Appendix
BULK VISCOUS INHOMOGENEOUS COSMOLOGICAL MODELS WITH ELECTROMAGNETIC FIELD IN LYRA GEOMETRY

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Received June 3, 2011

A new class of exact solutions of Einstein’s modified field equations in inhomogeneous space-time for bulk viscous fluid distribution with electromagnetic field is obtained in the context of normal gauge for Lyra’s manifold. We have obtained solutions by considering the time dependent displacement field. The source of magnetic field is due to an electric current produced along the z-axis. Only $F_{12}$ is a non-vanishing component of the electromagnetic field tensor. The coefficient of bulk viscosity is assumed to be a power function of the mass density. It has been found that the displacement vector behaves like the cosmological constant in the normal gauge treatment and the solutions are consistent with the recent observations of Type Ia supernovae. Physical and geometric aspects of the models are also discussed in the presence of magnetic field.

Key words: Inhomogeneous Universe, bulk viscosity, Lyra’s manifold, magnetic field.

PACS: 98.80.Jk, 98.80-k.

1. INTRODUCTION AND MOTIVATION

The inhomogeneous cosmological models play a significant role in understanding some essential features of the universe, such as the formation of galaxies during the early stages of evolution and process of homogenization. The early attempts at the construction of such models have been done by Tolman [1] and Bondi [2] who considered spherically symmetric models. Inhomogeneous plane-symmetric models were considered by Taub [3] and later by Tomimura [4], Szekeres [5], Collins and Szafron [6], Szafron and Collins [7]. Senovilla [8] obtained a new class of exact solutions of Einstein’s equations without big bang singularity, representing a cylindrically symmetric, inhomogeneous cosmological model filled with perfect fluid which is smooth and regular everywhere satisfying energy and causality conditions. Later, Ruiz and Senovilla [9], Dadhich et al. [10], Patel et al. [11], Singh et al. [12] and Pradhan et al. [13] have investigated inhomogeneous cosmological models in various contexts.

The occurrence of magnetic fields on a galactic scale is a well-established fact today, and its importance for a variety of astrophysical phenomena is generally acknowledged as pointed out by Zeldovich et al. [14]. Also Harrison [15] suggests

that magnetic field could have a cosmological origin. As a natural consequences, we should include magnetic fields in the energy-momentum tensor of the early universe. The choice of anisotropic cosmological models in Einstein system of field equations leads to the cosmological models more general than Robertson-Walker model [16]. The presence of primordial magnetic field in the early stages of the evolution of the universe is discussed by many authors [17]. Strong magnetic field can be created due to adiabatic compression in clusters of galaxies. A large-scale magnetic field gives rise to anisotropies in the universe. The anisotropic pressure created by the magnetic fields dominates the evolution of the shear anisotropy and decays slowly as compared to the case when the pressure is held isotropic [18]. Such fields can be generated at the end of an inflationary epoch [19]. Anisotropic magnetic field models have significant contribution in the evolution of galaxies and stellar objects. Bali and Ali [20] obtained a magnetized cylindrically symmetric universe with an electrically neutral perfect fluid as the source of matter. Chakrabarty et al. [21], Pradhan and Ram [22] and Pradhan et al. [23] have investigated magnetized cosmological models in various contexts.

A realistic treatment of the problem requires the consideration of material distribution other than the perfect fluid. It is well known that in an earlier stage of the universe when the radiation in the form of photons as well as neutrinos decoupled from matter, it behaved like a viscous fluid. Misner [24] has studied the effect of viscosity on the evolution of cosmological models. A number of authors have discussed cosmological solutions with bulk viscosity in various context [25–33].

In 1917, Einstein introduced the cosmological constant into his field equations of general relativity in order to obtain a static cosmological model since, as is well known, without the cosmological term his field equations admit only non-static solutions. After the discovery of the red-shift of galaxies and explanation thereof Einstein regretted the introduction of the cosmological constant. Recently, there has been much interest in the cosmological term in the context of quantum field theories, quantum gravity, super-gravity theories, Kaluza-Klein theories and the inflationary-universe scenario. Shortly after Einstein’s general theory of relativity Weyl [34] suggested the first so-called unified field theory based on a generalization of Riemannian geometry. With its backdrop, it would seem more appropriate to call Weyl’s theory a geometrized theory of gravitation and electromagnetism (just as the general theory was a geometrized theory of gravitation only), instead a unified field theory. It is not clear as to what extent the two fields have been unified, even though they acquire (different) geometrical significance in the same geometry. The theory was never taken seriously in as much as it was based on the concept of non-integrability of length transfer; and, as pointed out by Einstein, this implies that spectral frequencies of atoms depend on their past histories and therefore have no absolute significance. Nevertheless, Weyl’s geometry provides an interesting example of non-Riemannian
connections, and recently Folland [35] has given a global formulation of Weyl mani-
folds clarifying considerably many of Weyl's basic ideas thereby.

In 1951 Lyra [36] proposed a modification of Riemannian geometry by intro-
ducing a gauge function into the structure-less manifold, as a result of which the 
cosmological constant arises naturally from the geometry. This bears a remarkable 
resemblance to Weyl's geometry. But in Lyra's geometry, unlike that of Weyl, the 
connection is metric preserving as in the Riemannian case; in other words, length 
transfers are integrable. Lyra also introduced the notion of a gauge and in the “nor-
mal” gauge the curvature scalar in identical to that of Weyl. In consecutive investiga-
tions Sen [37], Sen and Dunn [38] proposed a new scalar-tensor theory of gravitation 
and constructed an analog of the Einstein field equations based on Lyra’s geometry. 
It is, thus, possible [37] to construct a geometrized theory of gravitation and electro-
magnetism much along the lines of Weyl's “unified” field theory, however, without 
the inconvenience of non-integrability length transfer.

Halford [39] has pointed out that the constant vector displacement field $\phi_i$ in 
Lyra’s geometry plays the role of cosmological constant $\Lambda$ in the normal general 
relativistic treatment. It is shown by Halford [40] that the scalar-tensor treatment 
based on Lyra’s geometry predicts the same effects within observational limits as the 
Einstein’s theory. Several authors [41] have studied cosmological models based on 
Lyra’s manifold with a constant displacement field vector. However, this restriction 
of the displacement field to be constant is merely one for convenience and there is 
no a priori reason for it. Beesham [42] considered FRW models with time depen-
dent displacement field. Singh and Singh [43], Singh and Desikan [44] have studied 
Bianchi-type I, III, Kantowaski-Sachs and a new class of cosmological models with 
time dependent displacement field and have made a comparative study of Robertson-
Walker models with constant deceleration parameter in Einstein’s theory with cosmo-
logical term and in the cosmological theory based on Lyra’s geometry. Soleng [45] 
has pointed out that the cosmologies based on Lyra’s manifold with constant gauge 
vector $\phi$ will either include a creation field and are equal to Hoyle’s creation field cos-
mology [46] or contain a special vacuum field, which together with the gauge vector 
term, may be considered as a cosmological term. In the latter case the solutions are 
equal to the general relativistic cosmologies with a cosmological term.

Recently, Pradhan et al. [47], Casama et al. [48], Rahaman et al. [49], Bali 
and Chandnani [50], Kumar and Singh [51], Yadav et al. [52], Rao, Vinatha and 
Santhi [53], Pradhan [54] and Singh and Kale [55] have studied cosmological models 
based on Lyra’s geometry in various contexts. With these motivations and following 
the technique of Pradhan et al. [56], in this paper, we have obtained exact solutions 
of Einstein’s modified field equations in inhomogeneous space-time within the frame 
work of Lyra’s geometry in the presence of magnetic field and bulk viscous fluid for 
time varying displacement vector. This paper is organized as follows. In Section 1
the introduction and motivation for the present work is discussed. The metric and the field equations are presented in Section 2. In Section 3 the solutions of field equations for two cases §3.1 and §3.2 are derived for time varying displacement field \( \beta(t) \) in presence of magnetic field and their geometric and physical properties are also described. Finally, in Section 4 discussion and concluding remarks are given.

2. THE METRIC AND FIELD EQUATIONS

We consider the metric in the form
\[
ds^2 = dx^2 - dt^2 + B^2 dy^2 + C^2 dz^2,
\]
where \( B \) and \( C \) are both functions of \( x \) and \( t \). The energy-momentum tensor as taken has the form
\[
T^i_j = (\rho + p) u^i u^j + pg^i_j + E^i_j,
\]
where \( E^i_j \) is the electromagnetic field given by
\[
E^i_j = F^i_k F^k_j - \frac{1}{4} F^k_l F^l_m g^i_j,
\]
and
\[
\bar{p} = p - \xi u^i.
\]
Here \( \rho, p, \bar{p} \) and \( \xi \) are, respectively, the energy density, isotropic pressure of the cosmic fluid, effective pressure and bulk viscous coefficient; \( F^i_j \) is the components of electromagnetic field tensor; and \( u^i \) is the flow vector satisfying the condition
\[
g^i_j u^i u^j = -1.
\]
The co-ordinates are considered to be co-moving so that \( u^1 = 0 = u^2 = u^3 \) and \( u^4 = 1 \). If we consider that the current flows along the \( z \)-axis, then \( F_{12} \) is the only non-vanishing component of \( F^i_j \).
The field equations (in gravitational units \( c = 1, G = 1 \), in normal gauge for Lyra's manifold, obtained by Sen [37] as
\[
R_{ij} - \frac{1}{2} g_{ij} R + \frac{3}{2} \phi_i \phi_j - \frac{3}{4} g_{ij} \phi_k \phi^k = -8\pi T_{ij},
\]
where \( \phi_i \) is the displacement field vector defined as
\[
\phi_i = (0, 0, 0, \beta(t)),
\]
where other symbols have their usual meaning as in Riemannian geometry.

For the line-element (1), the field Eq. (6) with Eqs. (2) and (7) lead to the
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following system of equations

\[
\begin{align*}
\frac{\dot{B}}{B} + \frac{\ddot{C}}{C} + \frac{B\dot{C}}{BC} - \frac{B'C'}{BC} + \frac{3}{4}B^2 = -8\pi \left( \bar{\rho} + \frac{F_{12}^2}{2B^2} \right), \\
\frac{\ddot{C}}{C} - \frac{C''}{C} + \frac{3}{4}B^2 = -8\pi \left( \bar{\rho} + \frac{F_{12}^2}{2B^2} \right), \\
\frac{\ddot{B}}{B} - \frac{B''}{B} + \frac{3}{4}B^2 = -8\pi \left( \rho + \frac{F_{12}^2}{2\mu A^2 B^2} \right), \\
\frac{B''}{B} + \frac{C''}{C} + \frac{B'C'}{BC} - \frac{B\dot{C}}{BC} - \frac{3}{4}B^2 = -8\pi \left( \rho + \frac{F_{12}^2}{2\mu A^2 B^2} \right), \\
\frac{\dot{B}}{B} + \frac{\dot{C}}{C} = 0.
\end{align*}
\]  

Here, and also in the following expressions, a dot and a dash indicate ordinary differentiation with respect to \( t \) and \( x \) respectively.

From (8)-(10), we obtain

\[
\frac{B'C'}{BC} - \frac{B\dot{C}}{BC} = \frac{C''}{C} + \frac{\dot{B}}{B},
\]

and

\[
8\pi \frac{F_{12}^2}{B^2} = \frac{C''}{C} - \frac{\ddot{C}}{C} + \frac{B''}{B} + \frac{\dot{B}}{B}.
\]

The energy conservation equation \( T_{ij}^\nu = 0 \) leads to

\[
\rho + (\rho + \bar{\rho}) \left( \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right) = 0.
\]

and

\[
(R_i^i - \frac{1}{2} g_{ij} R)_{,j} + \frac{3}{2} (\phi_i \phi_j)_{,j} - \frac{3}{4} (g_{ij} \phi_k \phi^k)_{,j} = 0.
\]

Equation (16) leads to

\[
\begin{align*}
\frac{3}{2} \phi_i \left[ \frac{\partial \phi_j}{\partial x^j} + \phi^j \Gamma^i_{lj} \right] + \frac{3}{2} \phi^j \left[ \frac{\partial \phi_i}{\partial x^i} + \phi_i \Gamma^j_{lj} \right] - \frac{3}{4} g_{ij} \phi_k \left[ \frac{\partial \phi_k}{\partial x^j} + \phi^j \Gamma^k_{lj} \right] - \frac{3}{4} g_{ij} \phi_k \left[ \frac{\partial \phi_k}{\partial x^j} + \phi^j \Gamma^k_{lj} \right] = 0.
\end{align*}
\]

Equation (17) is identically satisfied for \( i = 1, 2, 3 \) but for \( i = 4 \), it is reduced to

\[
\frac{3}{2} \beta \dot{\beta} + \frac{3}{2} \beta^2 \left( \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right) = 0.
\]
3. SOLUTION OF FIELD EQUATIONS IN PRESENCE OF MAGNETIC FIELD

We have the six independent equations (8)-(12) and (18), in seven unknowns $B, C, \rho, p, \xi, \beta$ and $F_{12}$. For the complete determinacy of the system, we need one extra condition which is narrated hereinafter. The research on exact solutions is based on some physically reasonable restrictions used to simplify the field equations. Let us consider functional separability of the metric coefficients as given by

$$B = f(x)g(t), \quad C = h(x)k(t).$$

Eqs. (12) and (19) reduce to

$$\frac{f'f}{h'h} = -\frac{k'k}{g'g} = a \text{ (constant)},$$

which leads to

$$\frac{f'}{f} = \frac{h'}{h},$$

and

$$\frac{k}{k} = -\frac{g'}{g}.$$

Eqs. (21) and (22) lead to

$$f = bh^a, \quad k = dg^{-a},$$

where $b$ and $d$ are constants of integrations. Using (19) in (13), we obtain

$$\frac{ah'^2}{h^2} - \frac{h''}{h} = \frac{\ddot{g}}{g} - \frac{ag^2}{g^2} = \ell \text{ (say)},$$

which gives

$$\frac{h''}{h} - \frac{ah'^2}{h^2} = -\ell,$$

and

$$\frac{\ddot{g}}{g} - \frac{ag^2}{g^2} = \ell.$$

Here two possible cases arise.

3.1. CASE I: WHEN $a > 1, \ell > 0$

In this case Eqs. (25) and (26) lead to

$$h = K_2 \cosh^{-\frac{1}{a\alpha}}(K_1 - \alpha x), \quad g = K_4 \sec^{-\frac{1}{a\alpha}}(\alpha t + K_3),$$

$$f = bK_2^a \cosh^{-\frac{1}{a\alpha}}(K_1 - \alpha x), \quad k = dK_4^{-a} \sec^{-\frac{1}{a\alpha}}(\alpha t + K_3),$$

where $K_1, K_2, K_3, K_4$ are arbitrary constants.
where $K_1, K_2, K_3, K_4$ are integrating constants, $\kappa = \sqrt{\frac{a-1}{c}}$ and $\ell \kappa = \alpha$. Accordingly, we obtain

$$B = fg = b K_2^2 K_4 \cosh^{-\frac{a}{\alpha}}(K_1 - \alpha x) \sec^{\frac{1}{\alpha}}(at + K_3),$$

and

$$C = hk = d K_2 K_4^{-k} \cosh^{-\frac{a}{\alpha}}(K_1 - \alpha x) \sec^{-\frac{a}{\alpha}}(at + K_3).$$

In this case, after suitable transformation of coordinates, the metric (1) reduces to the form

$$ds^2 = (dX^2 - dT^2) + \cosh^{-\frac{a}{\alpha}}(\alpha X) \sec^{-\frac{a}{\alpha}}(\alpha T) dY^2 + \cdots$$

$$+ \cosh^{-\frac{a}{\alpha}}(\alpha X) \sec^{-\frac{2a}{\alpha}}(\alpha T) dZ^2.$$  

### 3.1.1. Some Physical and Geometric Properties of the Model in Presence of Magnetic Field

Equation (18) gives

$$\dot{\beta} \beta = - \left( \frac{B}{B} + \frac{C}{C} \right),$$

as $\beta \neq 0$.

which leads to

$$\frac{\dot{\beta}}{\beta} = \frac{\alpha(1-a)}{\kappa} \tan(\alpha T).$$

Equation (32) on integration gives

$$\beta = \cos^{\frac{(a-1)}{\alpha}}(\alpha T).$$

Using (28), (29) and (33) in (8) and (11), the expressions for pressure $p$ and density $\rho$ for the model (29) are given by

$$8\pi \bar{p} = 8\pi(p - \xi \theta) = \left( \frac{\alpha}{\kappa} + \frac{1}{\kappa^2} \right) \tanh^2(\alpha X) - \left( \frac{a^2}{\kappa^2} \frac{\alpha - a}{\kappa} \right) \tan^2(\alpha T) + \frac{(a-1)\alpha}{\kappa}$$

$$- \frac{4\pi F_{12}^2}{\cosh^{-\frac{2a}{\alpha}}(\alpha X) \sec^{-\frac{2}{\alpha}}(\alpha T)} - \frac{3}{4} \cos^2 \frac{(a-1)}{\alpha} (\alpha T),$$

$$8\pi \bar{\rho} = \frac{(a+1)\alpha}{\kappa} - \frac{a}{\kappa^2} \tan^2(\alpha T) - \left( n \frac{a\alpha}{\kappa} + \frac{3a^2}{\kappa^2} \right) \tanh^2(\alpha X) -$$

$$- \frac{4\pi F_{12}^2}{\cosh^{-\frac{2a}{\alpha}}(\alpha X) \sec^{-\frac{2}{\alpha}}(\alpha T)} + \frac{3}{4} \cos^2 \frac{(a-1)}{\alpha} (\alpha T).$$
where \( n = \frac{a}{\kappa} + \frac{1}{\kappa^2} - \frac{a^2}{\kappa^3}, \kappa > 0 \). Here \( \theta \) is the scalar expansion calculated for the flow vector \( \mathbf{u} \) as

\[
\theta = \left(1 - \frac{a}{\alpha}\right) \tan(\alpha T).
\]

(36)

For the specification of \( \xi \), we assume that the fluid obeys an equation of state of the form

\[
p = \gamma \rho,
\]

(37)

where \( \gamma(0 \leq \gamma \leq 1) \) is a constant.

Thus, given \( \xi(t) \) we can solve the system for the physical quantities. Therefore, let us assume the following \( \text{ad hoc} \) law, [28–30]

\[
\xi(t) = \xi_0 \rho^m,
\]

(38)

where \( \xi_0 \) and \( m \) are real constants. For large value of \( \rho \), \( m \) is quite small and Santos et al. [31] suggested to get more realistic models if \( m \) lies in the regime \( 0 \leq m \leq \frac{1}{2} \).

