5. Batch Arrival Retrial G-Queue with Fluctuating Modes of Service, Impatient Customers and Random Breakdown

Batch arrival retrial G-queue with fluctuating modes of service where in the positive customers become impatient at certain instant is considered. There are two different types of failures – the failure due to the arrival of negative customer and the failure due to random breakdown. The positive customer being in service is eliminated from the system by the arrival of a negative customer. The customer interrupted by random breakdown waits in the system for the completion of work and the server after repair completion continues the interrupted service before accepting new customer. Generating function technique is employed to obtain joint distributions of the server state and orbit length. Expected system size, expected orbit size, availability of the server and failure frequency of the server are derived. Stochastic decomposition law is verified and numerical results are presented.

5.1 Model Description and Analysis

Consider a single server queueing system with positive and negative customers. Positive customers arrive according to Poisson process with rate $\lambda^+$ in groups of random size $Y$ having distribution function $P(Y = k) = C_k$, $k \geq 1$ and first two moments $m_1$ and $m_2$. Negative customers arrive singly according to Poisson process with rate $\lambda^-$. If the arriving batch of positive customers finds the server free, then one of the arrivals from the batch begins his service and the rest join the orbit. Upon arrival of a batch, if the server is found to be blocked the batch becomes impatient and balks the system with probability $1 - b$ or joins the orbit with probability $b$. Customers in the retrial queue attempts to seek the server in a retrial time generally distributed with distribution function $A(x)$, density function $a(x)$ and Laplace transform $A^*(\bullet)$ and the hazard rate function $\eta(x) = \frac{a(x)}{1 - A(x)}$. 
If a primary customer arrives first, the retrial customer cancels its attempt and returns to its position in the retrial group with probability \( e \) or quits the system (renege) with probability \( 1 - e \).

The server provides \( M \) modes of service with different service rates. Customers opt mode \( i \) service with probability \( p_i \) (\( 1 \leq i \leq M \)). The service time of mode \( i \) is generally distributed with distribution \( B_i(x) \), density function \( b_i(x) \), Laplace transform \( B_i^*(\bullet) \), first two moments \( \mu_{i,1}, \mu_{i,2} \) and the hazard rate function \( \mu_i(x) = \frac{b_i(x)}{1 - B_i(x)} \).

There are two different types of server breakdown. One is due to negative arrival and the other is random breakdown. Negative arrival makes the server down and removes the positive customer being in service from the system. The repair time of the server failed during \( i^{th} \) mode service due to negative arrival is generally distributed with distribution function \( R_i^{(1)}(x) \), density function \( r_i^{(1)}(x) \), Laplace transform \( R_i^{(1)}(\bullet) \), first two moments \( \beta_{i,1}^{(1)}, \beta_{i,2}^{(1)} \) and the hazard rate function \( \beta_i^{(1)}(x) = \frac{r_i^{(1)}(x)}{1 - R_i^{(1)}(x)} \), \( i = 1, 2, \ldots, M \).

The time until random failure of the server busy in \( i^{th} \) mode of service is exponentially distributed with rate \( \alpha_i \). Upon failure of the server, the customer in service either remains in the service position with probability \( \tau_i \) until the server is up or enters the retrial orbit with probability \( 1 - \tau_i \) and keeps returning at times exponentially distributed with rate \( \omega_i \). The repair time of the server, failed during \( i^{th} \) mode service is generally distributed with distribution function \( R_i^{(2)}(x) \), density function \( r_i^{(2)}(x) \), Laplace transform \( R_i^{(2)}(\bullet) \), first two moments \( \beta_{i,1}^{(2)}, \beta_{i,2}^{(2)} \) and the hazard rate function \( \beta_i^{(2)}(x) = \frac{r_i^{(2)}(x)}{1 - R_i^{(2)}(x)} \), \( i = 1, 2, \ldots, M \).
After repair completion, if the interrupted customer is available, the server provides the remaining service to the customer. Otherwise the server waits for the same customer to return. This waiting time is referred as reserved time. The server is not allowed to accept new customer until the interrupted customer leaves the system.

The diagrammatic representation of the model under consideration is given in Fig. 5.1.

5.2 Queue Size Distribution

For \( i = 1, 2, \ldots, M \), define the server state as

\[
C(t) = \begin{cases} 
0, & \text{server is idle} \\
i, & \text{server is busy in mode } i \\
M + i, & \text{server failed due to negative arrival is under repair} \\
2M + i, & \text{server failed due to random breakdown is under repair} \\
3M + i, & \text{server is in reserved time}
\end{cases}
\]

Let \( N(t) \) denote the number of customers in the orbit. The state of the interrupted customer \( J^*(t) \) is defined as

\[
J^*(t) = \begin{cases} 
0, & \text{if the interrupted customer remains in service position} \\
1, & \text{if the interrupted customer not in service position}
\end{cases}
\]

Define supplementary variables as

\[
\begin{align*}
\xi_0(t) &= \text{elapsed retrial time} \\
\xi_1(t) &= \text{elapsed service time} \\
\xi_2(t) &= \text{elapsed repair time} \\
\xi_3(t) &= \text{elapsed reserved time}
\end{align*}
\]

