one sided bending angle is defined by \( \varepsilon = \psi - \phi \)

To conform exactly to the method in its original form, we change in (4.2.3), \( \varphi \to \varphi - \pi/2 \).
so that the solution becomes

\[
\begin{align*}
u &= \sin \varphi \frac{R}{R} + \frac{a^2}{8b^2 R^3} \{ (3b^2 - 2R^2)(\pi - 2\varphi) \cos \varphi + (4b^2 - 2R^2 + b^2 \cos 2\varphi) \sin \varphi \} \\
&\quad + \frac{a^4}{256b^4 R^5} \{ 4(39b^4 - 37b^2 R^2 + 8R^4)(\pi - 2\varphi) \cos \varphi \\
&\quad + 6b^2(3b^2 - 2R^2)(\pi - 2\varphi) \cos 3\varphi + \{ 199b^4 - 18b^2 \pi^2 - 176b^2 R^2 \\
&\quad + 24b^2 R^2 \pi^2 - 8\pi^2 R^4 + 72b^4 \varphi^2 + 96b^2 R^2 \varphi^2 - 32R^4 \varphi^2 \\
&\quad + (86b^4 - 56b^2 R^2) \cos 2\varphi + 2b^4 \cos 4\varphi \} \sin \varphi \}.
\end{align*}
\]

(5.4.4)

We want to calculate the bending angle \( \delta \) when \( \varphi = 0 \). This happens at the \( r \) value, from Eq.(5.4.4),

\[
r = \frac{128b^4 R^5}{\pi a^2 \{ 16a^2 R^4 + b^4(87a^2 + 48R^2) - 16b^2 (5a^2 R^2 + 2R^4) \}}
\]

(5.4.5)

To that direction, differentiating Eq.(5.4.4) with respect to \( \varphi \), evaluating \( \frac{du}{d\varphi} \) at \( \varphi = 0 \) and putting the above value of \( r \) into it, we get

\[
A(r,0) = (-r^2) \frac{du}{d\varphi} \bigg|_{\varphi=0} = \frac{U}{V},
\]

(5.4.6)

where

\[
\begin{align*}
U &= 64b^4 R^5 [8a^4 R^4 (4 + \pi^2) - 8b^2 a^2 R^2 \{ a^2 (11 + 3\pi^2) + 8\pi^2 \} \\
&\quad + b^4 \{ a^4 (61 + 18\pi^2) + 32a^2 R^2 - 256R^4 \}] \\
V &= \pi^2 a^4 \{ 16a^2 R^4 + b^4 (87a^2 + 48R^2) - 16b^2 (5a^2 R^2 + 2R^4) \}^2.
\end{align*}
\]

(5.4.7)

(5.4.8)

Next we take

\[
\sqrt{B(r)} \simeq 1 - \frac{a^2}{2r^2} - \frac{a^4}{8r^4}
\]

(5.4.9)

and put into it the value of \( r \) from Eq.(5.4.5). Then, for small \( \psi \), we get \( \tan \psi \simeq \psi \). Putting the values of \( r \), \( |A| \) and \( \sqrt{B(r)} \) from Eqs.(5.4.5), (5.4.6), (5.4.9) respectively into Eq.(5.4.3), and expanding, we find the one way deflection as

\[
\psi = \left( \frac{3\pi}{8R^2} - \frac{\pi}{4b^2} \right) a^2 + \left( \frac{3\pi}{16b^4} + \frac{93\pi}{128R^4} - \frac{3\pi}{4b^2 R^2} \right) a^4 + O(a^6).
\]

(5.4.10)
The value of $b$ remains the same, viz., $b = R_0$. But now the maximum of $u$ occurs at $\varphi = \pi/2$. So putting $\varphi = \pi/2$ in the orbit equation (5.3.6), and denoting $u = u_{\text{max}} = \frac{1}{R_0}$, we obtain the same equation (5.3.11). Using it, we obtain the total deflection

$$\alpha(R_0) = 2\psi = \frac{\pi a^2}{4R_0^2} + \frac{9\pi a^4}{64R_0^4} + O\left(\frac{a}{R_0}\right)^6,$$

(5.4.11)

which is what we wanted to show. Again, we could do all the above calculations in terms of $\ell$ starting with the metric (5.2.3), and obtain exactly Eq.(5.2.5).

### 5.5 Conclusions

The work in this chapter puts three independent methods in one place and exemplifies the roles of proper length and coordinate markers in wormhole geometry. Light bending in wormhole geometry does not seem to have been widely studied yet. Hence its study should be interesting. The following are our conclusions:

1. We saw how the bending expression reveals the most important feature of wormhole geometry, viz., the two asymptotically flat mouths, in one of which light bends towards the source (attractive gravity) while in the other it bends away from the source (repulsive gravity). These facts are discovered only in terms of the proper length $\ell \in (-\infty, +\infty)$. They will not be discovered if one adheres only to the coordinate markers including $r_S \in (a, \infty)$. Thus, while the coordinate independent language in terms of $r_S$ is certainly appropriate for Schwarzschild black hole, it seems to be less so for geometries peculiar to wormholes. We had argued in Sec.2 that in the latter case, the coordinate invariant expression should be Eq.(5.2.4) rather than Eq.(5.2.9).