For small density, \( m \) may even be equal to unity as used in Murphy’s work [33] for simplicity. Also if \( m = 1 \), Eq. (38) may correspond to a radiative fluid [57].

On using Eq. (38) in (34), we obtain

\[
8\pi (p - \xi_0 \rho^m \theta) = \left( \frac{\alpha}{\kappa} + \frac{1}{\kappa^2} \right) \tanh^2(\alpha X) - \left( \frac{a^2}{\kappa^2} - \frac{a\alpha}{\kappa} \right) \tan^2(\alpha T) + \left( \frac{a - 1}{\kappa} \right) - \frac{4\pi F_2^2}{\cosh \frac{2\alpha}{\kappa} (\alpha X) \sec \frac{2\alpha}{\kappa} (\alpha T)} - \frac{3}{4} \cos \frac{2(a - 1)}{\alpha \kappa} (\alpha T),
\]

(39)

For simplicity and realistic models for physical importance, we consider the following two cases:

**Model I: Solution for** \( m = 0 \).

When \( m = 0 \), (38) reduces to \( \xi = \xi_0 \). With the use of (36) and (37), (39) leads to

\[
8\pi p = 8\pi \gamma \rho = 8\pi \xi_0 \left( \frac{1 - a}{\kappa} \right) \tan(\alpha T) + \left( \frac{\alpha}{\kappa} + \frac{1}{\kappa^2} \right) \tanh^2(\alpha X) - \left( \frac{a^2}{\kappa^2} - \frac{a\alpha}{\kappa} \right) \times \tan^2(\alpha T) + \left( \frac{(a - 1)}{\kappa} \right) - \frac{4\pi F_2^2}{\cosh \frac{2\alpha}{\kappa} (\alpha X) \sec \frac{2\alpha}{\kappa} (\alpha T)} - \frac{3}{4} \cos \frac{2(a - 1)}{\alpha \kappa} (\alpha T).
\]

(40)

**Model II: Solution for** \( m = 1 \).

When \( m = 1 \), (38) reduces to \( \xi = \xi_0 \rho \). With the use of (36) and (37), (39) leads to
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\[ 8\pi \rho = \frac{8\pi \rho}{\gamma} = \frac{1}{\gamma - \xi_0 \left(1 - \frac{a}{\kappa}\right)} \left[ \left( \frac{\alpha + 1}{\kappa^2} \right) \tanh^2(\alpha X) - \left( \frac{a^2}{\kappa^2} - \frac{a\alpha}{\kappa} \right) \tan^2(\alpha T) + \right. \]
\[ + \left. \frac{(a - 1)\alpha}{\kappa} \frac{4\pi F_{12}^2}{\cosh^{\frac{2\kappa}{a\alpha}}(\alpha X) \sec^{\frac{2}{a\alpha}}(\alpha T)} - \frac{3}{4} \cos \frac{2(a - 1)b}{a\alpha}(\alpha T) \right], \tag{41} \]

The non-vanishing component \( F_{12} \) of the electromagnetic field tensor \( F_{ij} \) is obtained from (14)

\[ 4\pi F_{12}^2 = \frac{\sec^{\frac{2\kappa}{a\alpha}}(\alpha T)}{\cosh^{\frac{2\kappa}{a\alpha}}(\alpha X) \sec^{\frac{2}{a\alpha}}(\alpha T)} \left[ \frac{2a\alpha}{\kappa} + \left( n - \frac{a\alpha}{\kappa} \right) \tanh^2(\alpha X) + \left( n + \frac{a\alpha}{\kappa} \right) \tan^2(\alpha T) \right]. \tag{42} \]

The component of charge current density is given by

\[ J^2 = -\tanh(\alpha X) \frac{\sec^{\frac{1}{a\alpha}}(\alpha X) \sec^{\frac{1}{a\alpha}}(\alpha T)}{\sqrt{8\pi T}} \left[ \alpha(n\kappa - a\alpha) \sech^2(\alpha X) - L \right], \tag{43} \]

where

\[ L = \left[ \frac{2a\alpha}{\kappa} + \left( n - \frac{a\alpha}{\kappa} \right) \tanh^2(\alpha X) + \left( n + \frac{a\alpha}{\kappa} \right) \tan^2(\alpha T) \right]. \]

Halford [39] has pointed out that the displacement field \( \phi_4 \) in Lyra's manifold plays the role of cosmological constant \( \Lambda \) in the normal general relativistic treatment. From (33) it is observed that the displacement vector \( \beta(T) \) is a periodic and decreasing function of time and it approaches to a small positive value at late time, which is corroborated with Halford as well as with the recent observations [58, 59] leading to the conclusion that \( \Lambda(T) \) is a decreasing function of \( T \). It is observed from (40) and (41) that the energy density in both models is also a decreasing function of time and it is positive under appropriate condition.

The expressions for the Hubble parameter \( H \), shear scalar \( \sigma^2 \), deceleration parameter \( q \) and proper volume \( V^3 \) for the model (30) are given by

\[ H = 3 \left( \frac{1 - a}{\kappa} \right) \tan(\alpha T), \tag{44} \]
\[ \sigma^2 = \frac{(a^2 - a + 1)}{\kappa^2} \tan^2(\alpha T), \tag{45} \]
\[ q = -1 - \frac{a\kappa}{a - 1} \cosec^2(\alpha T), \tag{46} \]
\[ V^3 = \sqrt{-g} = \cosh^{\frac{(a + 1)b}{a\alpha}}(\alpha X) \sec^{\frac{(a + 1)b}{a\alpha}}(\alpha T). \tag{47} \]
From (45) and (46) we obtain
\[ \frac{\sigma^2}{\tilde{g}^2} = \frac{(a^2 - a + 1)}{(1 - a)^2} = \text{constant.} \] (48)

The rotation \( \omega \) is identically zero.

The non-vanishing components of conformal curvature tensor are obtained as

\[ C^{12}_{12} = \frac{1}{6} \left[ \left( \frac{3a^2}{\kappa^2} + \frac{2a\alpha}{\kappa} - \frac{a}{\kappa^2} - n \right) \tanh^2(\alpha X) + \left( \frac{-a^2}{\kappa^2} + \frac{2a\alpha}{\kappa} - \frac{a}{\kappa^2} + n \right) \tanh^2(\alpha T) + \frac{2a}{\kappa} \right], \] (49)

\[ C^{13}_{13} = C^{24}_{24} = \frac{1}{6} \left[ \left( \frac{a^2}{\kappa^2} - \frac{a\alpha}{\kappa} - \frac{a}{\kappa^2} + 2n \right) \tanh^2(\alpha X) + \left( \frac{a^2}{\kappa^2} - \frac{a\alpha}{\kappa} - \frac{a}{\kappa^2} - \frac{2a}{\kappa^2} - \frac{2a}{\kappa} \right) \tanh^2(\alpha T) - \frac{4a}{\kappa} \right], \] (50)

\[ C^{14}_{14} = C^{23}_{23} = \frac{1}{6} \left[ \left( \frac{2a}{\kappa^2} - \frac{a\alpha}{\kappa} - n \right) \tanh^2(\alpha X) + \left( \frac{2a^2}{\kappa^2} - \frac{a\alpha}{\kappa} + \frac{2a}{\kappa^2} + n \right) \tanh^2(\alpha T) + \frac{2a}{\kappa} \right], \] (51)

\[ C^{34}_{14} = \frac{a}{\kappa^2} \tanh(\alpha X) \tan(\alpha T), \] (52)

\[ C^{34}_{13} = -\frac{a}{\kappa^2} \tanh(\alpha X) \tan(\alpha T). \] (53)

Generally the model (30) represents an expanding, shearing and non-rotating universe in which the flow vector is geodetic. The model (30) has initial singularity at \( X = 0, T = 0 \). The model will start expanding at \( T > \frac{\pi}{6} \) and the expansion will be maximum at \( T = 0, T = \frac{\pi}{6} \). Since \( \frac{3}{6} = \text{constant} \), the model does not approach isotropy. As \( T \) increases the proper volume also increases. The physical quantities \( p \) and \( \rho \) decrease as \( F_{12} \) increases. It is observed from Eq. (46) that \( q < 0 \) when \( \kappa \alpha > 0 \), which implies an accelerating model of the universe. Recent observations of type Ia supernovae [58, 59] reveal that the present universe is in accelerating phase and deceleration parameter lies somewhere in the range \(-1 < q \leq 0 \). It follows that our model of the universe is consistent with recent observations. When \( \alpha = 0 \), the deceleration parameter \( q \) approaches the value \((-1)\) as in the case of de Sitter uni-
verse. The space-time is non-degenerate Petrov-type I, in general. If we put $\xi = 0$ in above results, we get the results recently obtained by Pradhan et al. [56].

3.2. CASE II: WHEN $a < 1$, $\ell < 0$

In this case (25) and (26) lead to

$$h = c_2 \cosh^{\frac{3}{2}} (d \zeta x + c_1), \quad g = b_2 \sec^{-\frac{3}{2}} (b_1 - d \zeta t),$$

$$f = bh^a = b c_2 \cosh^{\frac{3}{2}} (d \zeta x + c_1), \quad k = d g^{-a} = d b_2^{-a} \sec^{\frac{3}{2}} (b_1 - d \zeta t),$$

where $b_1, b_2, c_1, c_2$ are constants of integration, $a - 1 = -d, \ell = -1, \zeta = \sqrt{1 \ell, \Delta B > 0}.$ Accordingly, we obtain

$$B = fg = bh^a \cosh^{\frac{3}{2}} (d \zeta x + c_1) \sec^{-\frac{3}{2}} (b_1 - d \zeta t),$$

and

$$C = hk = dh^a b_2^{-a} \cosh^{\frac{3}{2}} (d \zeta x + c_1) \sec^{\frac{3}{2}} (b_1 - d \zeta t).$$

In this case, after suitable transformation of coordinates, the metric (1) reduces to the form

$$ds^2 = (dX^2 - dT^2) + \cosh^{\frac{2a}{w^2}} (wX) \sec^{\frac{2a}{w^2}} (wT) dY^2 + \cosh^{-\frac{2a}{w^2}} (wX) \sec^{-\frac{2a}{w^2}} (wT) dZ^2,$$

where $\zeta d = w.$

3.2.1. Some Physical and Geometric Properties of the Model in Presence of Magnetic Field

In this case Eq. (18) leads to

$$\frac{\dot{\beta}}{\beta} = \frac{\ell d \zeta (a-1)}{w} \tan(wT),$$

which on integration gives

$$\beta = \cos\frac{(a-1)}{w^2} (wT).$$

The expressions for pressure $p$ and density $\rho$ for the model (58) are given by

$$8\pi p = 8\pi (p - \xi \theta) = \left( \frac{a^2}{w^2} + \ell \right) \tanh^2 (wX) - \left( \frac{a^2 \ell^2}{w^2} - \ell \right) \tan^2 (wT) + (a - 1) \ell - \frac{4\pi F_{12}^2}{\cosh^{\frac{2a}{w^2}} (wX) \sec^{\frac{2a}{w^2}} (wT)} - \frac{3}{4} \cos\frac{(a-1)}{w^2} (wT),$$

$$8\pi \rho = 8\pi \rho_{m} = \frac{-4\pi F_{12}^2}{\cosh^{\frac{2a}{w^2}} (wX) \sec^{\frac{2a}{w^2}} (wT)} - \frac{3}{4} \cos\frac{(a-1)}{w^2} (wT),$$
Here $\theta$ is the scalar expansion calculated for the flow vector $u^i$ as

$$\theta = \frac{(1-a)\ell}{w^2} \tan(wT),$$

(62)

On using Eq. (38) in (61) we obtain

$$8\pi (p - \rho^2 m^2 \theta) = \left( \frac{\ell^2}{w^2} + \ell \right) \tanh^2(wX) - \left( \frac{\alpha^2 \ell^2}{w^2} - \alpha \ell \right) \tan^2(wT) + (a - 1) \ell - \frac{4\pi F^2_{12}}{\cosh w^2(wX) \sec w^2(wT)} - \frac{3}{4} \cos \frac{2\ell (a - 1)}{w^2} (wT).$$

(63)

For simplicity and realistic models for physical importance, we consider the following two cases:

**MODEL I: SOLUTION FOR** $m = 0$

When $m = 0$, Eq. (38) reduces to $\xi = \xi_0$. With the use of (36) and (37), (63) leads

$$8\pi p = 8\pi \gamma \xi_0 \left[ \frac{1 - a}{w^2} \tan(wT) + \left( \frac{\ell^2}{w^2} + \ell \right) \tanh^2(wX) - \left( \frac{\alpha^2 \ell^2}{w^2} - \alpha \ell \right) \right] \times \tan^2(wT) + (a - 1) \ell - \frac{4\pi F^2_{12}}{\cosh w^2(wX) \sec w^2(wT)} - \frac{3}{4} \cos \frac{2\ell (a - 1)}{w^2} (wT).$$

(64)

**MODEL II: SOLUTION FOR** $m = 1$

When $m = 1$, (38) reduces to $\xi = \xi_0 \rho$. With the use of (36) and (37), (63) leads

$$8\pi p = \frac{8\pi p}{\gamma} = \frac{1}{\gamma - \xi_0 \frac{(1-a)\ell}{w^2} \tan(wT)} \left[ \left( \frac{\ell^2}{w^2} + \ell \right) \tanh^2(wX) - \left( \frac{\alpha^2 \ell^2}{w^2} - \alpha \ell \right) \times \tan^2(wT) + (a - 1) \ell - \frac{4\pi F^2_{12}}{\cosh w^2(wX) \sec w^2(wT)} - \frac{3}{4} \cos \frac{2\ell (a - 1)}{w^2} (wT) \right].$$

(65)

The non-vanishing component $F_{12}$ of the electromagnetic field tensor $F_{ij}$ is obtained
from (14)

\[ 4\pi F_{12}^2 = \cosh^\frac{2\ell}{w^2}(wX) \sec^\frac{2\ell}{w^2}(wT) \left[ 2a\ell + \left( (1 + a)\ell + (1 - a^2)\frac{\ell^2}{w^2} \right) \times \right. \\
\left. \times \tan^2(wT) + \left( (1 - a)\ell + (1 - a^2)\frac{\ell^2}{w^2} \right) \tanh^2(wX) \right]. \quad (66) \]

The component of charge current density is given by

\[ J^2 = -\frac{\tanh(wX) \sech^\frac{2\ell}{w^2}(wX) \sec^\frac{2\ell}{w^2}(wT)}{\sqrt{8\pi \hbar}} \times \] 
\[ \times \left[ w \left( (1 - a)\ell + (1 - a^2)\frac{\ell^2}{w^2} \right) \sech^2(wX) - \frac{\ell \hbar}{w} \right], \quad (67) \]

where

\[ \hbar = \left[ 2a\ell + \left( (1 + a)\ell + (1 - a^2)\frac{\ell^2}{w^2} \right) \tan^2(wT) \right. \\
\left. + \left( (1 - a)\ell + (1 - a^2)\frac{\ell^2}{w^2} \right) \tanh^2(wX) \right]. \quad (68) \]

From Eq. (60), it is observed that the displacement vector \( \beta(T) \) is a periodic function of time. It is observed that the displacement vector \( \beta(T) \) is a decreasing function of time and it approaches to a small positive value at late time, which is corroborated with Halford as well as with the recent observations [58,59] leading to the conclusion that \( \Lambda(T) \) is a decreasing function of \( T \). It is observed from Eqs. (65) and (66) that the energy density in both models is also a decreasing function of time and it is positive under appropriate condition.

The expressions for the Hubble parameter \( H \), shear scalar \( \sigma^2 \), deceleration parameter \( q \) and proper volume \( V^3 \) for the model (58) are given by

\[ H = 3 \frac{(1 - a)\ell}{w^2} \tan(wT), \quad (69) \]
\[ \sigma^2 = \frac{(a^2 - a + 1)\ell^2}{w^4} \tan^2(wT), \quad (70) \]
\[ q = 1 + \frac{w^2}{\ell(a - 1)} \cosec^2(wT), \quad (71) \]
\[ V^3 = \sqrt{-g} = \cosh^\frac{\ell(a - 1)}{w^4}(wX) \sec^\frac{\ell(a - 1)}{w^4}(wT). \quad (72) \]
From Eqs. (70) and (71) we obtain
\[ \frac{\sigma^2}{\beta^2} = \frac{(a^2 - a + 1)}{(1 - a)^2} = \text{constant}. \] (73)

The rotation \( \omega \) is identically zero.

The non-vanishing components of conformal curvature tensor are obtained as
\[ C_{12}^{12} = C_{34}^{34} = \frac{1}{6} \left[ \left( \frac{2a^2 \ell^2}{w^2} - \frac{a \ell^2}{w^2} - \frac{\ell^2}{w^2} - (a + 1)\ell \right) \tanh^2(wX) + \left( \frac{\ell^2}{w^2} - \frac{2a^2 \ell^2}{w^2} - \frac{a \ell^2}{w^2} + (2a + 1)\ell \right) \tan^2(wT) + 2\ell \right], \] (74)
\[ C_{13}^{13} = C_{24}^{24} = \frac{1}{6} \left[ \left( \frac{a^2 \ell^2}{w^2} - \frac{a \ell^2}{w^2} - \frac{2a \ell^2}{w^2} - (a + 2)\ell \right) \tan^2(wT) + \left( \frac{2a \ell^2}{w^2} - \frac{a^2 \ell^2}{w^2} - \frac{\ell^2}{w^2} \right) \tanh^2(wX) - 2\ell \right], \] (75)
\[ C_{14}^{14} = C_{23}^{23} = \frac{1}{6} \left[ \left( \frac{a^2 \ell^2}{w^2} + \frac{2a \ell^2}{w^2} + \frac{\ell^2}{w^2} - (a - 1)\ell \right) \tan^2(wT) - \left( \frac{\ell^2}{w^2} + \frac{a^2 \ell^2}{w^2} - \frac{2a \ell^2}{w^2} + (a + 1)\ell \right) \tanh^2(wX) \right], \] (76)
\[ C_{13}^{13} = \frac{a \ell^2}{w^2} \tanh(wX) \tan(wT), \] (77)
\[ C_{14}^{14} = \frac{a \ell^2}{w^2} \tanh(wX) \tan(wT). \] (78)

Generally the model (57) represents an expanding, shearing and non-rotating universe in which the flow vector is geodetic. The model (57) has initial singularity at \( X = 0, T = 0 \). The model will start expanding at \( T > \frac{\pi}{w} \) and the expansion will be maximum at \( T = 0, T = \frac{\pi}{w} \) and expansion will be maximum at \( T = \frac{3\pi}{2w} \). The expansion stops at \( T = 0, T = \frac{\pi}{w} \). Since \( \frac{H}{\dot{r}} = \text{constant} \), the model does not approach isotropy. As \( T \) increases the proper volume also increases. The physical quantities \( p \) and \( \rho \) decrease as \( F_{12} \) increases. It is observed from Eq. (71) that \( q > 0 \) always. So in this case the model is in decelerating phase. The space-time is non-degenerate Petrov-type I, in general. If we put \( \xi = 0 \) in above results, we get the results recently obtained [56].
4. DISCUSSION AND CONCLUDING REMARKS

By revisiting the solutions of Pradhan et al. [56], in this paper, we have obtained a new class of exact solutions of Einstein's modified field equations for inhomogeneous space-time with a bulk viscous fluid distribution within the framework of Lyra's geometry in presence of magnetic field. The solutions are obtained by using the functional separability of the metric coefficients. The source of the magnetic field is due to an electric current produced along the z-axis. $F_{12}$ is the only non-vanishing component of electromagnetic field tensor. In both cases the electromagnetic field tensors are given by equations (42) and (66). It is observed that in the presence of magnetic field, the rate of expansion of the universe is faster than that in absence of magnetic field (although the results in absence of magnetic field are not reported in the present paper). The idea of primordial magnetism is appealing because it can potentially explain all the large-scale fields seen in the universe today, specially those found in remote proto-galaxies. As a result, the literature contains many studies examining the role and the implications of magnetic fields for cosmology. In the presence of a magnetic field both the models (30) and (57) represent an expanding, shearing and non-rotating universe in which the flow vector is geodetic.