The state of the system at time \( t \) can be described by the Markov process \( \{X(t), t \geq 0\} = \{C(t), J^*(t), N(t), \xi_0(t), \xi_1(t), \xi_2(t), \xi_3(t), t \geq 0\} \).
Fig. 5.1 Batch Arrival Retrial G-Queue with Fluctuating Modes of Service, Impatient Customers and Random Breakdown
Define the following probabilities

\[ I_0(t) = \Pr\{C(t) = 0, N(t) = 0\} \]
\[ I_n(x, t) \, dx = \Pr\{C(t) = 0, N(t) = n, x < \xi_0(t) \leq x + dx\}, n \geq 1 \]

For \( n \geq 0, i = 1, 2, ..., M, j = 0, 1 \)

\[ P_{i,n}(x, t) \, dx = \Pr\{C(t) = i, N(t) = n, x < \xi_1(t) \leq x + dx\} \]
\[ W_{i,n}^{(1)}(x, t) \, dx = \Pr\{C(t) = M + i, N(t) = n, x < \xi_2(t) \leq x + dx\} \]
\[ W_{i,j,n}^{(2)}(x, y, t) \, dy \, dx = \Pr\{C(t) = 2M + i, N(t) = n, x < \xi_1(t) \leq x + dx, y < \xi_2(t) \leq y + dy\} \]
\[ Q_{i,n}(x, y, t) \, dy \, dx = \Pr\{C(t) = 3M + i, N(t) = n, x < \xi_1(t) \leq x + dx, y < \xi_3(t) \leq y + dy\} \]

5.3 Governing Equations

The steady state equations for the model under study are given below.

\[ \lambda^+ I_0 = \sum_{i=1}^{M} \left[ \int_{0}^{\infty} P_{i,0}(x) \mu_i(x) \, dx + \int_{0}^{\infty} W_{i,0}^{(1)}(x) \beta_i^{(1)}(x) \, dx \right] \quad (5.1) \]
\[ \frac{d}{dx} I_n(x) = - (\lambda^+ + \eta(x)) I_n(x), \quad n \geq 1 \quad (5.2) \]
\[ \frac{d}{dx} P_{i,n}(x) = - (\lambda^+ b + \lambda^- + \alpha_i + \mu_i(x)) P_{i,n}(x) + (1 - \delta_{0,n}) \lambda^+ b \sum_{k=1}^{n} C_k P_{i,n-k}(x) \]
\[ + \int_{0}^{\infty} W_{i,0,n}^{(2)}(x, y) \beta_i^{(2)}(y) \, dy + \omega_i \int_{0}^{\infty} Q_{i,n}(x, y) \, dy, \]
\[ n \geq 0, i = 1, 2, ..., M \quad (5.3) \]
\[ \frac{d}{dx} W_{i,n}^{(1)}(x) = - (\lambda^+ b + \beta_i^{(1)}(x)) W_{i,n}^{(1)}(x) + (1 - \delta_{0,n}) \lambda^+ b \sum_{k=1}^{n} C_k W_{i,n-k}^{(1)}(x), \]
\[ n \geq 0, i = 1, 2, ..., M \quad (5.4) \]
\[
\frac{d}{dy} W_{i,0,n}^{(2)}(x, y) = - (\lambda^+ b + \beta_i^{(2)}(y)) W_{i,0,n}^{(2)}(x, y)
+ (1 - \delta_{0,n}) \lambda^+ b \sum_{k=1}^{n} C_k W_{i,n-k}^{(2)}(x, y), \\
\quad n \geq 0, i = 1, 2, \ldots, M \quad (5.5)
\]

\[
\frac{d}{dy} W_{i,1,n}^{(2)}(x, y) = - (\lambda^+ b + \beta_i^{(2)}(y)) W_{i,1,n}^{(2)}(x, y)
+ (1 - \delta_{0,n}) \lambda^+ b \sum_{k=1}^{n} C_k W_{i,n-k}^{(2)}(x, y), \\
\quad n \geq 0, i = 1, 2, \ldots, M \quad (5.6)
\]

\[
\frac{d}{dy} Q_{i,n}(x, y) = - (\lambda^+ b + \omega_i) Q_{i,n}(x, y) + (1 - \delta_{0,n}) \lambda^+ b \sum_{k=1}^{n} C_k Q_{i,n-k}(x, y), \\
\quad n \geq 0, i = 1, 2, \ldots, M \quad (5.7)
\]

with boundary conditions

\[
I_n(0) = \sum_{i=1}^{M} \left[ \int_{0}^{\infty} P_{i,n}(x) \mu_i(x) dx + \int_{0}^{\infty} W_{i,n}^{(1)}(x) \beta_i^{(1)}(x) dx \right], \quad n \geq 1 \quad (5.8)
\]