2. We have demonstrated how the three methods yield the same result for light bending $\alpha$ in the simplest example of nonsingular Ellis wormhole, the entire manifold being covered by a single coordinate chart. The same $\alpha$ from different methods is not an unexpected result. Nevertheless, all the three methods are interesting in their own right and one or the other should be useful in more complicated geometries. We have seen that the throat ($r_S = a$) is not like a horizon but like a photon sphere. This also corrects a statement in Ref.[1] that there is no photon sphere in Ellis wormhole. The wormhole is traversable by definition [4] since no
horizon occurs at the throat (here $g_{tt} = 1$); also the tidal forces, travel time etc are finite.

(3) The Rindler-Ishak method is relatively a new addition to the first two methods. The speciality of the method is that it uses the metric as well as the path equation of light, unlike the first two methods that use only the latter. The metric could, in general, contain extra terms that do not appear in the path equation. This happens, for instance, in the case of Schwarzschild-de Sitter (SdS) metric; the cosmological constant $\Lambda$ appears in the metric but cancels out of the path equation. The method then yields a bending that depends on $\Lambda$ in addition to Schwarzschild contributions [6]. There are authors both against and in favor of this $\Lambda$-effect on light bending in the SdS geometry (see [6] for a detailed account). But since the method perfectly delivers correct effects to any order in the Schwarzschild, and now in the wormhole geometry, it is unlikely to be wrong just in the $\Lambda$ part of the effect. Besides, the method is particularly enlightening where a $\Lambda$-like term appears also in wormhole geometry\(^6\).

(4) We argued that, for light bending by a wormhole, the unique coordinate invariant language should preferably be $\ell$. This requirement is purely theoretical grounded only on the geometric picture of wormholes. One would be wrong to imagine that the proper radius $\ell_0$ is somehow also the uniquely observed one. Inference about the radius of a distant gravitating object employs a quite different technique.

A look at the light deflection in the Solar system scenario would be helpful to see what

\[ d\ell^2 = -\left(1 - \frac{r_0}{r}\right) dt^2 + \left(1 - \frac{r_0}{r}\right)^{-1} dr^2 + r^2 d\Omega^2, \quad r_0 \leq r \leq a, \]

\[ dr^2 = -\left(1 - \frac{2m}{r} - \frac{\Lambda r^3}{3}\right) dt^2 + \left(1 - \frac{2m}{r} - \frac{\Lambda r^3}{3}\right)^{-1} dr^2 + r^2 d\Omega^2, \quad a \leq r < \infty \]

where $m = (r_0 a)^{1/2}/3$, the "cosmological constant" $\Lambda = r_0^{1/2}/a^{5/2}$, the constant $r_0$ is the throat radius, $a$ is the radius where matching is done with the exterior de Sitter vacuum. The major difference with the SdS metric is that in it $\Lambda$ is totally uncorrelated to the mass parameter $m$, while in the above solution, $\Lambda$ is determined by the wormhole parameters $a$ and $r_0$. The photon sphere still appears at $r_{ph} = 3m$. Following the Rindler-Ishak method (see also A. Bhattacharya et al [e-print arxiv:gr-qc/0910.1112]), we get

\[ \alpha = \alpha_1 + \alpha_2 - \alpha_3 = \frac{4m}{R_0} + \frac{15\pi m^2}{4R_0^3} - \frac{\Lambda R_0^3}{12m}, \quad R_0 > 3m. \]

The conversion of $R_0$ into $\ell_0$ is not possible in an algebraically closed form. Now suppose $R_0 > a > 3m$, (say, $R_0 = 4m$) then

\[ \alpha = 1 + \frac{15\pi}{64} - \frac{1}{4} \left(\frac{r_0}{3a}\right)^{3/2}. \]

Since by construction, $r_0 < a$, the repulsive last term is always smaller than the attractive first term.