It is observed that the displacement vectors $\beta(t)$ in both cases coincide with the nature of the cosmological constant $\Lambda$ which has been supported by the work of several authors as discussed in the physical behavior of the model in previous section. In recent time $\Lambda$-term has attracted theoreticians and observers for many a reason. The nontrivial role of the vacuum in the early universe generates a $\Lambda$-term that leads to inflationary phase. Observationally, this term provides an additional parameter to accommodate conflicting data on the values of the Hubble constant, the deceleration parameter, the density parameter and the age of the universe (for example, see Refs. [60] and [61]). In recent past there has been an upsurge of interest in scalar fields in general relativity and alternative theories of gravitation in the context of inflationary cosmology [62–64]. Therefore the study of cosmological models in Lyra's geometry may be relevant for inflationary models. There seems to be a good possibility of Lyra's geometry to provide a theoretical foundation for relativistic gravitation, astrophysics and cosmology. As discussed in previous section, we have generalized the results recently obtained by Pradhan et al. [56].

Acknowledgments. The authors are grateful to Prof. A. Pradhan for his fruitful discussions and suggestions.
REFERENCES
Anisotropic Bianchi Type-III String Cosmological Models in Normal Gauge for Lyra's Manifold with Electromagnetic Field

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Received 09 June 2012

Abstract. The present study deals with a spatially homogeneous and anisotropic Bianchi type-III cosmological models representing massive strings in normal gauge for Lyra's manifold with electromagnetic field. The energy-momentum tensor for such string as formulated by Letelier (1983) is used to construct massive string cosmological models for which we assume that the expansion (θ) in the model is proportional to the eigenvalue $\sigma_1$ of the shear tensor $\sigma^i_j$. This condition leads to $B = \xi_1(AC)^{m_1}$, where $A$, $B$ and $C$ are the metric coefficients and $\xi_1$, $m_1$ are arbitrary constants. Our models are in accelerating phase which is consistent to the recent observations. It has been found that the displacement vector $\beta$ behaves like cosmological term $\Lambda$ in the normal gauge treatment and the solutions are consistent with recent observations of SNe Ia supernovae. It is also observed that in early stage of the evolution of the universe string dominates over the particle whereas the universe is dominated by massive string at the late time. Some physical and geometric behaviour of the models have also been discussed in presence and absence of magnetic field.

PACS codes: 98.80.Cq, 04.20.-q

1 Introduction and Motivation

In Einstein's general theory, the curvature of a space-time is influenced by matter, and provides the geometrical description of matter. Einstein (1917) succeeded in geometrizing gravitation by expressing gravitational potential in terms of metric tensor. Weyl, in 1918, was inspired by it and he was the first to unify gravitation and electromagnetism in a single space-time geometry. He
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has shown how one introduce a vector field in the Riemannian space-time with an intrinsic geometrical significance. But this theory was not accepted as it was based on non-integrability of length transfer. Lyra [1] introduced a gauge function, i.e., a displacement vector in Riemannian space-time which removes the non-integrability condition of a vector under parallel transport. In this way Riemannian geometry was given a new modification by him and the modified geometry was named as Lyra's manifold. In consecutive investigations Sen [2], Sen and Dunn [3] have proposed a new scalar-tensor theory of gravitation and constructed an analog of the Einstein field equations based on Lyra’s geometry. It is, thus, possible [2] to construct a geometrized theory of gravitation and electromagnetism much along the lines of Weyl’s “unified” field theory, however, without the inconvenience of non-integrability length transfer.

Halford [4] has pointed out that the constant vector displacement field \( \phi_i \) in Lyra’s manifold plays the role of cosmological constant \( \Lambda \) in the normal general relativistic treatment. It is shown by Halford [5] that the scalar-tensor treatment based on Lyra’s geometry predicts the same effects, within observational limits as the Einstein’s theory. The Sen [2] theory and its more generalizations (Sen and Dun [3]; Sen and Vanstone [6]) have received considerable attention in cosmological context. Several investigators [6-27] have studied cosmological models based on Lyra’s manifold in different contexts. Soleng [7] has pointed out that the cosmologies based on Lyra’s manifold with constant gauge vector \( \phi \) will either include a creation field and be equal to Hoyle’s creation field cosmology [28-30] or contain a special vacuum field which together with the gauge vector term may be considered as a cosmological term. In the latter case the solutions are equal to the general relativistic cosmologies with a cosmological term.

In recent years, there has been considerable interest in string cosmology. Cosmic strings are topologically stable objects which might be found during a phase transition in the early universe (Kibble [31]). Cosmic strings play an important role in the study of the early universe. These arise during the phase transition after the big bang explosion as the temperature goes down below some critical temperature as predicted by grand unified theories [31-35]. It is believed that cosmic strings give rise to density perturbations which lead to the formation of galaxies [36]. These cosmic strings have stress-energy and couple to the gravitational field. Therefore it is interesting to study the gravitational effects that arise from strings. The pioneering work in the formulation of the energy-momentum tensor for classical massive strings was done by Letelier [37] who considered the massive strings to be formed by geometric strings with particle attached along its extension. Letelier [38] first used this idea in obtaining cosmological solutions in Bianchi-I and Kantowski–Sachs space-times. Stachel [39] has studied massive string. During the last ten years, many authors [40-60] and references therein have discussed the string cosmological models in different contexts.

Since the observed universe is almost homogeneous and isotropic, space-time is
usually described by a Friedman–Lemaître–Robertson–Walker (FLRW) cosmology. But it is also believed that in the early universe the FLRW model could not give a correct matter description. The anomalies found in the cosmic microwave background (CMB) and the large structure observations stimulated a growing interest in anisotropic cosmological model of the universe. Observations by the Differential Radiometers on NASA’s Cosmic Background Explorer registered anisotropy in various angle scales. It is conjectured, that these anisotropies hide in their hearts the entire history of the cosmic evolution down to recombination, and they are considered to be indicative of the universe geometry and the matter composing the universe. It is expected, that much more will be known about anisotropy of cosmic microwave’s background after the investigations of the microwave’s anisotropy probe. There is a general agreement among cosmologists that cosmic microwave’s background anisotropy in the small angle scale holds the key to the formation of the discrete structure. The theoretical argument [61] and the modern experimental data support the existence of an anisotropic phase, which turns into an isotropic one.

The advantages of the anisotropic models are that they have a significant role in the description of the evolution of the early phase of the universe and they help in finding more general cosmological models than the isotropic FRW models. The field equations of general relativity for spatially homogeneous Bianchi type-III space-time have been studied by Ram [62]. Recently, Agarwal et al. [63] have studied B-I and B-II models respectively with cosmic string in normal gauge for Lyra’s manifold. Recently, several authors [64–82] have studied cosmological models based on Lyra’s geometry in various contexts. Recently Yadav & Yadav [83] have studied Bianchi type-III bulk viscous and barotropic perfect fluid cosmological models in Lyra’s geometry. Motivated by the above discussions, in this paper, the field equations in normal gauge for Lyra’s manifold where gauge function $\beta$ is taken as time dependent, have been solved for massive string in presence of perfect fluid distribution of matter in B-III space-time. The paper has the following structure. The metric and the field equations are presented in Section 2. In Section 3, we deal with an exact solution of the Einstein’s modified field equations with cloud of strings. In Section 4, we describe some physical and geometric properties of the model in presence of electromagnetic field. In Section 5, we deal the solution of the field equation and their physical aspects in absence of electromagnetic field. Section 6 is about entropy in the universe. Finally, in Section 7, we summarize the results.

2 The Metric and Field Equations

We consider anisotropic Bianchi type-III line element, given by

$$ds^2 = -dt^2 + A^2 dx^2 + B^2 e^{-2ax} dy^2 + C^2 dz^2, \quad (1)$$
where a is constant and the metric potentials A, B and C are functions of t alone. This ensures that the model is spatially homogeneous.

The energy-momentum tensor for a cloud of massive string with perfect fluid and electromagnetic field has the form

$$T^i_j = (\rho + p)v_iv^j + \rho g^i_j - \lambda x^i x^j + E^i_j$$

(2)

where $p$ is the isotropic pressure; $\rho$ is the rest energy density for a cloud of strings with particles attached to them; $\lambda$ is the string tension density; $v^i$ is the four-velocity of the particles, and $x^i$ is a unit space-like vector representing the direction of strings. The vectors $v^i$ and $x^i$ satisfy the conditions

$$v_i v^i = -x_i x^i = -1, \quad \gamma^i x_i = 0.$$  

(3)

In a co-moving co-ordinate system, we have

$$v^i = (0, 0, 0, 1), \quad x^i = (0, 0, 1/C, 0).$$  

(4)

In Eq. (2), $E^i_j$ is the electromagnetic field given by

$$E^i_j = \frac{1}{4\pi} \left[ g^{lm} F_{il} F_{jm} - \frac{1}{4} F_{lm} F^{lm} g_{ij} \right].$$  

(5)

We assume that electromagnetic field is in $xy$ plane. Therefore, the current is flowing along $z$ axis. Thus, $F_{12}$ is the only non-vanishing component of electromagnetic field tensor ($E^i_j$). Subsequently, Maxwell second equation

$$[F^{ik} (-g)^{\frac{1}{2}}]_{ik} = 0,$$

leads to

$$F_{12} = K e^{-\alpha z},$$

(7)

where $K$ is a constant so that electromagnetic field depends upon the space coordinate $x$ only. From Eqs. (4), (5), and (6), it follows that $F_{14} = 0$. Now the non-vanishing component of $E_{ij}$ corresponding to the line-element (1) is given by

$$E^1_1 = E^2_2 = \frac{1}{8\pi} \frac{K^2}{A^2 B^2}, \quad E^3_3 = E^4_4 = -\frac{1}{8\pi} \frac{K^2}{A^2 B^2}. $$

(8)

If the particle density of the configuration is denoted by $\rho_p$, then

$$\rho = \rho_p + \lambda.$$  

(9)

The field equations (in gravitational units $c = 1$, $8\pi G = 1$), in normal gauge for Lyra’s manifold, obtained by Sen [4] as

$$[R^i_j - \frac{1}{2} g^i_j R + \frac{3}{2} \phi^i \phi^j - \frac{3}{4} g^i_j \phi^k \phi^k] = -8\pi T^i_j,$$

(10)
where $\phi_i$ is the displacement field vector defined as
\[ \phi_i = (0, 0, 0, \beta(t)), \]  
and other symbols have their usual meaning as in Riemannian geometry.

In a co-moving co-ordinate system, the Einstein's modified field equation (6) with (2) for the metric (1) subsequently leads to the following system of equations:

\[ \frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} + \frac{\dot{D}}{D} + \frac{\dot{E}}{E} + \frac{\dot{F}}{F} + \frac{\dot{G}}{G} + \frac{\dot{H}}{H} + \frac{\dot{I}}{I} + \frac{\dot{J}}{J} + \frac{\dot{K}}{K} + \frac{\dot{L}}{L} + \frac{\dot{M}}{M} + \frac{\dot{N}}{N} + \frac{\dot{O}}{O} + \frac{\dot{P}}{P} + \frac{\dot{Q}}{Q} + \frac{\dot{R}}{R} + \frac{\dot{S}}{S} + \frac{\dot{T}}{T} + \frac{\dot{U}}{U} + \frac{\dot{V}}{V} + \frac{\dot{W}}{W} + \frac{\dot{X}}{X} + \frac{\dot{Y}}{Y} + \frac{\dot{Z}}{Z} = 0, \]  

(12)

Here, and also in what follows, a dot indicates ordinary differentiation with respect to $t$.

The energy conservation equation $T^i_{ij} = 0$ leads to
\[ \rho + (\rho + p) \left( \frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right) - \lambda \frac{\dot{A}}{A} = 0, \]  

(17)

and conservation of R. H. S. of Eq. (10) leads to
\[ \left( \frac{R_i}{A^2} - \frac{1}{2} g_{ij} R \right)_{ij} + \frac{3}{2} (\phi_i \phi^i)_{ij} - \frac{3}{4} (g_{ij} \phi_k \phi^k)_{ij} = 0. \]  

(18)

Equation (18) reduces to
\[ \frac{3}{2} \phi_i \left[ \frac{\partial \phi^j}{\partial x^j} + \phi^j \Gamma^i_{jj} \right] + \frac{3}{2} \phi_i \left[ \frac{\partial \phi^j}{\partial x^j} - \phi_i \Gamma^i_{jj} \right] - \frac{3}{4} g_{ij} \phi_i \left[ \frac{\partial \phi_k}{\partial x^j} + \phi^j \Gamma^i_{kj} \right] - \frac{3}{4} g_{ij} \phi_k \left[ \frac{\partial \phi_i}{\partial x^j} - \phi_i \Gamma^i_{kj} \right] = 0. \]  

(19)

Equation (19) is identically satisfied for $i = 1, 2, 3$.

For $i = 4$, Eq. (19) reduces to
\[ \frac{3}{2} \beta \left[ \frac{\partial (g_{4i} \phi_i)}{\partial x^4} + \phi^i \Gamma^4_{4j} \right] + \frac{3}{2} g_{4i} \phi_i \left[ \frac{\partial \phi_i}{\partial t} - \phi_i \Gamma^4_{4j} \right] - \frac{3}{4} g_{4i} \phi_i \left[ \frac{\partial \phi_i}{\partial x^4} + \phi^i \Gamma^4_{4j} \right] - \frac{3}{4} g_{4i} \phi_i \left[ \frac{\partial \phi_i}{\partial t} - \phi_i \Gamma^4_{4j} \right] = 0. \]  

(20)
which leads to
\[\frac{3}{2}\beta^2 + \frac{3}{2}\beta^2 \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right) = 0. \]
(21)

Thus, equation (17) combined with (21) is the resulting equation when the energy conservation equation is satisfied in the given system. It is important to mention here that the conservation equation in Lyra's manifold is not satisfied as in general relativity. Actually, conservation equation in Lyra’s manifold is satisfied only on giving some special condition on displacement vector \( \beta \) as shown above. This type of special condition was firstly given by Bali and Chandnani [73] and Pradhan [57].

The spatial volume for the model (1) is given by
\[V^3 = ABCe^{-ax}.\]
(22)

We define \( V = (ABCe^{-ax})^{\frac{1}{3}} \) as the average scale factor so that the Hubble’s parameter is anisotropic and may be defined as
\[H = \frac{\dot{V}}{V} = \frac{1}{3} \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right) = \frac{1}{3}(H_1 + H_2 + H_3),\]
(23)

where \( H_1 = \dot{A}/A, H_2 = \dot{B}/B \) and \( H_3 = \dot{C}/C \) are the directional Hubble parameters in the directions of \( x, y \) and \( z \) respectively.

The velocity field \( v^i \) as specified by (4) is irrotational. The deceleration parameter \( q \), the scalar expansion \( \theta \), the shear \( \sigma_{ij} \) and the average anisotropy parameter \( A_m \) are defined by
\[q = -\frac{V\dot{V}}{V^2},\]
(24)
\[\theta = \frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C},\]
(25)
\[\sigma^2 = \frac{1}{3} \left[ \frac{\dot{A}^2}{A^2} + \frac{\dot{B}^2}{B^2} + \frac{\dot{C}^2}{C^2} - \frac{\dot{A}\dot{B}}{AB} - \frac{\dot{B}\dot{C}}{BC} - \frac{\dot{C}\dot{A}}{CA}\right],\]
(26)
\[A_m = \frac{1}{3} \sum_{i=1}^{3} \left(\frac{\Delta H_i}{H}\right)^2,\]
(27)

where \( \Delta H_i = H_i - H(i = 1, 2, 3) \).

3 Solutions of the Field Equations

Equations (12)-(16) and (21) are six equations in seven unknown parameters \( \bar{A}, A, B, C, p, \rho, \lambda \) and \( \beta \). Therefore, one additional constraint relating these parameters is required to obtain explicit solutions of the system. We assume that
the expansion ($\theta$) in the model is proportional to the eigen value $\sigma^2$ of the shear tensor $\sigma^i_j$. This condition leads to the following relation between the metric potentials:

$$B = \ell_1 (AC)^{m_1},$$

(28)

where $\ell_1$ and $m_1$ are arbitrary constants. The motive behind assuming this condition is explained with reference to Thorne [84]; the observations of the velocity-red-shift relation for extragalactic sources suggest that Hubble expansion of the universe is isotropic today within $\approx 30$ per cent [85, 86]. To put more precisely, the red-shift studies place the limit

$$\frac{\sigma}{H} \leq 0.3$$

on the ratio of shear $\sigma$ to Hubble constant $H$ in the neighbourhood of our Galaxy today. Collins et al. [87] have pointed out that for spatially homogeneous metric, the normal congruence to the homogeneous expansion satisfies that the condition $\sigma/\theta$ is constant.

Equations (16) lead to

$$A = mB,$$

(29)

where $m$ is an integrating constant. Eqs. (13) and (14) reduce to

$$\frac{\dot{B}}{B} - \frac{\dot{A}}{A} + \frac{\dot{B}C}{BC} - \frac{\dot{A}C}{AC} = 0.$$  

(30)

Using (29) in (30), we obtain

$$(1 - m) \left( \frac{\dot{B}}{B} \mp \frac{\dot{B}C}{BC} \right) = 0.$$  

(31)

As $m \neq 1$, Eq. (31) gives

$$\left( \frac{\dot{B}}{B} \mp \frac{\dot{B}C}{BC} \right) = 0,$$

(32)

which on integration reduces to

$$BC = k_1,$$

(33)

where $k_1$ is an integrating constant.