\[
P_{i,0}(0) = p_i [\lambda^+ C_1 I_0 + \int_{0}^{\infty} I_1(x) \eta(x) dx + \lambda^+ (1 - e) C_1 \int_{0}^{\infty} I_1(x) dx], \\
i = 1, 2, \ldots, M \quad (5.9)
\]

\[
P_{i,n}(0) = p_i [\lambda^+ C_{n+1} I_0 + \int_{0}^{\infty} I_{n+1}(x) \eta(x) dx
+ \lambda^+ (1 - e) \sum_{k=1}^{n} C_k \int_{0}^{\infty} I_{n-k+2}(x) dx + \lambda^+ e \sum_{k=1}^{n} C_k \int_{0}^{\infty} I_{n-k+1}(x) dx], \\
\quad n \geq 1, i = 1, 2, \ldots, M \quad (5.10)
\]

\[
W_{i,n}^{(1)}(0) = \lambda^- \int_{0}^{\infty} P_{i,n}(x) dx, \quad n \geq 0, i = 1, 2, \ldots, M \quad (5.11)
\]

\[
W_{i,0,n}^{(2)}(x, 0) = \tau_i \alpha_i P_{i,n}(x), \quad n \geq 0, i = 1, 2, \ldots, M \quad (5.12)
\]

\[
W_{i,1,n}^{(2)}(x, 0) = (1 - \tau_i) \alpha_i P_{i,n}(x), \quad n \geq 0, i = 1, 2, \ldots, M \quad (5.13)
\]
Qi,n(x, 0) = \int_{0}^{\infty} W_{i,1,n}^{(2)}(x, y) \beta_{i}^{(2)}(y) \, dy, \quad n \geq 0, \; i = 1, 2, \ldots, M \quad (5.14)

5.4 Steady State Distributions

Define the probability generating functions

\[ I(x, z) = \sum_{n=1}^{\infty} I_{n}(x) z^{n} ; \quad P_{i}(x, z) = \sum_{n=0}^{\infty} P_{i,n}(x) z^{n} ; \]

\[ W_{i}^{(1)}(x, z) = \sum_{n=0}^{\infty} W_{i,n}^{(1)}(x) z^{n} ; \quad W_{i,j}^{(2)}(x, y, z) = \sum_{n=0}^{\infty} W_{i,j,n}(x, y) z^{n} \]

and \[ Q_{i}(x, y, z) = \sum_{n=0}^{\infty} Q_{i,n}(x, y) z^{n}, \quad i = 1, 2, \ldots, M ; j = 0, 1. \]

Multiplying equations (5.2) to (5.14) by \( z^{n} \) with appropriate values of \( n \) and summing over \( n, n = 0, 1, \ldots \), we obtain the following equations

\[ \left( \frac{d}{dx} + \lambda^{+} + \eta(x) \right) I(x, z) = 0 \quad (5.15) \]

\[ \left( \frac{d}{dx} + \lambda^{+} \right) P_{i}(x, z) = \int_{0}^{\infty} W_{i,0}^{(2)}(x, y, z) \beta_{i}^{(2)}(y) \, dy + \omega_{i} \int_{0}^{\infty} Q_{i}(x, y, z) \, dy, \quad i = 1, 2, \ldots, M \quad (5.16) \]

\[ \left( \frac{d}{dx} + \lambda^{+} \right) W_{i}^{(1)}(x, z) = 0, \quad i = 1, 2, \ldots, M \quad (5.17) \]

\[ \left( \frac{\partial}{\partial y} + \lambda^{+} \right) W_{i,0}^{(2)}(x, y, z) = 0, \quad i = 1, 2, \ldots, M \quad (5.18) \]

\[ \left( \frac{\partial}{\partial y} + \lambda^{+} \right) W_{i,1}^{(2)}(x, y, z) = 0, \quad i = 1, 2, \ldots, M \quad (5.19) \]

\[ \left( \frac{\partial}{\partial y} + \lambda^{+} \right) Q_{i}(x, y, z) = 0, \quad i = 1, 2, \ldots, M \quad (5.20) \]
\begin{align*}
I(0, z) &= \sum_{i=1}^{M} \left[ \int_{0}^{x} P_i(x, z) \mu_i(x) \, dx + \int_{0}^{x} W_i^{(1)}(x, z) \beta_i^{(1)}(x) \, dx \right] - \lambda^+ I_0 
\tag{5.21} \\
P_i(0, z) &= \frac{P_i}{z^2} \left[ \lambda^+ z \lambda C(z) I_0 + z \int_{0}^{x} I(x, z) \eta(x) \, dx + \lambda^+ (1 - e) \lambda C(z) \int_{0}^{x} I(x, z) \, dx \right] \\
&\quad + \lambda^+ e z \lambda C(z) \int_{0}^{x} I(x, z) \, dx], 
\quad i = 1, 2, \ldots, M \tag{5.22} \\
W_i^{(1)}(0, z) &= \lambda^- \int_{0}^{x} P_i(x, z) \, dx, 
\quad i = 1, 2, \ldots, M \tag{5.23} \\
W_{i,0}^{(2)}(x, 0, z) &= \tau_i \alpha_i P_i(x, z), 
\quad i = 1, 2, \ldots, M \tag{5.24} \\
W_{i,1}^{(2)}(x, 0, z) &= (1 - \tau_i) \alpha_i P_i(x, z), 
\quad i = 1, 2, \ldots, M \tag{5.25} \\
Q_i(x, 0, z) &= \int_{0}^{x} W_i^{(2)}(x, y, z) \beta_i^{(2)}(y) \, dy, 
\quad i = 1, 2, \ldots, M \tag{5.26} \\
\end{align*}