\(^6\)There is a new wormhole solution developed by Lemos, Lobo and de Oliveira [7] using the matching technique. It is given by

\[ dr^2 = -\left(1 - \frac{r_0}{r}\right) dt^2 + \left(1 - \frac{r_0}{r}\right)^{-1} dr^2 + r^2 d\Omega^2, \quad r_0 \leq r \leq a, \]

\[ dr^2 = -\left(1 - \frac{2m}{r} - \frac{\Lambda R_0^3}{12m}\right) dt^2 + \left(1 - \frac{2m}{r} - \frac{\Lambda R_0^3}{12m}\right)^{-1} dr^2 + r^2 d\Omega^2, \quad a \leq r < \infty \]

where $m = (r_0 a)^{1/2}/3$, the "cosmological constant" $\Lambda = r_0^{1/2}/a^{5/2}$, the constant $r_0$ is the throat radius, $a$ is the radius where matching is done with the exterior de Sitter vacuum. The major difference with the SdS metric is that in it $\Lambda$ is totally uncorrelated to the mass parameter $m$, while in the above solution, $\Lambda$ is determined by the wormhole parameters $a$ and $r_0$. The photon sphere still appears at $r_{ph} = 3m$. Following the Rindler-Ishak method (see also A. Bhattacharya et al [e-print arxiv:gr-qc/0910.1112]), we get

\[ \alpha = \alpha_1 + \alpha_2 - \alpha_3 = \frac{4m}{R_0} + \frac{15\pi m^2}{4R_0^3} - \frac{\Lambda R_0^3}{12m}, \quad R_0 > 3m. \]

The conversion of $R_0$ into $\ell_0$ is not possible in an algebraically closed form. Now suppose $R_0 > a > 3m$, (say, $R_0 = 4m$) then

\[ \alpha = 1 + \frac{15\pi}{64} - \frac{1}{4} \left(\frac{r_0}{3a}\right)^{3/2}. \]

Since by construction, $r_0 < a$, the repulsive last term is always smaller than the attractive first term.
issues are involved in the measurement of distant radius. In the light bending by the Sun, one or the other coordinate radius is commonly used. The second order predictions of gravitational deflection of light would be numerically different, hence seemingly ambiguous, if we allow identification of coordinate radii with the Euclidean radius of Sun as is simply visible from a distance. The fundamental reason for such an ambiguity is that the measurements of solar radius usually employ Euclidean geometry as an approximation [8] whereas, in principle, the angle of gravitational deflection or other general relativity effects are based on the consideration of curved spacetime. But, comparing points in two different geometries i.e., in curved spacetime and flat spacetime is totally meaningless [9]. The Euclidean approximation works tolerably well only up to first order, that is, in weak field gravity caused by a source like Sun. The magnitude of the second order contribution, however, is of the same order as the error that arises due to such an approximation. Hence it looks as if the numerical value of gravitational deflection angle of light can never be unambiguously predicted at the level of second-Post Newtonian (PN) order unless we a priori fix the coordinate radius of the gravitating object without ambiguity. But there is a way out. The idea is to turn the problem upside down.

According to general relativity paradigm, coordinate length is not an observable quantity as such, but it can be indirectly “constructed” from the values of actual measurements of PN parameters \((\beta , \epsilon , \gamma , \delta)\) in any chosen coordinate system (see for details, [10]). These parameters as well as the unknown coordinate solar radius in that coordinate system can be constructed through least square curve fitting with the input of measured deflection angles. The idea is that the values of coordinates, \(R_{S0}, R_{H0}\) and \(R_0\), which refer to the same radial point, should be treated more like PN parameters due to the fact that the “flat geometry” spacetime points can not be algebraically identified in a curved spacetime [9]. Technically, however, the flat radial distances can be constructed by using metric gravity itself in terms of a large set of unknown PN parameters including the radial distance by fitting them with the observed deflection data [8]. This method has been adopted, for example, by Shapiro and his group in the radar echo delay observations [11]. The resulting PN parameter values can then be compared with the theoretical predictions of deflection in Einstein’s theory as well as in other competing theories (like Brans-Dicke theory) in the second-PN order involving both massive and massless particles [10].
Let us now look at the Ellis wormhole scenario. A practical astronomer cannot probe the repulsive side $\ell \in (-\infty, 0]$ unless light rays pass through the wormhole, which is impossible as argued in Sec. 5.1. Therefore she/he can measure the deflection angle $\alpha$ to increasingly higher precisions only in the accessible attractive side $\ell \in [0, +\infty)$ with light rays passing at the closest approach away from the throat or photon sphere. As we saw, the deflection $\alpha$ can always be expanded in any coordinate system, be it $\ell, r_C$ or other markers. But once one PN coordinate system (and the corresponding PN expansion) is chosen at will, the value of PN parameters including the radius in it can be technically constructed by curve fitting. The constructed parameters are then compared with the known values in the similar expansion in a given theory [8, pp.183, 190]. Accordingly, practical measurement scheme cannot single out any of the latter expansion, hence radius as preferred [14]. Any radius is fine, only the corresponding expansion coefficients have to be comparable with the fitted PN expansion. For practical implications about gravitational lensing by the zero mass Ellis geometry, see. Ref.[13].

5.6 Figure captions

Fig.5-1. We have chosen $a = 1$ for simplicity in Eq.(5.2.5). As we decrease the value of the closest approach distance $\ell_0$ from both right and left, we see that $\alpha(\ell_0)$ diverges to $\pm \infty$ as $\ell_0 \to \pm 0$.

5.7 References