From Eqs. (28) and (29), we obtain

$$B = \ell_2 C^\ell,$$

(34)

where $\ell_2 = \ell_1 \frac{1}{m_1}$, $\ell = \frac{-m_1}{1 - m_1}$. Using (34) in (33), we get

$$C^\ell \dot{C} = \frac{k_2}{\ell \ell_2},$$

(35)

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which on integration gives

\[ C = (\ell + 1)^\frac{\ell+1}{\ell+2} \left[ \frac{k_1}{\ell+2} t + k_2 \right] \frac{dt}{r^{\ell+1}}, \quad (36) \]

where \( k_2 \) is an integrating constant. Using (36) in (34) and (29), we obtain

\[ B = \ell_2(\ell + 1)^{\frac{\ell+1}{\ell+2}} \left[ \frac{k_1}{\ell+2} t + k_2 \right] \frac{dt}{r^{\ell+1}} \quad (37) \]

and

\[ A = m\ell_2(\ell + 1)^{\frac{\ell+1}{\ell+2}} \left[ \frac{k_2}{\ell+2} t + k_3 \right] \frac{dt}{r^{\ell+1}} \quad (38) \]

respectively.

Hence the metric (1) reduces to the form

\[ ds^2 = -dt^2 + \left[ m\ell_2(\ell + 1)^{\frac{\ell+1}{\ell+2}} \left( \frac{k_1}{\ell+2} t + k_2 \right) \frac{dt}{r^{\ell+1}} \right]^2 \, dx^2 \]

\[ + \left[ \ell_2(\ell + 1)^{\frac{\ell+1}{\ell+2}} e^{-\alpha t} \left( \frac{k_1}{\ell+2} t + k_2 \right) \frac{dt}{r^{\ell+1}} \right]^2 \, dy^2 \]

\[ + \left[ (\ell + 1)^{\frac{\ell+1}{\ell+2}} \left( \frac{k_1}{\ell+2} t + k_2 \right) r^{\ell+1} \right]^2 \, dz^2. \quad (39) \]

Using the suitable transformation

\[ m\ell_2(\ell + 1)^{\frac{\ell+1}{\ell+2}} x = X, \quad \ell_2(\ell + 1)^{\frac{\ell+1}{\ell+2}} y = Y, \quad (\ell + 1)^{\frac{\ell+1}{\ell+2}} z = Z, \quad (40) \]

the metric (39) reduces to

\[ ds^2 = -\alpha^2 dt^2 + T^2 e^{-\frac{\ell+1}{\ell+2} X} dX^2 + T^2 e^{\frac{\ell+1}{\ell+2} Y} dY^2 + T^2 e^{\frac{3\ell}{2}} dZ^2, \quad (41) \]

where

\[ \alpha = \frac{\ell+2}{k_1}, \quad M = (\ell + 1)^{\frac{\ell+1}{\ell+2}}, \quad N = m\ell_2 M, \quad L = \frac{\ell}{\ell+1}. \quad (42) \]

Eq. (21) gives either \( \beta = 0 \) or \( \frac{3}{2} \beta + \frac{3}{2} \beta \left( \frac{\overline{A}}{A} + \frac{\overline{B}}{B} + \frac{\overline{C}}{C} \right) = 0. \) Therefore

\[ \frac{\beta}{\overline{\beta}} = -\left( \frac{\overline{A}}{A} + \frac{\overline{B}}{B} + \frac{\overline{C}}{C} \right), \quad (43) \]

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Integrating Eq. (44), we obtain
\[ \beta = \kappa T^{-\frac{(L+1)}{a}}, \] (45)
where \( \kappa \) is an integrating constant.

Halford [4] has pointed out that the displacement field \( \phi_l \) in Lyra's manifold plays the role of cosmological constant \( \Lambda \) in the normal general relativistic treatment. From Eq. (45), it is observed that for \( \kappa > 0, L > 0 \) and \( a > 0 \), the displacement vector \( \beta(T) \) is a positive decreasing function of time and it approaches to a small positive value at late time (i.e. present epoch), which is corroborated with Halford as well as with the recent observations of SNe Ia. Recent cosmological observations of SNe Ia [88–95] suggest the existence of a positive cosmological constant \( \Lambda \) with the magnitude \( \Lambda(Gh/c^3) \approx 10^{-123} \). These observations on magnitude and red-shift of type Ia supernova suggest that our universe may be an accelerating one with induced cosmological density through the cosmological \( \Lambda \)-term. But this does not rule out the decelerating ones which are also consistent with these observations [96]. Thus the nature of \( \beta(T) \) in our derived model is supported by recent observations.

4 Some Physical and Geometric Properties of the Model

The expressions for the isotropic pressure (\( p \)), the proper energy density (\( \rho \)), the string tension (\( \lambda \)) and the particle density (\( \rho_p \)) for the model (41) are given by

\[ p = \frac{L^2(1 + \ell + \ell^2)}{\ell \alpha^2 T^2} \frac{L \ell (\ell + 1)}{N^2 T^2 L} - \frac{m^2 K^2}{N^4 T^4 L} - \frac{3}{4} \kappa T^{-\frac{(L+1)}{a}}, \quad (46) \]

\[ \rho = \frac{\ell (\ell + 2) L^2}{\ell \alpha^2 T^2} + \frac{\alpha^2}{N^2 T^2 L} - \frac{m^2 K^2}{N^4 T^4 L} + \frac{3}{4} \kappa T^{-\frac{(L+1)}{a}}, \quad (47) \]

\[ \lambda = \frac{L(3L - 2)}{\ell \alpha^2 T^2} - \frac{L^2(1 + \ell + \ell^2) - L \ell (\ell + 1)}{N^2 T^2 L} \frac{\alpha^2}{\ell^2 T^2} - \frac{2m^2 K^2}{N^4 T^4 L} + \frac{m^2 K^2}{N^4 T^4 L}, \quad (48) \]

\[ \rho_p = \frac{L^2(1 + \ell)}{\ell \alpha^2 T^2} \frac{L^2(3L - 2)}{\alpha^2 T^2} + \frac{a^2}{\ell^2 T^2} \frac{2m^2 K^2}{N^2 T^2 L} + \frac{m^2 K^2}{N^4 T^4 L} + \frac{3}{4} \kappa T^{-\frac{(L+1)}{a}}. \quad (49) \]

From Eqs. (47) and (49) we observe that the energy conditions \( \rho \geq 0, \rho_p \geq 0 \) are satisfied under conditions

\[ \frac{1}{T^2(1-\theta)} \left[ a^2 - \frac{m^2 K^2}{N^2 T^2 L} \right] \geq \frac{N^2 L^2 (\ell + 2)}{\ell \alpha^2}, \quad (50) \]
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Figure 1. The plot of isotropic pressure $p$ vs. $T$ and $L$. Here $\kappa = 2$, $\alpha = 0.1$, $a = 0.01$, $N = 10$, $m = 2$, $K = 0.1$.

and

$$\frac{1}{T^{2(L-1)}} \left[ 2\alpha^2 + \frac{m^2K^2}{N^2T^2L} + \frac{3\kappa^2N^2}{4T^{L-1}(L(1-\alpha)+1)} \right] \geq \frac{LN^2}{\ell^2c^2} \left( L(3\ell^2 + \ell - 1) + \ell(1 - \ell) \right). \quad (51)$$

We also observe that string tension density $\lambda \geq 0$ under conditions

$$\frac{1}{T^{2(L-1)}} \left( \alpha^2 + \frac{2m^2K^2}{N^2T^2L} \right) \leq \frac{LN^2}{\ell^2c^2} \left( L(2\ell^2 - \ell - 1) + \ell(1 - \ell) \right). \quad (52)$$

Figure 1 depicts isotropic pressure $p$ versus time $T$. From this figure, it is observed that $p > 0$ when $L \leq 0.5$ and $p < 0$ when $L > 0.5$. But we also see from the figure that in both situations the pressure tends to zero at late time.

From Eq. (47), it is noted that the proper energy density $\rho(t)$ is a positive decreasing function of time and it approaches a small positive value at present epoch. This behaviour is clearly depicted in Figures 2 as a representative case with appropriate choice of constants of integration and other physical parameters using reasonably well known situations.

From Eq. (48), it is found that the tension density $\lambda$ is negative. From Figure 3, it is observed that $\lambda$ is an increasing function of time but it is negative. $\lambda$ may be
Figure 2. The plot of energy density $\rho$ vs. $T$ and $L$. Here $\kappa = 2$, $\alpha = 0.1$, $a = 0.01$, $N = 10$, $m = 2$, $K = 0.1$.

Figure 3. The plot of particle density $\rho_p$ vs. $T$ and $L$. Here $\kappa = 2$, $\alpha = 0.1$, $a = 0.01$, $N = 10$, $m = 2$, $K = 0.1$. 
only positive under condition (52). It is pointed out by Letelier [13] that $\lambda$ may be positive or negative. When $\lambda < 0$, the string phase of the universe disappears, i.e. we have an anisotropic fluid of particles.

From Eq. (49), it can be seen that the particle density $\rho_p$ is a decreasing function of time and $\rho_p > 0$ under condition (51). This nature of $\rho_p$ is clearly shown in Figure 4.

All the physical quantities $\rho$, $\rho_0$, $\rho_p$ and $\lambda$ tend to infinity at $T = 0$ and $0$ at $T = \infty$. The model (41) therefore starts with a big-bang at $T = 0$ and it goes on expanding until it comes to rest at $T = \infty$. We also note that $T = 0$ and $T = \infty$ respectively correspond to the proper time $t = -\ell_2 k_2 / k_1$ and $t = \infty$. There is a point type singularity (MacCallum [97]) in the model at $T = 0$.

For early universe, i.e., when $T \to 0$, we note that when $L > 1$

$$\frac{\rho_p}{\lambda} = -\frac{1}{2},$$

and when $L < 1$

$$\frac{\rho_p}{\lambda} = \frac{L(-3\ell^2 - \ell + 1) + \ell(\ell - 1)}{L(2\ell^2 - \ell - 1) + \ell(1 - \ell)}.$$  

From Eqs. (48) and (49), we obtain the same relation as given by (54) when
$T \to \infty$, $L > 1$ and $\alpha > 0$. But for $T \to \infty$, $L > 1$ and $\alpha < 0$, we obtain

$$\frac{\rho_p}{\lambda} = 0.$$  \hspace{1cm} (55)

According to Refs. (see Kibble [31] and Krori et al. [98]), when $\rho_p/|\Lambda| > 1$, in the process of evolution, the universe is dominated by massive strings, and when $\rho_p/|\Lambda| < 1$, the universe is dominated by the strings. From Eq. (53) we observe that (for $L > 1$) when $T \to 0$, $\rho_p/|\Lambda| < 1$ implies that the universe is dominated by strings. We also note that when $T \to \infty$, for $L < 1$ and $\alpha > 0$, we obtain $\rho_p/\Lambda = -2$, which is less than 1 and hence the universe is dominated by the strings in this case.

Also from Eq. (54) we note that either for $(L < 1$ and $T \to 0)$ or for $(L > 1$ and $T \to \infty)$, with $\alpha > 0$ we obtain $\rho_p/|\Lambda| > 1$, when $L < \frac{2\ell(\ell - 1)}{5\ell^2 - 2}$.

Thus, in these cases, the universe is dominated by massive strings throughout the whole process of evolution of the universe at early as well as late time. But in these cases, when $L > \frac{2\ell(\ell - 1)}{5\ell^2 - 2}$, we obtain $\rho_p/|\Lambda| < 1$, i.e., the universe is dominated by strings in early as well as at late time.

The expressions for the scalar of expansion $\theta$, magnitude of shear $\sigma^2$, the average anisotropy parameter $A_m$, deceleration parameter $q$ and proper volume $V$ for the model (41) are given by

$$\theta = \frac{(2\ell + 1)L}{6\ell T},$$

$$\sigma^2 = \frac{1}{3} \left( \frac{(\ell - 1)L}{6\alpha T} \right)^2,$$

$$A_m = 2 \left( \frac{\ell - 1}{2\ell + 1} \right)^2,$$

$$q = -\frac{\ell \beta}{(2\ell + 1)},$$

$$V = \frac{N^2}{M} \frac{1}{\ell T} \frac{L(2\ell + 1)}{4}.$$  \hspace{1cm} (60)

The rate of expansion $H_i$ in the direction of $x$, $y$, and $z$ is given by

$$H_1 = H_2 = \frac{L}{\beta T},$$

$$H_3 = \frac{L}{\ell \beta T}.$$  \hspace{1cm} (62)

Hence the average generalized Hubble's parameter is given by

$$H = \frac{L(2\ell + 1)}{3\ell \beta T}.$$  \hspace{1cm} (63)
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From the above results, it can be seen that the spatial volume is zero at $T = 0$ and it increases with the increase of $T$. This shows that the universe starts evolving with zero volume at $T = 0$ and expands with cosmic time $T$. From equations (61) and (62), we observe that all the three directional Hubble parameters are zero at $T \to \infty$. The model has the point-type singularity at $T = 0$. The shear scalar diverges at $T = 0$. As $T \to \infty$, the scale factors $A(t), B(t)$ and $C(t)$ tend to infinity. The energy density becomes zero as $T \to \infty$. The expansion scalar and shear scalar all tend to zero as $T \to \infty$. The mean anisotropy parameter is uniform throughout the whole expansion of the universe when $\ell \neq -\frac{1}{2}$ but for $\ell = -\frac{1}{2}$ it tends to infinity. This shows that the universe is expanding with the increase of cosmic time but the rate of expansion and shear scalar decreases to zero and tends to isotropic. At the initial stage of expansion, when $\rho$ is large, the Hubble parameter is also large and with the expansion of the universe $H, \theta$ decreases as does $\rho$. Since $\sigma^2/\theta^2 = \text{const}$ provided $\ell \neq -\frac{1}{2}$, the model does not approach isotropy at any time. The cosmological evolution of Bianchi type-III space-time is expansionary, with all the three scale factors monotonically increasing function of time. The dynamics of the mean anisotropy parameter depends on the value of $\ell$.

From (59) we observe that

\[ (i) \quad \text{for } \ell < -\frac{1}{2}, \quad q > 0, \]

i.e., the model is decelerating and

\[ (ii) \quad \text{for } \ell > -\frac{1}{2}, \quad q < 0, \]

i.e., the model is accelerating. Thus this case implies an accelerating model of the universe. Recent observations of type Ia supernovae [88–95] reveal that the present universe is in accelerating phase and the deceleration parameter lies somewhere in the range $-1 < q < 0$. It follows that our model of the universe is consistent with the recent observations.

5 Solutions of the Field Equations and their Physical Aspects in Absence of Electromagnetic Field

When $K \to 0$, we get the solution of the Einstein field equations (12)–(16) in absence of electromagnetic field. In this case the expressions for the isotropic pressure ($p$), the proper energy density ($\rho$), the string tension ($\sigma$) and the particle density ($\rho_p$) for the model (41) are given by

\[ p = \frac{L^2(1 + \ell + \ell^2) - L\ell(\ell + 1)}{\ell^2a^2T^2} - \frac{3}{4}k^2T^{-\frac{2L+1}{\ell}}, \]  

(64)
\[ \rho = -\frac{\ell(\ell + 2) L^2}{T^2 \alpha^2 T^2} + \frac{3}{4} \kappa^2 T \frac{2(\ell + 1)}{\alpha}, \quad (65) \]
\[ \lambda = \frac{L(3L - 2)}{\alpha^2 T^2} - \frac{L^2 (1 + \ell + \ell^2)}{\alpha^2 T^2} - \frac{\ell \ell (\ell + 1)}{N^2 T^2}, \quad (66) \]
\[ \rho_p = \frac{L^2 (1 - \ell) - L \ell (\ell + 1)}{\alpha^2 T^2} + \frac{2}{N^2 T^2} + \frac{3}{4} \kappa^2 T \frac{2(\ell + 1)}{\alpha}. \quad (67) \]

From Eqs. (65) and (67) we observe that the energy conditions \( \rho \geq 0, \rho_p \geq 0 \) are satisfied under conditions

\[ \frac{1}{T^2 (L-1)} \left[ \alpha^2 + \frac{3 \kappa^2 N^2}{4T^2 \alpha (L-1)} \right] \geq \frac{N^2 L^2 (L + 2)}{\alpha^2 T^2}, \quad (68) \]

and

\[ \frac{1}{T^2 (L-1)} \left[ 2 \alpha^2 + \frac{3 \kappa^2 N^2}{4T^2 \alpha (L-1)} \right] \geq \frac{L N^2}{\alpha^2 T^2} \left[ L(3 \ell^2 + \ell - 1) + \ell(1 - \ell) \right]. \quad (69) \]

We also observe that string tension density \( \lambda \geq 0 \) under conditions

\[ T \leq \frac{\ell \alpha}{\sqrt{N \sqrt{L(3 \ell^2 - 2) - 1}}}. \quad (70) \]

All the physical quantities \( p, \rho, \rho_p \) and \( \lambda \) tend to infinity at \( T = 0 \) and \( T = \infty \). The model (41) therefore starts with a big-bang at \( T = 0 \) and it goes on expanding until it comes to rest at \( T = \infty \). We also note that \( T = 0 \) and \( T = \infty \) respectively correspond to the proper time \( t = t_0 \) and \( t = \infty \), where \( t_0 = -k_0 \alpha \). There is a point type singularity (MacCallum [96]) in the model at \( T = 0 \). Both \( \rho_p \) and \( \lambda \) tend to zero asymptotically.

From Eq. (64) and Figure 5, we observe that isotropic pressure \( p \) has the same character as in previous case. From Eq. (65), we note that \( \rho(t) \) is a decreasing function of time and \( \rho > 0 \) under condition (68). This behaviour is clearly depicted in Figure 6.

From Eq. (67), we note that the particle density \( \rho_p \) is a decreasing function of time and \( \rho_p > 0 \) under condition (69). This nature of \( \rho_p \) is clearly shown in Figure 7. It is worth mentioning here that \( \rho_p > 0 \) for \( L > 0.5 \) as demonstrated by the figure. From Figure 8, we see that the string tension density \( \lambda \) is negative. It can only be positive under condition (70).