Solving the partial differential equations (5.15), (5.17) to (5.20), we get

\begin{align*}
I(x, z) &= I(0, z) \, e^{-\lambda^- x} (1 - A(x)) 
\tag{5.27} \\
W_i^{(1)}(x, z) &= W_i^{(1)}(0, z) \, e^{-\lambda^- b (1 - C(z)) x} (1 - R_i^{(1)}(x)), 
\quad i = 1, 2, \ldots, M \tag{5.28} \\
W_{i,0}^{(2)}(x, y, z) &= W_{i,0}^{(2)}(x, 0, z) \, e^{-\lambda^- b (1 - C(z)) y} (1 - R_i^{(2)}(y)), 
\quad i = 1, 2, \ldots, M \tag{5.29} \\
W_{i,1}^{(2)}(x, y, z) &= W_{i,1}^{(2)}(x, 0, z) \, e^{-\lambda^- b (1 - C(z)) y} (1 - R_i^{(2)}(y)), 
\quad i = 1, 2, \ldots, M \tag{5.30} \\
Q_i(x, y, z) &= Q_i(x, 0, z) \, e^{-\lambda^- b (1 - C(z)) y} \, e^{(1 - \tau_i) \alpha_i P_i(x, z) \lambda C(z)}, 
\quad i = 1, 2, \ldots, M \tag{5.31} \\
\end{align*}

Using equations (5.25) and (5.30) in equation (5.26) and on simplifying we have

\begin{align*}
Q_i(x, 0, z) &= (1 - \tau_i) \alpha_i P_i(x, z) \beta_i^{(2)}(z), 
\quad i = 1, 2, \ldots, M \tag{5.32} \\
\end{align*}

where \( h(z) = \lambda^+ b - \lambda^+ b C(z) \)
Substituting the expressions of \( W_{i0}^{(2)}(x, y, z) \) and \( Q_i(x, y, z) \) in equation (5.16) and on solving, we get

\[
P_i(x, z) = P_i(0, z) e^{-g_i(z)x} (1 - B_i(x)), \quad i = 1, 2, \ldots, M \tag{5.33}
\]

where \( g_i(z) = h(z) + \lambda^{-} + \alpha_i - \alpha_i R_i^{(2)\nu}(h(z)) \left( \frac{h(z) \tau_i + \omega_i}{h(z) + \omega_i} \right) \)

Inserting the expressions of \( P_i(x, z) \) and \( W_i^{(1)}(x, z) \) in equation (5.21), we obtain

\[
I(0, z) = \sum_{i=1}^{M} P_i(0, z) \left[ B_i^{\ast}(g_i(z)) + \lambda^{-} B_i^{\ast}(g_i(z)) R_i^{(1)\nu}(h(z)) \right] - \lambda^{-} I_0 \tag{5.34}
\]

where \( B_i^{\ast}(g_i(z)) = \frac{1 - B_i^{\ast}(g_i(z))}{g_i(z)} \)

Using equations (5.27) and (5.34) in equation (5.22) and on simplifying we get

\[
P_i(0, z) = I_0 \lambda^{+} p_i \left[ z C(z) - [z A^\ast(\lambda^{+}) + (1 - A^\ast(\lambda^{+})) C(z) (ze + 1 - e)] \right] / D(z), \quad i = 1, 2, \ldots, M \tag{5.35}
\]

where

\[
D(z) = z^2 - (z A^\ast(\lambda^{+}) + (1 - A^\ast(\lambda^{+})) C(z) (ze + 1 - e)) \sum_{i=1}^{M} p_i [B_i^{\ast}(g_i(z))
\]

\[
+ \lambda^{-} B_i^{\ast}(g_i(z)) R_i^{(1)\nu}(h(z))] \]

Inserting the expression of \( P_i(0, z) \), the equation (5.34) yields

\[
I(0, z) = I_0 \lambda^{+} [z C(z) \sum_{i=1}^{M} p_i [B_i^{\ast}(g_i(z)) + \lambda^{-} B_i^{\ast}(g_i(z)) R_i^{(1)\nu}(h(z))] - z^2] / D(z), \quad i = 1, 2, \ldots, M \tag{5.36}
\]

Substituting equations (5.33) and (5.35) in equations (5.23) to (5.25) and simplifying we get the expressions \( W_i^{(1)}(0, z) \), \( W_{i0}^{(2)}(x, 0, z) \), \( W_{i1}^{(2)}(x, 0, z) \) and \( Q_i(x, 0, z) \) in terms of \( I_0 \).
Using equation (5.36) in equation (5.27) and integrating with respect to x from 0 to $\infty$, we get

$$I(z) = I_0 \left(1 - A^+ (\lambda^+)\right) \left[ z C(z) \sum_{i=1}^{M} p_i [B_i^+ (g_i(z)) \right. \\
+ \left. \lambda^- B_i^+ (g_i(z)) R_i^{(1)+} (h(z))] - z^2 \right] / D(z), \quad (5.37)$$

Using equation (5.35) in equation (5.33) and integrating with respect to x from 0 to $\infty$, we get

$$P_i(z) = I_0 \lambda^+ [z C(z) - (z A^+ (\lambda^+)) \\
+ (1 - A^+ (\lambda^+)) C(z) (z e + 1 - e))] p_i \frac{B_i^+ (g_i(z))}{D(z)}, \quad i = 1, 2, \ldots, M \quad (5.38)$$

Equation (5.28) yields

$$W_i^{(1)}(z) = I_0 \lambda^+ \lambda^- [z C(z) - (z A^+ (\lambda^+)) \\
+ (1 - A^+ (\lambda^+)) C(z) (z e + 1 - e))] p_i \frac{B_i^+ (g_i(z))}{D(z)} \frac{R_i^{(1)+} (h(z))}{h(z)}, \quad i = 1, 2, \ldots, M \quad (5.39)$$

where $\frac{R_i^{(1)+} (h(z))}{h(z)} = \frac{1 - R_i^{(1)+} (h(z))}{h(z)}$

From equations (5.29), (5.30) and (5.31), we obtain

$$W_{i,0}^{(2)}(z) = I_0 \lambda^+ [z C(z) - (z A^+ (\lambda^+)) \\
+ (1 - A^+ (\lambda^+)) C(z) (z e + 1 - e))] p_i \tau_i \alpha_i \frac{B_i^+ (g_i(z))}{D(z)} \frac{R_i^{(2)+} (h(z))}{h(z)}, \quad i = 1, 2, \ldots, M \quad (5.40)$$

$$W_{i,1}^{(2)}(z) = I_0 \lambda^+ [z C(z) - (z A^+ (\lambda^+)) \\
+ (1 - A^+ (\lambda^+)) C(z) (z e + 1 - e))] p_i (1-\tau_i) \alpha_i \frac{B_i^+ (g_i(z))}{D(z)} \frac{R_i^{(2)+} (h(z))}{D(z)}, \quad i = 1, 2, \ldots, M \quad (5.41)$$
\[ Q_i(z) = I_0 \lambda^+ [z C(z) - (z A^+ (\lambda^+) \]
\[ + (1-A^+ (\lambda^+)) C(z)((z e + 1 - e)) p_i (1 - \tau_i) \alpha_i \overline{B}_i (g_i(z)) R_i^{(2)\nu} (h(z)) / \]
\[ \left[ (h(z) + \omega_i) D(z) \right], \]
\[ i = 1, 2, ..., M \quad (5.42) \]

where \( R_i^{(2)\nu} (h(z)) = \frac{1 - R_i^{(2)\nu} (h(z))}{h(z)} \)

Using normalizing condition, \( I_0 \) is obtained as

\[ I_0 = \frac{[2 - (A^+ (\lambda^+) + (m_i + e)(1 - A^+ (\lambda^+))) - \frac{\lambda^+ b m_i}{\lambda^-} \sum_{i=1}^{M} p_i T_i (1 - B_i^+ (\lambda^-))]}{[1 - e + e A^+ (\lambda^+) + \lambda^+ (m_i A^+ (\lambda^+) + (1 - e)(1 - A^+ (\lambda^+))) - m_i b A^+ (\lambda^+)) \sum_{i=1}^{M} p_i (1 - B_i^+ (\lambda^-)) (\beta_{i,1}^{(1)} + \frac{T_i}{\lambda^-})]} \quad (5.43) \]

where \( T_i = 1 + \alpha_i (\beta_{i,1}^{(2)} + \frac{1 - \tau_i}{\omega_i}) \)

The probability generating function \( P_q(z) \) of the number of customers in the orbit is

\[ P_q(z) = I_0 + I(z) + \sum_{i=1}^{M} (P_i(z) + W_i^{(1)}(z) + W_{i,0}^{(2)}(z) + W_{i,1}^{(2)}(z) + Q_i(z)) \]

\[ = I_0 \{ A^+ (\lambda^+) b (1 - C(z)) z [z - C(z) \sum_{i=1}^{M} p_i (B_i^*(g_i(z)) \]
\[ + \lambda^- \overline{B}_i (g_i(z)) R_i^{(1)*} (h(z))] + [z C(z) - z A^+ (\lambda^+) \]
\[ - C(z)(1 - A^+ (\lambda^+))(z e + 1 - e)][1 + (b (1 - C(z)) - 1) \]
\[ \sum_{i=1}^{M} p_i (B_i^*(g_i(z)) + \lambda^- \overline{B}_i (g_i(z)) R_i^{(1)*} (h(z)))]/[b(1 - C(z)) D(z)] \quad (5.44) \]
The probability generating function \( P_S(z) \) of the number of customers in the system is