For early universe, \( i.e., \, T \to 0 \) we note that for \( L > 1, \, \rho_p/\lambda = -2 \), and for \( L < 3 \), we obtain the same relation as given by (54). It is also observed after detailed study that the same characteristics for \( \rho_p/\lambda \) has been found out as already discussed in the previous section in presence of electromagnetic field. So it is not reported here again.
Figure 5. The plot of isotropic pressure $p$ vs. $T$ and $L$. Here $\kappa = 2$, $\alpha = 0.1$, $\alpha = 0.01$, $N = 10$.

Figure 6. The plot of energy density $\rho$ vs. $T$ and $L$. Here $\kappa = 2$, $\alpha = 0.1$, $\alpha = 0.01$, $N = 10$. 
Figure 7. The plot of tension density $\lambda$ vs. $T$ and $Li$. Here $\kappa = 2$, $\alpha = 0.1$, $\sigma = 0.01$, $N = 10$.

Figure 8. The plot of particle density $\rho_p$ vs. $T$ and $L$. Here $\kappa = 2$, $\alpha = 0.1$, $\sigma = 0.01$, $N = 10$. 

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6 Entropy in the Universe

In this section we discuss entropy in our derived universe. In thermodynamics, the expression for entropy is given by

$$dS = d(pV^3) + p(dV^3),\quad (71)$$

where $V^3 = ABCe^{-ax}$ is the proper volume in our case. To solve the entropy problem of the standard model, it is necessary to treat $dS > 0$ for at least a part of evolution of the universe. Hence Eq. (71) reduces to

$$TdS = \rho + (\rho + p)\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right) - \frac{\dot{A}}{A} > 0.\quad (72)$$

The conservation equation $T^I_{\cdot,\cdot} = 0$ for (1) leads to

$$\dot{\beta} + (\rho + p)\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right) - \frac{3}{2}\beta^2 - \frac{3}{2}\beta^2\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right) = 0.\quad (73)$$

Therefore, Eqs. (72) and (73) lead to

$$\frac{3}{2}\beta^2 + \frac{3}{2}\beta^2\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right) < 0,\quad (74)$$

which gives $\beta < 0$. Since for a physical model of the universe $\beta(t) > 0$, thus we observe that the entropy affects the displacement vector because the entropy $dS > 0$ leads to $\beta(t) < 0$. It is remarkable to mention here that we obtain the displacement vector $\beta(t)$ in our derived universe as decreasing function of time and always positive but entropy affects the displacement vector.

7 Concluding Remarks

A spatially homogeneous and anisotropic Bianchi type-III perfect fluid cosmological models representing massive strings in normal gauge of Lyra’s manifold in presence and absence of electromagnetic field have been studied in this paper. We have presented an alternative and straightforward approach to solve the Einstein’s typical, non-linear field equations by considering the expansion in the model is proportional to the shear as Collins et al. [87] have shown that the normal congruence to the homogeneous expansion satisfies that the condition $\kappa = \text{constant}$. In presence and absence of magnetic field the derived models start with a big bang singularity. All the physical quantities $\rho$, $\rho_p$, and $\lambda$ tend to infinity at $T = 0$ and $0$ at $T = \infty$. We have obtained a Point Type singularity in both models. In general, the models represent an expanding, shearing and non-rotating universe.
It is observed that the displacement vector $\beta(t)$ matches with the nature of the cosmological constant $\Lambda$ which has been supported by the work of several authors as discussed in the physical behaviour of the models in Sections 4 and 5. In recent time $\Lambda$-term has attracted theoreticians and observers for many a reason. The nontrivial role of the vacuum in the early universe generates a $\Lambda$-term that leads to inflationary phase. Observationally, this term provides an additional parameter to accommodate conflicting data on the values of the Hubble constant, the deceleration parameter, the density parameter and the age of the universe (for example, see Refs. [99] and [100]). Assuming that $\Lambda$ owes its origin to vacuum interaction, as suggested particularly by Sakharov [101], it follows that it would, in general, be a function of space and time coordinates, rather than a strict constant. In a homogeneous universe $\Lambda$ will be at most time dependent [102]. In recent past there was an upsurge of interest in scalar fields in general relativity and alternative theories of gravitation in the context of inflationary cosmology [103-105]. Therefore the study of cosmological models in Lyra's manifold may be relevant for inflationary models. There seems a good possibility of Lyra’s manifold to provide a theoretical foundation for relativistic gravitation, astrophysics and cosmology. However, the importance of Lyra’s manifold for astrophysical bodies is still an open question. In fact, it needs a fair trial for experiment.

Acknowledgement

The first author (A. Pradhan) would like to thank the Institute of Mathematical Sciences (IMSc.), Chennai, India for providing facility and hospitality under associateship scheme where part of this work was done.

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Viscous Fluid Cosmological Models in Bianchi Type-I Space-Time

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Abstract—Exact solutions of Einstein’s field equations are obtained in a spatially homogeneous and anisotropic Bianchi type-I space-time in presence of a dissipative fluid. Following the technique (Pradhan & Amirhashchi in Mod. Phys. Lett. A 26: 2261 2011), we consider the scale factor $a(t) = \sqrt{t^2 + 1}$, which yield time dependent DP. To get the deterministic solution a barotropic equation of state together with the shear viscosity is proportional to expansion scalar, is also assumed. It is observed that the initial nature of singularity is not changed due to the presence of viscous fluid. The basic equation of thermodynamics have been deduced and the thermodynamic aspects of the models have been discussed. For different option of $n$, we can generate a class of bulk viscous models in Bianchi type-I space-time. The physical and geometric properties of cosmological models are also discussed.

Index Terms—Bianchi type-I universe, Viscous fluid, Variable deceleration parameter

I. INTRODUCTION

Anisotropic cosmological models play significant role in understanding the behaviour of the universe at its early stages of evolution. Observations by the Differential Radiometers on NASA’s Cosmic Background Explorer registered anisotropy in various angle scales. The theoretical argument [1] and the modern experimental data support the existence of an anisotropic phase, which turns into an isotropic one. The anisotropy of the universe within the framework of Bianchi type space-times is found to be useful in generating arbitrary ellipsoidiality of the Universe, and to fine tune the observed CMBR anisotropies. Koivisto and Mota [2], [3] have investigated cosmological models with anisotropic EoS and have also shown that the present SN Ia data allows large anisotropy. The motivation for this investigation comes from the hints of statistical anisotropy of our Universe that several observations seem to suggest. Recently, Akarsu and Kilinc [4] have described some features of the Bianchi type-I universes in presence of fluid that yields an anisotropic EoS. Hence, for a realistic cosmological model one should consider spatially homogeneous and anisotropic space-times and then show whether they can evolve to the observed amount of homogeneity and isotropy. The only spatially homogeneous but anisotropic models other than Bianchi type models are the Kantowski-Sachs locally symmetric family. See Ellis & van Elst [5] for generalized, particularly anisotropic, cosmological models and Ellis [6] for a concise review on Bianchi type models. Recently Pradhan and Pandey [7], [8], Pradhan and Singh [9], [10], Pradhan and Choiouhan [11], Pradhan et al. [12], Pradhan [13], Bali et al. [14], Singh et al. [15], Singh and Kumar [16], [17], Kumar and Singh [18], Saha [19—21] and Yadav et al. [22] have studied Bianchi type-I cosmological models in different context.

The study of role of dissipative effects in the evolution of the universe during early stages has taken considerable interest of researchers as in the early universe viscosity may arise due to various processes such as decoupling of neutrinos during the radiation era, creation of superstring during the quantum era, particle collision involving gravitation, particle creation process and the formation of galaxies [1], [23], [24]. To describe the relativistic theory of viscosity, Eckart [25] made the first attempt. The theories of dissipation in Eckart formulation suffers from serious short-coming viz., causality and stability regardless of the choice of equation of state. Israel and Stewart [26] and Pavon [27] developed a fully relativistic formulation of the theory taking into account second order deviation terms in the theory, which is termed as “transient” or “extended” irreversible thermodynamics (EIT). In irreversible thermodynamics, the entropy is no longer conserved, but grows according to the second law of thermodynamics. Bulk viscosity arises typically in the mixtures either of different species or of species but with different energies. The solution of the full causal theory are well behaved for all time. Several authors (Singh and Kale [28] and Yadav [29] and references therein) have investigated cosmological models with dissipative effect. Pradhan [30—33], Pradhan et al. [34], Rao et al. [35], Singh et al. [36], Bali and Pradhan [37] have studied bulk viscous fluid cosmological models in different physical context.

With these motivations and following the technique of Pradhan and Amirhashchi [38], in this paper, we have investigated a new spatially homogeneous and anisotropic Bianchi type-I cosmological model by considering a dissipative fluid. The Einstein’s field equations are solved explicitly. The out line of the paper is as follows: In Sect. 2, the metric and the field equations are described. Section 3
II. THE METRIC AND FIELD EQUATIONS

We consider totally anisotropic Bianchi type-I metric in the form
\[ ds^2 = -dt^2 + A^2(t)dx^2 + B^2(t)dy^2 + C^2(t)dz^2, \tag{1} \]
where the metric potentials \( A, B \) and \( C \) are functions of cosmic time \( t \) alone. This ensures that the model is spatially homogeneous.

We define the following parameters to be used in solving Einstein’s field equations for the metric (1).

The average scale factor \( a \) of Bianchi type-I model (1) is defined as
\[ a = (ABC)^{1/3}, \tag{2} \]
A volume scale factor \( V \) is given by
\[ V = a^3 = ABC. \tag{3} \]
In analogy with FRW universe, we also define the generalized Hubble parameter \( H \) and deceleration parameter \( q \) as
\[ H = \frac{\dot{a}}{a} = \frac{1}{3} \left( \frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right), \tag{4} \]
\[ q = -\frac{\ddot{a}}{\dot{a}^2} = -\frac{\ddot{a}}{aH^2}, \tag{5} \]
where an over dot denotes derivative with respect to the cosmic time \( t \).

Also we have
\[ H = \frac{1}{3}(H_1 + H_2 + H_3), \tag{6} \]
where \( H_1 = \frac{\dot{A}}{A} \), \( H_2 = \frac{\dot{B}}{B} \) and \( H_3 = \frac{\dot{C}}{C} \) are directional Hubble factors in the directions of \( x- \), \( y- \) and \( z- \) axes respectively.

The Einstein’s field equations (in gravitational unit \( 8\pi G = c = 1 \)) are given by
\[ R_{ij} - \frac{1}{2}Rg_{ij} = -T_{ij}, \tag{7} \]
where \( T_{ij} \) is the stress energy tensor of matter which, in case of viscous fluid, has the form [39]
\[ T_{ij} = (\rho + p)u_iu_j + p\delta_{ij} - \eta\mu_{ij}, \tag{8} \]
with
\[ \dot{p} = p - \left( \xi - \frac{2}{3}\eta \right)u^3 - p(3\xi - 2\eta)H \tag{9} \]
and
\[ \mu_{ij} = u_{ij} + u_{ji} + u_iu^a u_{aj} + u_ju^a u_{ia}. \tag{10} \]
In the above equations, \( \xi \) and \( \eta \) stand for the bulk and shear viscosity coefficients respectively; \( \rho \) is the matter density; \( p \) is the isotropic pressure and \( u^i \) is the four-velocity vector satisfying \( u^iu_i = -1 \).

In a co-moving coordinate system, where \( u^i = \delta^i_0 \), the field equations (7), for the anisotropic Bianchi type-I space-time (1) and viscous fluid distribution (8), yield
\[ \frac{\dot{B}}{B} + \frac{\dot{C}}{C} + \frac{\dot{BC}}{BC} = -\ddot{\rho} + 2\eta \frac{\dot{A}}{A}, \tag{11} \]
\[ \frac{\dot{C}}{C} + \frac{\dot{A}}{A} + \frac{\dot{CA}}{CA} = -\ddot{\rho} + 2\eta \frac{\dot{B}}{B}, \tag{12} \]
\[ \frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{AB}}{AB} = -\ddot{\rho} + 2\eta \frac{\dot{C}}{C}, \tag{13} \]
\[ \frac{\dot{AB}}{AB} + \frac{\dot{BC}}{BC} + \frac{\dot{CA}}{CA} = \rho. \tag{14} \]

The usual definitions of the dynamical scalars such as the expansion scalar \( (\theta) \) and the shear scalar \( (\sigma) \) are considered to be
\[ \theta = u^i_i = \frac{3\dot{a}}{a}, \tag{15} \]
and
\[ \sigma^2 = \frac{1}{2}\sigma_{ij}\sigma^{ij} = \frac{1}{2} \left( \frac{\dot{A}}{A} \right)^2 + \left( \frac{\dot{B}}{B} \right)^2 + \left( \frac{\dot{C}}{C} \right)^2 - \frac{1}{6}\theta^2, \tag{16} \]
where
\[ \sigma_{ij} = u_{ij} + \frac{1}{3}(u_{ik}u^k u_{ij} + u_{jk}u^j u_{ik}) + \frac{1}{3}\theta(u_{ij} + u_i u_j). \tag{17} \]
The anisotropy parameter \( (A_m) \) is defined as
\[ A_m = \frac{1}{3} \sum_{i=1}^{3} \left( \frac{H_i - H}{H} \right)^2. \tag{18} \]
The energy conservation equation \( T^i_{ij} = 0 \), leads to
\[ \dot{\rho} = -(p + \rho)\theta + \xi\theta^2 + 4\eta\theta^2. \tag{19} \]
It follows from (19) that for contraction, that is, \( \theta < 0 \), we have \( \dot{\rho} > 0 \) so that the matter density increases or decreases depending on whether the viscous heating is greater or less than the cooling due to expansion.

III. SOLUTION OF FIELD EQUATIONS

We have a system of four independent equations (11) – (14) and seven unknown variables, namely \( A, B, C, p, \rho, \xi \) and \( \eta \). So for complete determinacy of the system, we need three appropriate relations among these variables that we shall consider in the following section and solve the field equations.

We follow the approach of Saha [19] to solve the field equations (11)–(14). Subtracting (11) from (12), (11) from (13), (12) from (13) and taking second integral of each, we get the following three relations
\[ \frac{A}{B} = d_1 \exp \left( x_1 \int e^{-3}e^{-2}f ndt dt \right), \tag{20} \]
\[ \frac{A}{C} = d_2 \exp \left( x_2 \int e^{-3}e^{-2}f ndt dt \right), \tag{21} \]
\[ \frac{B}{C} = d_3 \exp \left( x_3 \int e^{-3}e^{-3}f ndt dt \right), \tag{22} \]
is an arbitrary constant and the constant of integration is absorbed in $t$ without any loss of generality.

From (5) and (27), we get the time varying DP as

$$q(t) = \frac{2n}{(n+t)^2} - 1.$$  

(28)

From Eq. (28), we observe that $q > 0$ for $t < \sqrt{2n} - n$ and $q < 0$ for $t > \sqrt{2n} - n$. It is observed that for $0 < n < 2$, our model is evolving from deceleration phase to acceleration phase. Also, recent observations of SNe Ia, expose that the present universe is accelerating and the value of DP lies to some place in the range $-1 < q < 0$.

It follows that in our derived model, one can choose the value of DP consistent with the observation. Figure 1 graphs the deceleration parameter $q$ versus time which gives the behaviour of $q$ from decelerating to accelerating phase for different values of $n$. Thus our derived model has accelerated expansion at present epoch which is consistent with recent observations of Type Ia supernova (Riess et al. [40]; Perlmutter et al. [41]) and CMB anisotropies (Bennett et al. [42]; de Bernardis et al. [43]; Hanany et al. [44]).

Next, we assume that the coefficient of shear viscosity ($\eta$) is proportional to the expansion scalar ($\theta$) i.e. $\eta \propto \theta$, which leads to

$$\eta = \eta_0 \theta,$$

(29)

where $\eta_0$ is proportionality constant. Such relation has already been proposed in the physical literature as a physically plausible relation [9].

Finally to conveniently specify the source, we assume the perfect gas equation of state, which may be written as

$$p = \gamma \rho, 0 \leq \gamma \leq 1.$$  

(30)

Using Eqs. (15), (27) and (29) into (23)–(25), we get the following expressions for the scale factors

$$A = a_1 (t^n e^t)^{\frac{1}{3}} \exp \left( b_1 \int a^{-3} e^{-2 f n dt} dt \right),$$

(31)

$$B = a_2 (t^n e^t)^{\frac{1}{3}} \exp \left( b_2 \int a^{-3} e^{-2 f n dt} dt \right),$$

(32)

$$C = a_3 (t^n e^t)^{\frac{1}{3}} \exp \left( b_3 \int a^{-3} e^{-2 f n dt} dt \right).$$

(33)

The physical parameters such as directional Hubble factors ($H_i$), Hubble parameter ($H$), expansion scalar ($\theta$), spatial volume ($V$), anisotropy parameter ($A_m$) and shear scalar ($\sigma$) are given by

$$H_i = \frac{1}{2} \left( \frac{n}{t} + 1 \right) + b_i (t^n e^t)^{-\frac{1}{3} (1+2\gamma)})$$

(34)

$$H = \frac{1}{2} \left( \frac{n}{t} + 1 \right),$$

(35)

$$\theta = \frac{3}{2} \left( \frac{n}{t} + 1 \right),$$

(36)

$$V = (t^n e^t)^{\frac{3}{2}},$$

(37)

$$A_m = \frac{1}{3} \beta^2 (t^n e^t)^{-\frac{3}{2} (1+2\gamma)}. $$

(38)
Figure 2. The plot of anisotropy parameter $A_m$ versus $t$ for $\beta_1 = 1$, $\eta_0 = 0.1$

$$A_m = \begin{cases} \text{--} & n = 0.5 \\ \text{--} & n = 1 \\ \text{--} & n = 1.5 \end{cases}$$

Figure 3. The plot of energy density $\rho$ versus $t$ for $\beta_2 = 1$, $\eta_0 = 0.1$

$$\sigma^2 = \frac{1}{2} \beta_1 (t^{\alpha} e^t)^{-3(1+2\eta_0)}, \quad (39)$$

where

$$\beta_1 = b_1^2 + b_2^2 + b_3^2.$$ 

The energy density and effective pressure of the model read as

$$\rho = \frac{3}{4} \left( \frac{n}{t} + 1 \right)^2 + \beta_2 (t^{\alpha} e^t)^{-3(1+2\eta_0)}, \quad (40)$$

$$\bar{p} = \frac{n}{t^2} \frac{3}{4} (1-2\eta_0) \left( \frac{n}{t} + 1 \right)^2 - \beta_3 (t^{\alpha} e^t)^{-3(1+2\eta_0)}, \quad (41)$$

where

$$\beta_2 = b_1 b_2 + b_1 b_3 + b_2 b_3,$$

$$\beta_3 = b_1^2 + b_2^2 + b_3^2.$$ 

The expressions for isotropic pressure, bulk viscosity and shear viscosity are given by

$$p = \frac{3}{4} \gamma \left( \frac{n}{t} + 1 \right)^2 + \beta_2 \gamma (t^{\alpha} e^t)^{-3(1+2\eta_0)}, \quad (42)$$

$$\xi = \frac{1}{2} (1+\gamma) \left( \frac{n}{t} + 1 \right) - \frac{2n}{3} \left( \frac{n}{t} + 1 \right) \frac{2}{3} (\beta_2 \gamma + \beta_3) \left( \frac{t}{n+t} \right)^{-3(1+\gamma)}, \quad (43)$$

$$\eta = \frac{3}{2} \left( \frac{n}{t} + 1 \right). \quad (44)$$

It is already observed that the above set of solutions satisfy the energy conservation equation (19) identically. Therefore, the above solutions are exact solutions of Einstein's field equations (11)—(14). From Eqs. (36) and (37), we observe that the spatial volume is zero at $t = 0$ and the expansion scalar is infinite, which shows that the universe starts evolving with zero volume at $t = 0$ which is big bang scenario. From Eqs. (31)—(33), we see that the spatial scale factors are zero at the initial epoch $t = 0$ and hence the model has a point type singularity [45]. All the physical quantities pressure ($p$), energy density ($\rho$), bulk viscosity ($\xi$), shear viscosity ($\eta$), Hubble factor ($H$) and shear scalar ($\sigma$) diverge at $t = 0$. As $t \to \infty$, scale factors and volume becomes infinite whereas $p$, $\rho$ approach to zero.

Figure 2 depicts the variation of anisotropic parameter $A_m$ versus time $t$. From the figure, we observe that $A_m$ decreases with time and tends to zero as $t \to \infty$. Thus, the observed isotropy of the universe can be achieved in our derived model at present epoch.

From Eqs. (40) and (42) we observed that the energy density $\rho$ and isotropic pressure $p$ are always positive.
and decreasing functions of time. Figure 3 describes the variation of energy density \( p \) with cosmic time \( t \). From this figure we observe that \( p \) is a positive decreasing function of time.

Figure 4 plots the variation of bulk viscous coefficient \( \xi \) versus time \( t \). From this figure we observe that \( \xi \) is a positive decreasing function of time and it approaches to a constant quantity which is near to zero. This is in good agreement with physical behaviour of \( \xi \).

From Eqs. (36) and (39), we get

\[
\frac{\sigma^2}{\beta^2} = \frac{2}{9} \beta_1 \left( \frac{t}{t+n+1} \right)^2 (t^n e^t)^{-3(1+2\eta_0)}.
\]

(45)

The above equation shows that the ratio of shear and expansion scalars is a decreasing function of time and decays to zero as \( t \to \infty \). Hence the model approaches to isotropy for large value of time i.e. at present epoch. Also we observe that the rate of decay falls in the absence of shear viscosity.

From (36) and (40), one can get the following relation

\[
\frac{\rho}{\beta^2} = \frac{1}{3} + \frac{4}{9} \beta_2 \left( \frac{t}{t+n+1} \right)^2 (t^n e^t)^{-3(1+2\eta_0)}.
\]

(46)

The above relation shows that \( \xi \) is maximum at \( t = \infty \) and the maximum value is \( \frac{1}{3} \). It is important to note here that \( \lim_{t \to 0} \left( \frac{\xi}{\beta^2} \right) \) spread out to be constant. Therefore the model of the universe goes up homogeneity and matter is dynamically negligible near the origin. This is in good agreement with the result already given by Collins [46].

It is worth mentioned here that the anisotropy parameter decreases faster with time due to the presence of viscosity [see, Fig. 2, and Eq. (38)]. So it is concluded that viscosity played an important role in the process of isotropization of the large scale structure of the universe. We also conclude that the model represents shearin, non-rotating and expanding universe, which starts with a big bang and approaches to isotropy at present epoch.

IV. THERMODYNAMIC EQUATIONS

The energy in a comoving volume is \( U = \rho V \). The equation for production of entropy \( S \) in a comoving volume due to the dissipative effects in a fluid with temperature \( T \) is given by

\[
T \dot{S} = \dot{U} + p \dot{V} = 3V(3\xi + 2\eta A_m)H^2.
\]

(47)

In a cosmic fluid where the energy density and pressure of the cosmic fluid are functions of temperature only, \( p = \rho(T) \), \( p = p(T) \) and where the cosmic fluid has no net charge, we obtain easily (Grn (47))

\[
S = \frac{\rho}{T}(\rho + p).
\]

(48)

From (47) and (48), we get the following expression for the entropy production rate in viscous Bianchi type-I universe

\[
\frac{\dot{S}}{S} = \frac{3(3\xi + 2\eta A_m)H^2}{\rho + p}.
\]

(49)

For a fluid obeying the equation of state (30), (48) and (49) become

\[
S = \frac{V}{T}(1 + \gamma)\rho,
\]

(50)

\[
\frac{\dot{S}}{S} = \frac{3(3\xi + 2\eta A_m)H^2}{(1 + \gamma)\rho}.
\]

(51)

Equation (51) can be rewritten as

\[
\frac{\dot{S}}{S} = \frac{\xi + 4\eta(\sigma^2/\beta^2)}{(1 + \gamma)(\rho/\beta^2)},
\]

(52)

which gives the rate of change of entropy with time.

Let the entropy density be \( s \) so that

\[
s = \frac{S}{V} = \frac{(1 + \gamma)\rho}{T}.
\]

(53)

It defines the entropy density in terms of the temperature.

The first law of thermodynamics may be written as

\[
d(\rho V) + \gamma p dV = (1 + \gamma)Td\left(\frac{\rho V}{T}\right),
\]

(54)

which on integration, yields

\[
T \sim \rho^{(1+\gamma)}.
\]

(55)

From (53) and (55), one can get

\[
s \sim \rho^{(1+\gamma)},
\]

(56)

The entropy in a comoving volume then varies according to

\[
S \sim sV.
\]

(57)

These equations are not valid for a vacuum fluid with \( \gamma = -1 \). For a Zel’dovich fluid \((\gamma = 1)\), we get

\[
T \sim \rho^{1/2} \quad \text{and} \quad s \sim \rho^{1/2},
\]

(58)

so that the entropy density is proportional to the temperature.

Using (37) and (40), we find the respective temperature \( T \), entropy density \( s \) and total entropy \( S \) from (55)—(57), as

\[
T = T_0 \left[ \frac{3}{4} \left( \frac{n}{t} + 1 \right)^2 + \beta_2(t^n e^t)^{-3(1+2\eta_0)} \right]^{\frac{T_0+T}{T_0}}
\]

(59)

\[
s = s_0 \left[ \frac{3}{4} \left( \frac{n}{t} + 1 \right)^2 + \beta_2(t^n e^t)^{-3(1+2\eta_0)} \right]^{\frac{T_0+T}{T_0}}
\]

(60)

\[
S = S_0 \left[ \frac{3}{4} \left( \frac{n}{t} + 1 \right)^2 + \beta_2(t^n e^t)^{-3(1+2\eta_0)} \right]^{\frac{T_0+T}{T_0}}
\]

(61)

where \( T_0, s_0 \) and \( S_0 \) are positive constants.

From Eq. (61) we observe that the total entropy \( S \) increases with time.

The rate of change of entropy is obtained as

\[
\frac{\dot{S}}{S} = \left( 1 + \gamma \right) \left[ \frac{3}{4} \left( \frac{n}{t} + 1 \right)^2 + \beta_2(t^n e^t)^{-3(1+2\eta_0)} \right]^{\infty}
\]

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\[
\left( \frac{3}{8} (1 + \gamma) \left( \frac{n}{t} + 1 \right)^3 - \frac{3n(n+t)}{2t^2} + \frac{3}{2} (2\eta_0\beta_1 + \gamma\beta_2 + \beta_3) \left( \frac{n}{t} + 1 \right) (t^n e^t)^{-3(1+2m)} \right]. \tag{62}
\]

From above equation we observe that the relation \( \frac{\dot{S}}{S} > 0 \). This also implies that the total entropy increases with time. In other words, the universe is expanding faster than the rate at which entropy is produced. This is in line with the second law of thermodynamics.

V. CONCLUDING REMARKS

In this paper we have studied a spatially homogeneous and anisotropic Bianchi type-I space-time with bulk and shear viscosity in the context of general relativity. The Einstein’s field equations have been solved exactly by considering a scale factor \( a(t) = \sqrt{t^2 + \varepsilon^2} \) which yields a time dependent deceleration parameter (DP). In literature it is a plebeian practice to consider a constant deceleration parameter. Now for a Universe which was decelerating in past and accelerating at present epoch, the DP must show signature flipping. Therefore our consideration of DP to be variable is physically justified and consistent with recent observations.

For different choice of \( n \), we can generate a class of viscous fluid models in Bianchi type-I space-time. For \( n = 1 \), Eq. (27) reduces to \( a(t) = \sqrt{t^2} \) and it yields \( q(t) = \frac{t}{t^2 + \varepsilon^2} - 1 \). It is observed from Figures 1 to 4 that all the physical parameters are also in good harmony with current observations. All the values of physical parameters can be derived from their respective expressions by putting \( n = 1 \).

The derived models represent expanding, shearing and non-rotating universe, which approach to isotropy for large value of \( t \). This is consistent with the behaviour of the present universe as already discussed in introduction.

Due to dissipative processes, the mean anisotropy and the shear of the cosmological model of the universe for Bianchi type-I space-time tend to zero very rapidly in both cases. It has also been observed that shear coefficient \( \eta \) plays more important role than bulk coefficient \( \xi \) in the isotropization process of the universe. Therefore it may be possible that the isotropy observed in the present universe, is a consequence of the viscous effects in the cosmic fluid right from the beginning of the evolution of the universe.

The basic equations of thermodynamics for the present models of the universe in Bianchi type-I space-time have been derived. We have observed that the total entropy of the system increases with time. Since we know that from the state of thermodynamic equilibrium, the law deduced the principle of increase of entropy and explains the phenomenon of irreversibility in nature. Hence our results derived in the present paper are in good agreement with second law of thermodynamics.

In absence of viscosity i.e. if we set \( \xi \to 0 \) and \( \eta \to 0 \) (\( \eta_0 = 0 \)), we can obtain the solutions of Einstein’s field equations (7) for perfect fluid distribution in Bianchi type-I space-time. It is a straight forward calculation from our results described in Sect. 3, we do not need to mention here.

Finally, the exact solutions presented in this paper are new and may be useful for better understanding of the evolution of the universe in Bianchi type-I space-time with viscous effects as well as for perfect fluid distribution in general theory of gravitation.

ACKNOWLEDGMENTS

One of the authors (A. Pradhan) would like to thank the Institute of Mathematical Sciences (IMSc.), Chennai, India for providing facility & support during a visit under associateship scheme where part of this work was done.

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Viscous Fluid Cosmology with Time Dependent $q$ and $\Lambda$-term in Bianchi Type-I Space-Time and Late Time Acceleration

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Abstract

Exact solutions of Einstein's field equations are obtained in a spatially homogeneous and anisotropic Bianchi type-I space-time in presence of a dissipative fluid with constant and time dependent cosmological term $\Lambda$. Einstein's field equations are solved by considering a time dependent deceleration parameter which affords a late time acceleration in the universe. The cosmological constant $\Lambda(t)$ is found to be a decreasing function of time and it approaches a small positive value at the present epoch which is corroborated by consequences from recent supernovae Ia observations. To get the deterministic solution a barotropic equation of state together with the shear viscosity is proportional to expansion scalar, is also assumed. It is observed that the initial nature of singularity is not changed due to the presence of viscous fluid. The basic equation of thermodynamics have been deduced and the thermodynamic aspects of the models have been discussed. The physical and geometric properties of cosmological models are also discussed.

Keywords: Bianchi type-I universe, Viscous fluid, Variable deceleration parameter, Variable cosmological constant

PACS number: 98.80-k, 98.80.Cq

1 Introduction

The cosmological constant($\Lambda$) was introduced by Einstein in 1917 as the universal repulsion to make the Universe static in accordance with generally accepted picture of that time. In absence of matter described by the stress energy tensor $T_{ij}$, $\Lambda$ must be constant, since the Bianchi identities guarantee vanishing covariant divergence of the Einstein tensor, $G_{ij}^\Lambda = 0$, while $g_{ij}^\Lambda = 0$ by definition. If Hubble parameter and age of the universe as measured from high red-shift would be found to satisfy the bound $H_0 t_0 > 1$ (index zero labels values today), it would require a term in the expansion rate equation that acts as a cosmological constant. Therefore the definitive measurement of $H_0 t_0 > 1$ and wide range of observations would necessitate a non-zero cosmological constant today or the abandonment of the standard big bang cosmology [1]. However,
a constant $A$, as it was originally introduced by Einstein in 1917, cannot explain why the calculated value of vacuum energy density at Plank epoch following quantum field theory is 123 orders of magnitude larger than its value as observed or as predicted by standard cosmology at the present epoch [2]. In attempt to solve this problem, variable $A$ was introduced such that $A$ was larger in the early universe and then decayed with the evolution [3]. The idea that $A$ might be variable has been studied for more than two decades (see [4, 5] and references therein). Linde [6] has suggested that $A$ is a function of temperature and is related to the process of broken symmetries. Therefore, it could be a function of time in a spatially homogeneous, expanding universe [5]. In a paper on $A$-variability, Overduin and Cooperstock [7] suggested that $A_{ij}$ is shifted onto the right-hand side of the Einstein field equation and treated as part of the matter content. In general relativity, $A$ can be regarded as a measure of the energy density of the vacuum and can in principle lead to the avoidance of the big bang singularity that is characterized of other FRW models. However, the rather simplistic properties of the vacuum that follows from the usual form of Einstein equations can be made more realistic if that theory is extended, which in general leads to a variable $A$. Recently, Overduin [8, 9] has given an account of variable $A$-models that have a non-singular origin. Liu and Wesson [10] have studied universe models with variable cosmological constant. Podariu and Ratra [11] have examined the consequences of also incorporating constraints from recent measurements of the Hubble parameter and the age of the universe in the constant and time-variable cosmological constant models. In recent time the $A$-term has interested theoreticians and observers for various reasons.

A dynamic cosmological term $A(t)$ remains a focal point of interest in modern cosmological theories as it solves the cosmological constant problem in a natural way. There are significant observational evidence for the detection of Einstein’s cosmological constant, $\Lambda$ or a component of material content of the universe that varies slowly with time to act like $\Lambda$. In the context of quantum field theory, a cosmological term corresponds to the energy density of vacuum. The birth of the universe has been attributed to an excited vacuum fluctuation triggering off an inflationary expansion followed by the super-cooling. The release of locked up vacuum energy results in subsequent reheating. The cosmological term, which is measure of the energy of empty space, provides a repulsive force opposing the gravitational pull between the galaxies. If the cosmological term exists, the energy it represents counts as mass because mass and energy are equivalent. If the cosmological term is large enough, its energy plus the matter in the universe could lead to inflation. Unlike standard inflation, a universe with a cosmological term would expand faster with time because of the push from the cosmological term (Croswell [12]). In the absence of any interaction with matter or radiation, the cosmological constant remains a “constant”. However, in the presence of interactions with matter or radiation, a solution of Einstein equations and the assumed equation of covariant conservation of stress-energy with a time-varying $A$ can be found. This entails that energy has to be conserved by a decrease in the energy density of the vacuum component followed by a corresponding increase in the energy density of matter or radiation (see also Weinberg [13], Carroll et al. [14], Peebles [15], Sahni and Starobinsky [16], Padmanabhan [17, 18], Singh et al. [19], Pradhan and Pandey [20, 21], Pradhan and Singh [22], Pradhan et al. [23]–[26], Pradhan [27]–[29], Pradhan and Jotania [30, 31], Amirhashchi et al. [32, 33], Yadav et al. [34], Chawla et al. [35] and El-Nabulsi [36]). There is a plethora of astrophysical evidence today, from supernovae measurements (Perlmutter et al. [37, 38], Riess et al. [39]–[42], Garnavich et al. [43, 44], Schmidt et al. [45], Blakeslee et al. [46], Astier et al. [47]), the spectrum of fluctuations in the Cosmic Microwave Background (CMB) [48], baryon oscillations [49] and other astrophysical data, indicating that the expan-
sion of the universe is currently accelerating. The energy budget of the universe seems to be
dominated at the present epoch by a mysterious dark energy component, but the precise nature
of this energy is still unknown. Many theoretical models provide possible explanations for the
dark energy, ranging from a cosmological term \([50\) to super-horizon perturbations \([51, 52\) and
time-varying quintessence scenarios \([53\). These recent observations strongly favour a significant
and a positive value of \(\Lambda (GH/c^2) \approx 10^{123}\). In Refs. \([41, 42\), Riess et al. have
recently presented an analysis of 156 SNe including a few at \(z > 1.3\) from the Hubble Space
Telescope (HST) “GOOD ACS” Treasury survey. They conclude to the evidence for present
acceleration \(q_0 < 0 \) \((q_0 \approx -0.7)\). Observations (Knop et al. \([54\); Riess et al. \([41, 42\)) of Type
Ia Supernovae (SNe) allow us to probe the expansion history of the universe leading to the
conclusion that the expansion of the universe is accelerating.

Most studies in cosmology deal with a perfect fluid. Large entropy per baryon and the
remarkable degree of isotropy of the cosmic microwave background radiation, suggest that we
should analyze dissipative effects in cosmology. Further, there are several processes which are
expected to give rise to viscous effect. These are the decoupling of neutrinos during the radiation
era and the recombination era \([55\), decay of massive super string modes into massless modes
\([56\), gravitational string production \([57, 58\) and particle creation effect in grand unification era
\([59\). It is known that the introduction of bulk viscosity can avoid the big bang singularity.
Thus, we should consider the presence of a material distribution other than a perfect fluid to
have realistic cosmological models (see Grøn \([60\) for a review on cosmological models with bulk
viscosity). The first effort to create a theory of relativistic dissipative fluids, were made by
Eskart \([61\) and Landau & Lifshitz \([62\). Weinberg \([63\) established general expression for bulk
and shear viscosity, and used them to evaluate the cosmological entropy production rate. A uni­
form cosmological model filled with fluid which possesses pressure and second (bulk) viscosity
was developed by Murphy \([64\). The solutions that he found exhibit an interesting feature that
the big bang type singularity appears in the infinite past. Misner \([65\) analyzed Bianchi type-I
models in the presence of viscous term, and resolved that viscosity of neutrinos can essentially
reduce the initial anisotropy of the universe.