\[
P_S(z) = I_0 + I(z) + \sum_{i=1}^{M} [z (P_i(z) + W_{i,0}^{(2)}(z) + W_{i,1}^{(2)}(z) + Q_i(z)) + W_i^{(1)}(z)]
\]

\[
= I_0 \{ A^*(\lambda^+) \cdot b (1 - C(z)) z [z - C(z) \sum_{i=1}^{M} p_i (B_i^*(g(z))

\[
+ \lambda^- B_i^*(g(z)) R_i^{(1)*}(h(z))) + [z C(z) - z A^*(\lambda^+) - C(z)(1 - A^*(\lambda^+))(z e + 1 - e)] [b(1 - C(z)) \sum_{i=1}^{M} p_i [B_i^*(g(z))

\[
+ \lambda^- B_i^*(g(z)) R_i^{(1)*}(h(z))] + z \sum_{i=1}^{M} p_i (1 - B_i^*(g(z)))

\[
+ \lambda^- \sum_{i=1}^{M} p_i B_i^*(g(z))(1 - z - R_i^{(1)*}(h(z))) / [b(1 - C(z)) D(z)] \} \tag{5.45}
\]

### 5.5 Performance Measures

Let \( I, P, F, Q \) be the limiting values of \( I(z), \sum_{i=1}^{M} P_i(z), \sum_{i=1}^{M} (W_i^{(1)}(z) + W_{i,0}^{(2)}(z) + W_{i,1}^{(2)}(z)) \) and \( \sum_{i=1}^{M} Q_i(z) \) as \( z \to 1 \). Then we have

- the steady state probability that the system is empty as \( I_0 \)
- the steady state probability that the server is idle in non-empty system as

\[
I = (1 - A^*(\lambda^+)) [m_1 - 1 + \lambda^+ b m_1 \sum_{i=1}^{M} p_i \beta_{i,1}^{(1)} (1 - B_i^*(\lambda^-))

\[
+ \frac{\lambda^+ b m_1}{\lambda^-} \sum_{i=1}^{M} p_i T_i (1 - B_i^*(\lambda^-))] / G_1 \tag{5.46}
\]

where

\[
G_1 = 1 - e + e A^*(\lambda^+) + \lambda^+ (m_1 A^*(\lambda^+) + (1 - e) (1 - A^*(\lambda^+))

\[
- m_1 b A^*(\lambda^-)) \sum_{i=1}^{M} p_i (1 - B_i^*(\lambda^-))(\beta_{i,1}^{(1)} + \frac{T_i}{\lambda^-})
\]
• The steady state probability that the server is busy as

\[ P = \sum_{i=1}^{M} P_i(1) = \left[ \frac{\lambda^+}{\lambda^-} \left( m_1 A^+(\lambda^+) + (1 - e) (1 - A^+(\lambda^+)) \right) \right] \sum_{i=1}^{M} p_i (1 - B_i^+(\lambda^-)) / G_1 \]  

(5.47)

• The steady state probability that the server is under repair as

\[ F = \sum_{i=1}^{M} (W_i^{(1)}(1) + W_i^{(2)}(1) + W_i^{(2)}(1)) \]

\[ = \left[ \frac{\lambda^+}{\lambda^-} \left( m_1 A^+(\lambda^+) + (1 - e) (1 - A^+(\lambda^+)) \right) \right] \sum_{i=1}^{M} p_i (1 - B_i^+(\lambda^-)) \left( \frac{\beta_i^{(1)}}{\lambda^-} + \frac{\alpha_i \beta_i^{(2)}}{G_1} \right) / G_1 \]  

(5.48)

• The steady state probability that the server is under reserved time as

\[ Q = \sum_{i=1}^{M} Q_i(1) \]

\[ = \left[ \frac{\lambda^+}{\lambda^-} \left( m_1 A^+(\lambda^+) + (1 - e) (1 - A^+(\lambda^+)) \right) \right] \sum_{i=1}^{M} p_i (1 - B_i^+(\lambda^-)) \left( (1 - \tau_i) \frac{\alpha_i}{\omega_i} \right) / G_1 \]  

(5.49)

• The mean number of customers in the orbit \( L_q \) is given by

\[ L_q = \lim_{z \to 1} \frac{d}{dz} P_q(z) \]

\[ = \frac{Dr^{''}(1)Nr^{''}(1) - Nr^{''}(1)Dr^{''}(1)}{3Dr^{''}(1)^2} \]  