Anisotropic cosmological models play significant role in understanding the behaviour of
the universe at its early stages of evolution. Observations by the Differential Radiometers on
NASA's Cosmic Background Explorer registered anisotropy in various angle scales. The simplest
of anisotropic models, which, nevertheless, rather completely describe the anisotropic effects,
are Bianchi type-I (BI) homogeneous models whose spatial sections are flat but the expansion or
contraction rate is directional dependent. The advantages of these anisotropic models are that
they have a significant role in the description of the evolution of the early phase of the universe
and they help in finding more general cosmological models than the isotropic FRW models. The
isotropy of the present-day universe makes the BI model a prime candidate for studying the
possible effects of an anisotropy in the early universe on modern-day data observations. Recently,
Kalita et al. \([66\), Dey et al. \([67\), Oli \([68\) and Nourinezhad & Mehdiyoun \([69\) have studied
cosmological models in anisotropic Bianchi type space-times in different context. Motivated by
the above discussions, in this paper, we have investigated a new class of spatially homogeneous
and anisotropic Bianchi type-I cosmological models with time dependent deceleration parameter
and cosmological constant in presence of a dissipative fluid. The Einstein's field equations are
solved explicitly. The outline of the paper is as follows: In Sect. 2, the metric and the field
equations are described. Section 3 deals with the solutions of the field equations by considering time dependent deceleration parameter. In Subsect. 3.1, we obtain the solution with variable A-term and constant ξ and also discuss the thermodynamic equations. Subsection 3.2 deals with the models with variable A-term and ξ ∝ ρ and also discuss entropy production rate of model. In Subsect. 3.3, we describe the solution with constant A-term and time dependent ξ along with the thermodynamic equation and its aspects of the models. Finally, conclusions are summarized in the last Sect. 4.

2 The Metric and Field Equations

We consider a spatially homogeneous and anisotropic Bianchi type-I metric in the form

$$ds^2 = -dt^2 + A^2(t)dx^2 + B^2(t)dy^2 + C^2(t)dz^2,$$  \(1\)

where the metric potentials \(A\), \(B\) and \(C\) are functions of cosmic time \(t\) alone. This ensures that the model is spatially homogeneous.

We define the following parameters to be used in solving Einstein's field equations for the metric \(1\).

The average scale factor \(a\) of Bianchi type-I model \(1\) is defined as

$$a = (ABC)^{1/3}.$$  \(2\)

A volume scale factor \(V\) is given by

$$V = a^3 = ABC.$$  \(3\)

In analogy with FRW universe, we also define the generalized Hubble parameter \(H\) and deceleration parameter \(q\) as

$$H = \frac{\dot{a}}{a} = \frac{1}{3} \left( \frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right),$$  \(4\)

$$q = -\frac{\ddot{a}}{a^2} = -\frac{\dot{a}}{aH^2},$$  \(5\)

where an over dot denotes derivative with respect to the cosmic time \(t\).

Also we have

$$H = \frac{1}{3}(H_1 + H_2 + H_3),$$  \(6\)

where \(H_1 = \frac{\dot{A}}{A}, H_2 = \frac{\dot{B}}{B}\) and \(H_3 = \frac{\dot{C}}{C}\) are directional Hubble factors in the directions of \(x\)-, \(y\)- and \(z\)-axes respectively.

The Einstein's field equations (in gravitational unit \(8\pi G = c = 1\)) are given by

$$R_{ij} - \frac{1}{2}Rg_{ij} = -T_{ij},$$  \(7\)
where $T_{ij}$ is the stress energy tensor of matter which, in case of viscous fluid and cosmological constant, has the form [62]

$$T_{ij} = (\rho + \bar{p})u_iu_j + \bar{p}g_{ij} - \eta\mu_{ij} - \Lambda(t)g_{ij},$$

with

$$\bar{p} = p - \left(\frac{2}{3}\eta\right)u^i_i = p - (3\xi - 2\eta)H$$

and

$$\mu_{ij} = u_{ij} + u_{i\alpha}u_{j\gamma} + u_{j\alpha}u_{i\gamma}.$$  

In the above equations, $\xi$ and $\eta$ stand for the bulk and shear viscosity coefficients respectively; $\rho$ is the matter density; $p$ is the isotropic pressure and $u^i$ is the four-velocity vector satisfying $u^i_i = -1$.

In a co-moving coordinate system, where $u^i = \delta^i_0$, the field equations (7), for the anisotropic Bianchi type-I space-time (1) and viscous fluid distribution (8), yield

$$\frac{\ddot{B}}{B} + \frac{\dot{C}}{C} + \frac{\dot{B}\dot{C}}{BC} = -\bar{p} + 2\eta\frac{\dot{A}}{A} + \Lambda, \quad (11)$$

$$\frac{\ddot{C}}{C} + \frac{\dot{A}}{A} + \frac{\dot{A}\dot{C}}{CA} = -\bar{p} + 2\eta\frac{\dot{B}}{B} + \Lambda, \quad (12)$$

$$\frac{\ddot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{B}\dot{A}}{AB} = -\bar{p} + 2\eta\frac{\dot{C}}{C} + \Lambda, \quad (13)$$

$$\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}\dot{C}}{BC} + \frac{\dot{C}\dot{A}}{CA} = \rho + \Lambda. \quad (14)$$

Here, and also in what follows, a dot designates ordinary differentiation with respect to $t$.

Equations (11)-(14) can also be written as

$$p - \xi \theta - \Lambda = H^2(2q - 1) - \sigma^2, \quad (15)$$

$$\rho + \Lambda = 3H^2 - \sigma^2, \quad (16)$$

where $\sigma$ is shear scalar given by

$$\sigma^2 = \frac{1}{2} \sigma_{ij} \sigma^{ij} = \frac{1}{2} \left[ \left( \frac{\dot{A}}{A} \right)^2 + \left( \frac{\dot{B}}{B} \right)^2 + \left( \frac{\dot{C}}{C} \right)^2 \right] - \frac{1}{6} \theta^2, \quad (17)$$

where

$$\sigma_{ij} = u_{ij} + \frac{1}{2} (u_{ik} u^k u_j + u_{jk} u^k u_i) + \frac{1}{3} \theta (g_{ij} + u_i u_j). \quad (18)$$

The expansion scalar ($\theta$) and the anisotropy parameter ($A_m$) are defined as

$$\theta = u^i_i = \frac{3\dot{a}}{a} \quad (19)$$

$$A_m = \frac{1}{3} \sum_{i=1}^{3} \left( \frac{H_i - H}{H} \right)^2. \quad (20)$$
The energy conservation equation \( T^{ij}_{\ j} = 0 \), leads to the following expression:

\[
\dot{\rho} = -(p + \rho)\theta + \xi \theta^2 + 4\eta \sigma^2 - \dot{\Lambda}.
\] (21)

It follows from (21) that for contraction, that is, \( \theta < 0 \), we have \( \dot{\rho} > 0 \) so that the matter density increases or decreases depending on whether the viscous heating is greater or less than the cooling due to expansion.

The Raychaudhuri equation is obtained as

\[
\dot{\theta} = -\frac{1}{2} [\rho + 3(p - \xi \theta)] - \frac{1}{3} \theta^2 - 2\sigma^2 + \Lambda.
\] (22)

3 Solutions of Field Equations

We have a system of four independent equations (11) — (14) and eight unknown variables, namely \( A, B, C, p, \rho, \xi, \eta \) and \( \Lambda \). So for complete determinacy of the system, we need four appropriate relations among these variables that we shall consider in the following section and solve the field equations.

We follow the approach of Saha [19] to solve the field equations (11)—(14). Subtracting (11) from (12), (11) from (13), (12) from (13) and taking second integral of each, we get the following three relations

\[
\frac{A}{B} = d_1 \exp \left( x_1 \int a^{-3} e^{-2\int \eta dt} \right),
\] (23)

\[
\frac{A}{C} = d_2 \exp \left( x_2 \int a^{-3} e^{-2\int \eta dt} \right),
\] (24)

\[
\frac{B}{C} = d_3 \exp \left( x_3 \int a^{-3} e^{-2\int \eta dt} \right),
\] (25)

where \( d_1, x_1, d_2, x_2, d_3 \) and \( x_3 \) are constants of integration.

From (23)—(25), the metric functions can be explicitly written as

\[
A(t) = a_1 \exp \left( b_1 \int a^{-3} e^{-2\int \eta dt} \right),
\] (26)

\[
B(t) = a_2 \exp \left( b_2 \int a^{-3} e^{-2\int \eta dt} \right),
\] (27)

\[
C(t) = a_3 \exp \left( b_3 \int a^{-3} e^{-2\int \eta dt} \right),
\] (28)

where

\[
a_1 = \sqrt{d_1 d_2}, \quad a_2 = \frac{3}{2} \sqrt{d_1^{-1} d_3}, \quad a_3 = \frac{3}{2} \sqrt{d_2 d_3},
\]

\[
b_1 = \frac{x_1 + x_2}{3}, \quad b_2 = \frac{x_2 - x_1}{3}, \quad b_3 = \frac{-x_2 + x_3}{3}.
\]
These constants satisfy the following two relations
\[ a_1a_2a_3 = 1, \quad b_1 + b_2 + b_3 = 0. \] (29)

Thus the metric functions are found explicitly in terms of the average scale factor \( a \). It is clear from Eqs. (26)-(28) that once we get the value of the average scale factor \( a \), we can easily calculate the metric functions \( A, B, C \).

We define the deceleration parameter \( q \) as
\[ q \equiv -\frac{\ddot{a}}{a} \left( \frac{\dot{a}}{c} \right)^{-2} = -\frac{a\ddot{a}}{a^2} = -\left( \frac{\dot{H} + H^2}{H^2} \right) = b(t) \quad \text{say}, \] (30)
where \( a \) is the average scale factor of the universe defined by Eq. (2) and the dots indicate derivatives by proper time. The expansion of the universe is said to be “accelerating” if \( \dot{a} \) is positive (recent measurements suggest it is), and in this case the DP will be negative. The minus sign and the name “deceleration parameter” are historical; at the time of definition \( q \) was thought to be positive, now it is believed to be negative. Recent observations (Perlmutter et al. [37, 38]; Riess et al. [39, 41]) have suggested that the rate of expansion of the universe is currently accelerating, perhaps due to dark energy. This yields negative values of the deceleration parameter.

In literature it is common to use a constant deceleration parameter (Kumar and Singh [70]; Singh and Kumar [71, 72]; Pradhan and Jotania [31]; Akarsu and Kilinc [73, 74]; Agarwal et al. [75, 76]; Amirhashchi et al. [77]; Pradhan and Amirhashchi [78]; Kumar and Yadav [79]; Yadav [80]) to name only a few, as it duly gives a power law for metric function or corresponding quantity. The motivation to choose such time dependent DP is behind the fact that the universe is accelerated expansion at present as observed in recent observations of Type Ia supernova (Riess et al. [39]; Perlmutter et al. [38]; Tonry et al. [81]; Riess et al. [41]; Clocchiatti et al. [82]) and CMB anisotropies (Bennett et al. [83]; de Bernardis et al. [84]; Hanany et al. [85] and decelerated expansion in the past. Also, the transition redshift from deceleration expansion to accelerated expansion is about 0.5. Now for a Universe which was decelerating in past and accelerating at the present time, the DP must show signature flipping (see the Refs. Padmanabhan and Roychowdhury [86], Amendola [87], Riess et al. [40]). So, in general, the DP is not a constant but time variable.

The equation (30) may be rewritten as
\[ \frac{\ddot{a}}{a} + \frac{b\dot{a}^2}{a^2} = 0. \] (31)
In order to solve the Eq. (31), we assume \( b = b(a) \). It is important to note here that one can assume \( b = b(t) = b(a(t)) \), as \( a \) is also a time dependent function. It can be done only if there is a one to one correspondences between \( t \) and \( a \). But this is only possible when one avoid singularity like big bang or big rip because both \( t \) and \( a \) are increasing function.

The general solution of Eq. (31) with assumption \( b = b(a) \), is given by
\[ \int e^{\int \frac{b}{a} da} da = t + k, \] (32)
where $k$ is an integrating constant.

One can not solve Eq. (32) in general as $b$ is variable. So, in order to solve the problem completely, we have to choose $\int \frac{b}{a} da$ in such a manner that Eq. (32) be integrable without any loss of generality. Hence we consider

$$
\int \frac{b}{a} da = \ln L(a),
$$

which does not affect the nature of generality of solution. Hence from Eqs. (32) and (34), we obtain

$$
\int L(a) da = t + k.
$$

Of course the choice of $L(a)$, in Eq. (34), is quite arbitrary but, since we are looking for physically viable models of the universe consistent with observations, we consider

$$
L(a) = \frac{1}{\alpha \sqrt{1 + a^2}},
$$

where $\alpha$ is an arbitrary constant. In this case, on integrating, Eq. (34) gives the exact solution

$$
a(t) = \sinh (\alpha T).
$$

where $T = t + k$. We also note that $T = 0$ and $T = \infty$ respectively correspond to the proper time $t = -k$ and $t = \infty$. The relation (36) is recently used by Pradhan et al. [88] in studying dark energy models with anisotropic fluid in Bianchi type-\(V\) space-time. Recently, relation (36) is also used by Amirhashchi et al. [89] to study the evolution of dark energy models in a spatially homogeneous and isotropic FRW space-time filled with barotropic fluid and dark energy by considering a time dependent deceleration parameter.

Next, we assume that the coefficient of shear viscosity ($\eta$) is proportional to the expansion scalar ($\theta$) i.e. $\eta \propto \theta$, which leads to

$$
\eta = \eta_0 \theta,
$$

where $\eta_0$ is proportionality constant. Such relation has already been proposed in the physical literature as a physically plausible relation [21, 22, 72].

Finally to conveniently specify the source, we assume the perfect gas equation of state, which may be written as

$$
p = \gamma \rho, 0 \leq \gamma \leq 1.
$$

Using Eqs. (36) and (37) into (26)–(28), we get the following expressions for the scale factors

$$
A = a_1 \sinh(\alpha T) \exp \left[ b_1 \int \left( \sinh(\alpha T) \right)^{-3(1+2\eta_0\alpha)} dT \right],
$$

$$
B = a_2 \sinh(\alpha T) \exp \left[ b_2 \int \left( \sinh(\alpha T) \right)^{-3(1+2\eta_0\alpha)} dT \right],
$$

where $\kappa$ is an integrating constant.
Figure 1: The plot of anisotropy parameter $A_m$ versus $T$. Here $\alpha = \eta_0 = 0.1$

$$C = a_3 \sinh(\alpha T) \exp \left[ b_3 \int \left[ \left( \sinh(\alpha T) \right)^{-3(1+2\eta_0\alpha)} \right] dT \right].$$ (41)

The physical parameters such as directional Hubble factors ($H_i$), Hubble parameter ($H$), expansion scalar ($\theta$), spatial volume ($V$), deceleration parameter ($q$), anisotropy parameter ($A_m$) and shear scalar ($\sigma$) are given by

$$H_i = \alpha \coth(\alpha T) + b_i \sinh(\alpha T)^{-3(1+2\eta_0\alpha)}$$ (42)

$$H = \alpha \coth(\alpha T),$$ (43)

$$\theta = 3\alpha \coth(\alpha T),$$ (44)

$$V = \sinh^{3}(\alpha T),$$ (45)

$$q = -\tanh^{3}(\alpha T),$$ (46)

$$A_m = \frac{\beta_1}{3\alpha^2} \left[ \frac{\tanh^{2}(\alpha T)}{\sinh(\alpha T)^{6(1+2\eta_0\alpha)}} \right],$$ (47)

$$\sigma^2 = \frac{1}{2} \beta_1 \sinh(\alpha T)^{-6(1+2\eta_0\alpha)},$$ (48)

where

$$\beta_1 = b_1^2 + b_2^2 + b_3^2,$$

$$\beta_2 = b_1 b_2 + b_2 b_3 + b_3 b_1,$$

$$\beta_3 = b_3^2 + b_2^2 + b_2 b_3.$$ (49)
The shear viscosity of the model reads as

$$\eta = 3\eta_0 \alpha \coth(\alpha T).$$

(Equation 50)

Equations (15) and (16) lead to

$$p - 3\xi \alpha \coth(\alpha T) - \Lambda = -\alpha^2 \coth^2(\alpha T)[2 \tanh^2(\alpha T) + 1] - \frac{1}{2} \beta_1 \sinh(\alpha T)]^{-\xi(1+2\eta_0 \alpha)},$$

(Equation 51)

$$p + \Lambda = 3\alpha^2 \coth^2(\alpha T) - \frac{1}{2} \beta_1 \sinh(\alpha T)]^{-\xi(1+2\eta_0 \alpha)}.$$  

(Equation 52)

From Eqs. (45) and (44), we observe that the spatial volume is zero at $T = 0$ and the expansion scalar is infinite, which show that the universe starts evolving with zero volume at $T = 0$ which is a big bang scenario. From Eqs. (39)-(41), we observe that the spatial scale factors are zero at the initial epoch $T = 0$ and hence the model has a point type singularity (MacCallum [90]). We observe that proper volume increases with time.

The dynamics of the mean anisotropic parameter depends on the constant $\beta_1 = b_1^2 + b_2^2 + b_3^2$. From Eq. (47), we observe that at late time when $T \to \infty$, $A_m \to 0$. Thus, our model has transition from initial anisotropy to isotropy at present epoch which is in good harmony with current observations. Figure 1 depicts the variation of anisotropy parameter ($A_m$) versus cosmic time $T$. From the figure, we observe that $A_m$ decreases with time and tends to zero as $T \to \infty$. Thus, the observed isotropy of the universe can be achieved in our model at present epoch.

If we plot the deceleration parameter $q$ versus time $T$, it is observed that $q$ decreases very rapidly and approaching to $-1$ and then after it remains constant $-1$ (as de Sitter universe). This figure is not reported in the paper.