(5.50)

where

\[ Nr(z) \] and \( Dr(z) \) are the numerator and denominator of \( P_q(z) \)

\[ Nr^{''}(1) = -2 I_0 b m_1 (L_1 + L_2) \]
\[ Nr''(1) = I_0 \left[ -3 \, b \, m_1 \, (m_2 \, A^* (\lambda^+)) \right. \\
+ 2 \, m_1 \, L_1 \, (1 + \lambda^+ \sum_{i=1}^{M} p_i \, (1 - B_i^* (\lambda^-))(\beta_{i,i}^{(1)} + \frac{T_{i,i}}{\lambda^+})) \\
- 3 \, (m_1 \, A^*(\lambda^+) + (1 - e) \, (1 - A^*(\lambda^+)))[b \, m_2 \\
+ 2 \, \lambda^+ \, b^2 \, m_2^2 \sum_{i=1}^{M} p_i \, (1 - B_i^* (\lambda^-))(\beta_{i,i}^{(1)} + \frac{T_{i,i}}{\lambda^+}) \\
+ \sum_{i=1}^{M} p_i \, (1 - B_i^* (\lambda^-))(\lambda^{+2} \, b^2 \, m_2^2 \, \beta_{i,i}^{(1)} + \lambda^+ \, b \, m_2 \, \beta_{i,i}^{(1)}) \\
+ \frac{2 \lambda^{+2} \, b^2 \, m_2^2}{\lambda^-} \sum_{i=1}^{M} p_i \, T_{i,i} \, \beta_{i,i}^{(1)} \, (1 - B_i^* (\lambda^-) - \lambda^- f_{i,i}) \\
+ \sum_{i=1}^{M} p_i \, (\lambda^{+2} \, b^2 \, m_2^2 \, T_{i,i} \, f_{1,2} - K_{i,1} f_{1,1}) + \lambda^- \sum_{i=1}^{M} p_i \, K_{i,2}] \\
\\
Dr''(1) = -2 \, b \, m_1 \, [2 - (A^*(\lambda^+) + (m_1 + e) \, (1 - A^*(\lambda^+)))] \\
- \lambda^+ \, b \, m_1 \sum_{i=1}^{M} p_i \, (1 - B_i^* (\lambda^-))(\beta_{i,i}^{(1)} + \frac{T_{i,i}}{\lambda^+})] \\
\\
Dr''(1) = -3 \, b \, m_2 \, [2 - (A^*(\lambda^+) + (m_1 + e) \, (1 - A^*(\lambda^+)))] \\
- \lambda^+ \, b \, m_1 \sum_{i=1}^{M} p_i \, (1 - B_i^* (\lambda^-))(\beta_{i,i}^{(1)} + \frac{T_{i,i}}{\lambda^+})] \\
- 3 \, b \, m_1 \, [2 - m_2 \, (1 - A^*(\lambda^+)) - 2 \, (A^*(\lambda^+)) \\
+ (m_1 + e) \, (1 - A^*(\lambda^+)) \lambda^+ \, b \, m_1 \sum_{i=1}^{M} p_i \, (1 - B_i^* (\lambda^-))(\beta_{i,i}^{(1)} + \frac{T_{i,i}}{\lambda^+})] \\
- \sum_{i=1}^{M} p_i \, ((1 - B_i^* (\lambda^-))(\lambda^{+2} \, b^2 \, m_2^2 \, \beta_{i,i}^{(1)} + \lambda^+ \, b \, m_2 \, \beta_{i,i}^{(1)}) \\
+ \frac{2 \lambda^{+2} \, b^2 \, m_2^2}{\lambda^-} \, T_{i,i} \, \beta_{i,i}^{(1)} \, (1 - B_i^* (\lambda^-) - \lambda^- f_{i,i}) + \lambda^{+2} \, b^2 \, m_2^2 \, T_{i,i} \, f_{1,2} \\
- K_{i,1} f_{i,1} + \lambda^- \, K_{i,2}] \\
\\
L_1 = 1 - e + e \, A^*(\lambda^+) \\
\\
L_2 = \lambda^+ \sum_{i=1}^{M} p_i \, ((1 - B_i^* (\lambda^-))(\beta_{i,i}^{(1)} + \frac{T_{i,i}}{\lambda^+})(m_1 \, A^*(\lambda^+) (1-b) + (1-e)(1 - A^+))}
\[ K_{i,1} = - (\lambda^+ b m_2 + \alpha_i (\lambda^+ b^2 m_2^2 \beta_{i,2}^{(2)} + \lambda^+ b m_2 \beta_{i,1}^{(2)}) + 2 \lambda^+ b^2 m_2^2 \beta_{i,2}^{(2)} (1 - \tau_i) + (1 - \tau_i) (\lambda^+ b m_2 \omega_i + 2 \lambda^+ b^2 m_2^2)) \]

\[ K_{i,2} = - \frac{1}{\lambda^-} (\lambda^+ b^2 m_2^2 T_i^2 f_{i,2} - K_{i,1} f_{i,1}) - \frac{(1 - B_i^* (\lambda^-))}{\lambda^-} K_{i,1} \]

\[ f_{i,1} = \int_0^\infty x e^{-\lambda^- x} b_i(x) \, dx \]

\[ f_{i,2} = \int_0^\infty x^2 e^{-\lambda^- x} b_i(x) \, dx \]

- The mean number of customers in the system \( L_S \) is given by

\[ L_S = L_q + P + \sum_{i=1}^M (W_{i,0}^{(2)} (1) + W_{i,1}^{(2)} (1)) + Q \quad (5.51) \]

### 5.6 Reliability Indices

Let \( \mathcal{A}(t) \) be the system availability at time \( t \)

- Availability of the server in equilibrium state is given by

\[ \mathcal{A} = 1 - F \]

\[ = \left[ L_1 + (m_1 A^*(\lambda^+) + (1-e) (1 - A^*(\lambda^+))) \frac{\lambda^+}{\lambda^-} \sum_{i=1}^M p_i (1 - B_i^*(\lambda^-)) \right. \]

\[ \left. (T_i + \alpha_i \beta_{i,1}^{(2)}) - \lambda^+ m_1 b A^*(\lambda^+) \sum_{i=1}^M p_i (1 - B_i^*(\lambda^-)) (\beta_{i,1}^{(1)} + \frac{T_i}{\lambda^-})] / G_1 \quad (5.52) \]

- Failure frequency of the server is given by

\[ \mathcal{F} = \sum_{i=1}^M p_i (1)[\lambda^- + \alpha_i] \]

\[ = \left[ \frac{\lambda^+}{\lambda^-} (m_1 A^*(\lambda^+) + (1-e) (1 - A^*(\lambda^+))) \sum_{i=1}^M p_i (1 - B_i^*(\lambda^-)) (\lambda^- + \alpha_i)] / G_1 \quad (5.53) \]
5.7 Special Cases

Case (i)

If $\lambda^- = 0$ (no negative customers) then the model reduces to batch arrival retrial queue with fluctuating modes of service, impatient customers, server breakdown and repair.

Case (ii)

If $C(z) \to z$, $\lambda^- = 0$ and $M = 1$ (single arrival, no negative customer and single type service) then the model under study becomes $M/G/1$ retrial queue with impatient customers and server breakdown. In this case, taking $\lambda^+ = \lambda$ we have

\begin{align*}
I &= I_1 (1 - A^\ast(\lambda)) \frac{\lambda b}{\mu_1} T / L_4 \\
P &= I_1 \frac{\lambda}{\mu_1} L_3 / L_4 \\
F &= I_1 \frac{\lambda}{\mu_1} \frac{\beta_1}{\alpha} L_3 / L_4 \\
Q &= I_1 \frac{\lambda}{\mu_1} (1 - \tau) \frac{\alpha L_3}{(L_4 \omega)}
\end{align*}

where

\begin{align*}
I_1 &= \frac{L_3 - \lambda b}{L_3 + \lambda} \frac{\mu_1 T}{T (L_3 - b A^\ast(\lambda))} \\
T &= 1 + \alpha (\beta_1 + \frac{1 - \tau}{\omega}) \\
L_3 &= 1 - e + e A^\ast(\lambda) \\
L_4 &= L_3 - \lambda b \frac{\mu_1 T}{T}
\end{align*}

The above results coincide with the results of Wu et al. (2005).

5.8 Numerical Results

In this section, numerical results are considered to validate the proposed models. Set the default parameters as $M = 2$, $\lambda^+ = 1.2$, $\lambda^- = 0.2$, $m_1 = 1.5$, $m_2 = 1$, $p_1 = 0.6$, $p_2 = 0.4$, $b = 0.4$, $e = 0.4$, $\eta = 10$, $\mu_1 = 50$, $\mu_2 = 45$, $\omega = 1$.
$\alpha_1 = 14, \alpha_2 = 13, \beta_1^{(1)} = 15, \beta_2^{(1)} = 14, \beta_1^{(2)} = 15, \beta_2^{(2)} = 15, \tau_1 = 0.7, \tau_2 = 0.5, \omega_1 = 0.6$ and $\omega_2 = 0.5$.

In order to show the influence of various parameters on the system performance measures $I_0$ – the probability that the system is empty, $I$ – the probability that the server is idle in non-empty system, $P$ – the probability that the server is busy, $L_s$ – the mean system size, $A$ – the availability of the server and $F$ – the failure frequency of the server, three dimensional graphs are displayed in Fig. 5.2 and Fig. 5.3.

The combined effect of $\lambda^+$ and $\mu_1$ on $I_0, I, P, L_s, A$ and $F$ are presented in Fig. 5.2. It is observed that

- $I, P, L_s$ and $F$ are increasing functions of $\lambda^+$ and decreasing functions of $\mu_1$.
- $I_0$ and $A$ are decreasing functions of $\lambda^+$ and increasing functions of $\mu_1$.

The combined effect of $b$ and $e$ on the performance measures are given in Fig. 5.3. Figures reveal that

- $I_0$ and $A$ are decreasing functions of $b$ and $e$
- $I, P, L_s$ and $F$ are increasing functions of $b$ and $e$
Fig. 5.2 Performance Measures by varying \((\lambda^*, \mu_1)\)
Fig. 5.3  Performance Measures by varying (b, e)