Here, we solve the Eqs. (51) and (52) with (38) in the following cases:

### 3.1 Models with Variable $\Lambda$-Term and Constant $\xi$

Let us assume that the coefficient of bulk viscosity is constant, i.e. $\xi(t) = \xi_0 = \text{constant}$. Then the Eqs. (51) and (52) together with (38) leads the following expressions for energy density, pressure and cosmological constant:

$$\rho = \frac{1}{(1+\gamma)} \left[ 3\xi_0 \alpha \coth(\alpha T) + 2\alpha^2 \text{csch}^2(\alpha T) + \kappa_1 \sinh^{2\kappa_2}(\alpha T) + \kappa_3 \coth(\alpha T) \sinh^{\kappa_2}(\alpha T) \right],$$

(Equation 53)

$$p = \frac{\gamma}{(1+\gamma)} \left[ 3\xi_0 \alpha \coth(\alpha T) + 2\alpha^2 \text{csch}^2(\alpha T) + \kappa_1 \sinh^{2\kappa_2}(\alpha T) + \kappa_3 \coth(\alpha T) \sinh^{\kappa_2}(\alpha T) \right],$$

(Equation 54)

$$\Lambda = \frac{1}{(1+\gamma)} \left[ 2\alpha^2 + (1+3\gamma)\alpha^2 \coth^2(\alpha T) - 3\xi_0 \alpha \coth(\alpha T) + (\beta_3 + \gamma\beta_2) \sinh^{2\kappa_2}(\alpha T) - \kappa_3 \coth(\alpha T) \sinh^{\kappa_2}(\alpha T) \right],$$

(Equation 55)
From above relations (53)—(55), we can obtain four types of models:

- When $\gamma = 0$, we obtain empty model.
- When $\gamma = \frac{1}{3}$, we obtain radiating dominated model.
- When $\gamma = -1$, we have the degenerate vacuum or false vacuum or $\rho$ vacuum model (Cho [91]).
- When $\gamma = 1$, the fluid distribution corresponds with the equation of state $\rho = p$ which is known as Zeldovich fluid or stiff fluid model (Zeldovich [92]; Barrow [93]).

From Eq. (53), it is observed that the energy density $\rho$ is a decreasing function of time and $\rho > 0$ always. The energy density has been graphed versus time in Figure 2 for $\gamma = 0, \frac{1}{3}, 1$. It is apparent that the energy density remains positive in all three types of models. However, it decreases more sharply with the cosmic time in Zeldovich universe, compare to radiating dominated and empty fluid universes. Also it can be seen from the figure that $\rho$ decreases more sharply with time in radiating dominated universe, compare to empty universe.

Figure 3 describes the variation of cosmological term $\Lambda$ with time ($\rho$ and $\Lambda$ are in geometric units in entire paper) for $\gamma = 0, \frac{1}{3}, 1$. This is taken to be a representative case of physical
Figure 3: The plot of cosmological constant $\Lambda$ versus $T$ for $\alpha = \xi_0 = \eta_0 = 0.1$, $\beta_2 = b_1 = 1$, $\beta_3 = b_2 = b_3 = 0.5$

viability of the models. In all three types of models, we observe that $\Lambda$ is decreasing function of time $T$ and it approaches a small positive value at late time (i.e. at present epoch). However, it decreases more sharply with the cosmic time in empty universe, compare to radiating dominated and stiff fluid universes. The $\Lambda$-term also decreases more sharply in radiating dominated universe, compare to stiff fluid universe. A positive cosmological constant resists the attractive gravity of matter due to its negative pressure and drives the accelerated expansion of the universe. Recent cosmological observations [37]–[45] suggest the existence of a positive cosmological constant $\Lambda$ with the magnitude $\Lambda(G h/c^3) \approx 10^{-123}$. These observations on magnitude and red-shift of type Ia supernova suggest that our universe may be an accelerating one with induced cosmological density through the cosmological $\Lambda$-term. Thus, the nature of $\Lambda$ in our derived models are supported by recent observations.

3.1.1 Thermodynamic Equations

The energy in a comoving volume is $U = \rho V$. The equation for production of entropy $S$ in a comoving volume due to the dissipative effects in a fluid with temperature $T$ is given by

$$T \dot{S} = \dot{U} + p \dot{V} = 3V(3\xi + 2\eta A_m)H^2.$$  \hspace{1cm} (56)

In a cosmic fluid where the energy density and pressure of the cosmic fluid are functions of temperature only, $\rho = \rho(T)$, $p = p(T)$ and where the cosmic fluid has no net charge, we obtain easily (Grøn [60])

$$S = \frac{V}{T} (\rho + p).$$  \hspace{1cm} (57)
From (56) and (57), we get the following expression for the entropy production rate in viscous Bianchi type-I universe

\[
\frac{\dot{S}}{S} = \frac{3(3\xi + 2\eta A_m)H^2}{\rho + p}. \tag{58}
\]

For a fluid obeying the equation of state (38), Eqs. (57) and (58) reduce to

\[
S = \frac{V}{T}(1 + \gamma)\rho, \tag{59}
\]

\[
\frac{\dot{S}}{S} = \frac{3(3\xi + 2\eta A_m)H^2}{(1 + \gamma)\rho}. \tag{60}
\]

Equation (60) can be rewritten as

\[
\frac{\dot{S}}{S} = \frac{\xi + 4\eta(\sigma^2/\theta^2)}{(1 + \gamma)(\rho/\theta^2)}, \tag{61}
\]

which gives the rate of change of entropy with time.

Let the entropy density be \( s \) so that

\[
s = \frac{S}{V} = \frac{(1 + \gamma)\rho}{T}. \tag{62}
\]

It defines the entropy density in terms of the temperature.

The first law of thermodynamics may be written as

\[
d(\rho V) + \gamma p dV = (1 + \gamma)Td\left(\frac{\rho V}{T}\right), \tag{63}
\]

which on integration, yields

\[
T \sim \rho^{\left(\frac{1}{1 + \gamma}\right)}. \tag{64}
\]

From (62) and (64), one can get

\[
s \sim \rho^{\left(\frac{1}{1 + \gamma}\right)}. \tag{65}
\]

The entropy in a comoving volume then varies according to

\[
S \sim sV. \tag{66}
\]

These equations are not valid for a vacuum fluid with \( \gamma = -1 \). For a Zel'dovich fluid \( (\gamma = 1) \), we get

\[
T \sim \rho^{\frac{1}{2}} \text{ and } s \sim \rho^{\frac{3}{2}}, \tag{67}
\]

so that the entropy density is proportional to the temperature.

Using (45) and (53), we find the respective temperature \( (T') \), entropy density \( (s) \) and total entropy \( (S) \) from (64)–(66), as

\[
T = \frac{T_0}{\sqrt{2}} \left[ 3\xi_0\alpha \coth(\alpha T) + 2\alpha^2 \csch^2(\alpha T) + \kappa_1 \sinh^{2\kappa_2}(\alpha T) + \kappa_3 \coth(\alpha T) \sinh^{\kappa_3}(\alpha T) \right]^{\frac{1}{2}}, \tag{68}
\]

13
\[ s = \frac{s_0}{\sqrt{2}} \left[ 3\xi_0\alpha \coth(\alpha T) + 2\alpha^2 \csch^2(\alpha T) + \kappa_1 \sinh^{2\kappa_2}(\alpha T) + \kappa_3 \coth(\alpha T) \sinh^{\kappa_3}(\alpha T) \right]^{\frac{1}{2}}, \quad (69) \]

\[ S = \frac{s_0 \sinh^3(\alpha T)}{\sqrt{2}} \left[ 3\xi_0\alpha \coth(\alpha T) + 2\alpha^2 \csch^2(\alpha T) + \kappa_1 \sinh^{2\kappa_2}(\alpha T) + \kappa_3 \coth(\alpha T) \sinh^{\kappa_3}(\alpha T) \right]^{\frac{1}{2}}, \quad (70) \]

where \( T_0 \) and \( s_0 \) are positive constants.

Now the entropy production rate of the model is given by

\[ \frac{\dot{S}}{S} = \frac{3 \left[ 3\xi_0\alpha \coth^2(\alpha T) + 2\eta_0 \beta_1 \alpha \coth(\alpha T) \sinh^{2\kappa_2}(\alpha T) \right]}{3\xi_0\alpha \coth(\alpha T) + 2\alpha^2 \csch^2(\alpha T) + \kappa_1 \sinh^{2\kappa_2}(\alpha T) + \kappa_3 \coth(\alpha T) \sinh^{\kappa_3}(\alpha T)}. \quad (71) \]

From above equation (71) we observe that the rate of change of entropy with time, i.e. the relation \( \frac{\dot{S}}{S} > 0 \). This also implies that the total entropy increases with time in Bianchi type-I model presented in this paper. This is in good agreement with second law of thermodynamics.

![Figure 4: The plot of energy density \( \rho \) versus \( T \) for \( \alpha = \xi_0 = \eta_0 = 0.1, b_1 = 1, \beta_4 = b_2 = b_3 = 0.5 \)]
Let us consider that $\xi = \xi_0 \rho$. In this case we obtain the expressions for energy density, pressure, bulk viscosity and cosmological constant as follows:

\begin{align}
\rho &= \frac{1}{(1 + \gamma - 3\xi_0 \alpha \coth(\alpha T))} \times \left[ \beta_4 \sinh^{2\kappa_3}(\alpha T) - 2\alpha^2 + \kappa_3 \coth(\alpha T) \sinh^{\kappa_2}(\alpha T) \right], \\
p &= \frac{\gamma}{(1 + \gamma - 3\xi_0 \alpha \coth(\alpha T))} \times \left[ \beta_4 \sinh^{2\kappa_3}(\alpha T) - 2\alpha^2 + \kappa_3 \coth(\alpha T) \sinh^{\kappa_2}(\alpha T) \right], \\
\xi &= \frac{\xi_0}{(1 + \gamma - 3\xi_0 \alpha \coth(\alpha T))} \times \left[ \beta_4 \sinh^{2\kappa_3}(\alpha T) - 2\alpha^2 + \kappa_3 \coth(\alpha T) \sinh^{\kappa_2}(\alpha T) \right], \\
\Lambda &= 3\alpha^2 \coth^2(\alpha T) + \beta_2 \sinh^{2\kappa_3}(\alpha T) - \frac{1}{(1 + \gamma - 3\xi_0 \alpha \coth(\alpha T))} \times \left[ \beta_4 \sinh^{2\kappa_3}(\alpha T) - 2\alpha^2 + \kappa_3 \coth(\alpha T) \sinh^{\kappa_2}(\alpha T) \right],
\end{align}

where

$$\beta_4 = b_1 b_2 + b_3 b_1 - b_2^2 - b_3^2.$$
Figure 6: The plot of cosmological constant $\Lambda$ versus $T$ for $a = \xi_0 = \eta_0 = 0.1, \beta_2 = b_1 = 1, \beta_4 = b_2 = b_3 = 0.5$

In this case, the expressions for entropy and entropy production rate of model are given by

$$S = s_0 \sinh^3(\alpha T) \left[ \frac{\eta_0 \sinh^{2\alpha_2}(\alpha T) - 2\alpha_2^2 + \kappa_3 \coth(\alpha T) \sinh^{2\alpha_2}(\alpha T)}{1 + \gamma - 3\xi_0 \coth(\alpha T)} \right] \frac{1}{1 + \gamma}, \quad (76)$$

$$\frac{\dot{S}}{S} = \frac{3\alpha \coth(\alpha T)}{1 + \gamma} \left[ 3\xi_0 \coth(\alpha T) + \frac{2\eta_0 \beta_1 \sinh^{2\alpha_2}(\alpha T) \{1 + \gamma - 3\xi_0 \coth(\alpha T)\}}{\beta_4 \sinh^{2\alpha_2}(\alpha T) - 2\alpha_2^2 + \kappa_3 \coth(\alpha T) \sinh^{2\alpha_2}(\alpha T)} \right]. \quad (77)$$

Figures 4 and 6 depict the variation of energy density $\rho$ and cosmological term $\Lambda$ versus time for $\gamma = 0, \frac{1}{3}, 1$ respectively. The nature of $\rho$ and $\Lambda$ in this model are same as described in previous section 3.1.

Figure 5 plots the variation of bulk viscosity coefficient $\xi$ with time $T$. From this figure we observe that $\xi$ is a positive decreasing function of time and it approaches to a constant quantity which is near to zero for all three types of models $\gamma = 0, \frac{1}{3}, 1$. This is in good agreement with physical behaviour of $\xi$. However, it decreases more sharply with the cosmic time in empty universe, compare to radiating dominated and Zeldovich universes. It is also observed from the figure that $\xi$ decreases more sharply with time in radiating dominated universe, compare to empty universe.

From Eq. (77), we observe that $\frac{\dot{S}}{S} > 0$. This implies that the total entropy increases with time. This is reproducible with second law of thermodynamics.
3.3 Models with Constant $\Lambda$-Term and $\xi(T)$

In this case, the expressions for energy density, isotropic pressure and bulk viscosity coefficient are respectively, given by

$$\rho = 3\alpha^2 \coth^2(\alpha T) + \beta_2 \sinh^{2\xi_2}(\alpha T) - \Lambda, \quad (78)$$

$$p = \gamma \left[ 3\alpha^2 \coth^2(\alpha T) + \beta_2 \sinh^{2\xi_2}(\alpha T) - \Lambda \right], \quad (79)$$

$$\xi = \frac{1}{3} \left[ (1 + 3\gamma)\alpha^2 \coth(\alpha T) - 2\alpha^2 \tanh(\alpha T) + (\beta_2 \gamma + \beta_3) \tanh(\alpha T) \sinh^{2\xi_2}(\alpha T) - 6\eta_0 \alpha \beta_5 \sinh^{4\xi_2}(\alpha T) - (1 + \gamma) \tanh(\alpha T) \Lambda \right]. \quad (80)$$

In this case, the expressions for entropy and entropy production rate of model are given by

$$S = s_0 \sinh^3(\alpha T) \left[ 3\alpha \coth^2(\alpha T) + \beta_2 \sinh^{2\xi_2}(\alpha T) - \Lambda \right] \frac{1}{1 + \gamma}, \quad (81)$$

$$\frac{\dot{S}}{S} = \frac{3\alpha \coth(\alpha T)}{(1 + \gamma)} \left[ 3\xi_0 \alpha \coth(\alpha T) + \frac{2\eta_0 \beta_1 \sinh^{2\xi_2}(\alpha T)}{3\alpha^2 \coth^2(\alpha T) + \beta_2 \sinh^{2\xi_2}(\alpha T) - \Lambda} \right]. \quad (82)$$

From Eq. (78), it is observed that the energy density $\rho$ is a decreasing function of time and $\rho > 0$ always. Figure 7 depicts the variation of energy density with cosmic time for $\gamma = 0, \frac{1}{3}, 1$. 

Figure 7: The plot of energy density $\rho$ versus $T$ for $\alpha = \eta_0 = 0.1, \Lambda = 0$
It is apparent that the energy density remains positive in all three types of models. However, it decreases more sharply with the cosmic time in Zeldovich universe, compared to radiating dominated and empty fluid universes. Also it can be seen from the figure that $\rho$ decreases more sharply with time in radiating dominated universe, compared to empty universe.

Figure 8 plots the variation of bulk viscosity coefficient $\zeta$ with time $T$. From this figure we observe that $\zeta$ is a positive decreasing function of time and it approaches to a constant quantity which is near to zero for all three types of models $\gamma = 0, \frac{1}{3}, 1$. This is in good agreement with physical behaviour of $\zeta$. However, it decreases more sharply with the cosmic time in empty universe, compare to radiating dominated and Zeldovich universes. It is also observed from the figure that $\zeta$ decreases more sharply with time in radiating dominated universe, compare to empty universe.

From Eq. (82), we observe that $\frac{\dot{\xi}}{\xi} > 0$. This implies that the total entropy increases with time. This is reproducible with second law of thermodynamics.

### 4 Concluding Remarks

In this paper, a class of cosmological models are presented with variable deceleration parameter $q$ and cosmological term $\Lambda$ in spatially homogeneous and anisotropic Bianchi type-I space-time in presence of bulk and shear viscosity. To find the explicit solution, we have considered a time dependent deceleration parameter which yields a scale factor as $a(t) = \sinh(\alpha T)$. The
apprehension of the global evolution of the observationally amenable universe, mathematically
coded in the dynamics of its scale factor $a$, is of utmost importance in explaining practically
all cosmological phenomena. One of the most intriguing aspects of this evolution is the recently
established late-time transition from a decelerated to an accelerating regime of the expansion
of the Universe. In this case, it is observed that as $T \to \infty$, $q = -1$. This is the case of
de Sitter universe. For $T \to 0$, $q = 0$. This shows that in the early stage the universe was
decelerating where as the universe is accelerating at present epoch which is corroborated from
the recent supernovae Ia observation [37—41, 81, 82]. The parameter $H_1, \dot{H}, \theta$, and $\sigma$
diverge at the initial singularity. There is a Point Type singularity [90] at $T = 0$ in the model. The rate
of expansion slows down and finally tends to zero as $T \to 0$. The pressure, energy density and
scalar field become negligible where as the scale factors and spatial volume become infinitely
large as $T \to \infty$, which would give essentially an empty universe.

The main features of the models are as follows:

- The models are based on exact solutions of the Einstein's field equations for the anisotropic
  Bianchi-I space-time in presence of a dissipative fluid with variable $\Lambda$-term.
- The model represents expanding, accelerating, shearing and non-rotating universe.
- The cosmological constant has been assumed to represent the energy density of vacuum,
  which has a potentially important contribution in the dynamics of the evolution of uni­
  verse. The cosmological constant is observed to have a small and positive value at late
times. The nature of decaying vacuum energy density $\Lambda(t)$ in our derived models are
  supported by recent cosmological observations. These observations on magnitude and
  red-shift of type Ia supernova [37—[41] suggest that our universe may be an accelerating
  one with induced cosmological density through the cosmological $\Lambda$-term.
- In literature it is a plebeian practice to consider constant deceleration parameter. Now for
  a Universe which was decelerating in past and accelerating at present epoch, the DP must
  show signature flipping as already discussed in Section 2. Therefore, our consideration
  of DP to be variable is physically justified. Our derived model is accelerating at present
  epoch.
- The solutions described in subsections 3.1, 3.2 and 3.3 satisfy the energy conservation Eq.
  (21) as well as Raychaudhuri Eq. (22) identically. Therefore, exact and physically viable
  Bianchi type-I models have been obtained for three types of models i.e. empty, radiating
dominated and stiff fluid models of the universe.
- The shear viscosity is observed to be responsible for the faster removal of initial anisotropies
  in the universe. This can be seen from the expression of anisotropy parameter (47). Hence,
  the isotropy observed in the present universe, is a possible consequence of viscous effects
  in the cosmic fluid.
- In the case of variable $\Lambda$ and $\xi = \eta = 0$, we obtain the recent solution of Ref. [26]).
  Further, if we take $\Lambda = 0$, we retrieve the the solutions recently obtained by Chawla and
  Mishra [94]). Hence our solutions generalize the recent work of Refs. [26, 94]).
Acknowledgments

The authors (R. Zia & A. Pradhan) would like to thank the Inter-University Centre for Astronomy and Astrophysics (IUCAA), Pune, India for providing facility & support during a visit where part of this work was done. The corresponding author (A. Pradhan) gratefully acknowledges the financial support by University Grants Commission, New Delhi, India under grant (Project F.No. 41-899/2012(SR)).

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