CHAPTER - II

NULL FLUID COLLAPSE IN HIGHER DIMENSIONS

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2.1 Introduction:

There has been a considerable interest in recent times to obtain solutions of the Einstein field equations in higher dimensions. It is supposed that underlying spacetime in the large energy limit of the Planck energy may be of higher dimensions than the usual four-dimensions. At this level, all the basic forces of nature are assumed to unify and hence it would be necessary in this context to obtain solutions of the gravitational field equations in higher dimensions. In fact such consideration would be relevant when the usual four dimensional manifold picture of spacetime becomes in applicable. Perhaps this would happen as we approach singularity whether in gravitational collapse or in cosmology. Several authors have considered the Kasner vaccum solution in five dimensional Kaluza-Klein theory and the problem of dimensional reduction also. There also have been investigations of entropy and classification of homogeneous cosmologies in Kaluza-Klein theory. For field of localised sources, higher dimensional versions of Schwarzschild, Reissner-Nordstrom, Vaidya and Bonnor-Vaidya solutions have been investigated by several authors.

Ever since mid sixties i.e. 1960, gravitational collapse continues to occupy centre stage in gravitational research. The problem is what initial conditions give to formation of black hole or naked singularity. There have been some very interesting investigations in this direction. Vaidya solution was used in studying gravitational collapse and it has been pointed out that it may lead to formation of naked singularity. Several workers have done an extensive analysis of gravitational collapse using the Vaidya metric in the context of naked singularity. It is an important problem, whether collapse always leads to singularity hidden behind a black hole even horizon or it is naked. Husain (1996) has presented collapse of a Type II null fluid with an equation of state, some nonstatic spherically symmetric exact solutions of the Einstein equation. It turns out that the metrics representing null fluid collapse have multiple apparent horizons and in the large limit asymptotically flat metrics have short hair and may be thought of lying between Schwarzschild, and the Reissner-Nordstrom metrics.

In this chapter we have investigated the generalized Vaidya family in higher dimensions presenting a large family of inhomogeneous nonstatic spherically symmetric solutions of the Einstein equation for null fluid. It covers higher dimensional versions of many previously known solutions such as Vaidya, charged Vaidya, and Husain solutions and also some new solutions representing global monopole or string dust.

2.2 The Metric and Generalised Vaidya family:

Let us consider (n+2) - dimensional spherical spacetime represented by the metric

$$ds^2 = 2\,dudr + \left(\frac{1-m(u,r)}{(n-1)r^{n-1}}\right)du^2 - r^2\,dw_n^2 \tag{2.1}$$

where

$$dw_n^2 = d\theta_1^2 + \sin^2\theta_1\,d\theta_2^2 + \ldots \sin^2\theta_{n-1}\,d\theta_n^2\ , \tag{2.2}$$

and m be an arbitrary function of retarded time u and radial coordinate r, i.e.

(2.3) $$m = m(u,r) \quad .$$

When

(2.4) $$m = m(u) \quad ,$$

one obtains Vaidya solution in higher dimension. The usual Vaidya solution follows for n = 2 i.e. in four dimensional spacetime. Let us now define the coordinates:

(2.5) $$x^0 = u \quad ,$$

(2.6) $$x^1 = r \quad ,$$

(2.7) $$x^{i+1} = \theta_i$$

(2.8) $$i = 1, 2 \ldots\ldots n$$

One may obtain the Einstein tensor in a standard way and which reads

$$(2.9) \qquad G_{oo} = \frac{n\,\dot{m}}{(n-1)r^n} - \frac{nm'}{(n-1)r^n}\left[1 - \frac{2m}{(n-1)r^{n-1}}\right],$$

$$(2.10) \qquad G_{01} = -\frac{nm'}{(n-1)r^n},$$

$$(2.11) \qquad G_{12} = \frac{m''}{(n-1)r^{n-3}},$$

$$(2.12) \qquad G^2_2 = G^3_3 = \ldots\ldots = G^{n+1}_{n+1},$$

where an overhead dot stands derivative with respect to u and dash for the derivative with respect to r.

Let us consider the energy momentum tensor of a Type II fluid as

$$(2.13) \qquad T_{ik} = \mu\, l_i\, l_k + (\rho + p)(l_i\, \eta_k + l_k\, \eta_i) - p\, g_{ik}$$

where

(2.14) $$\ell_i \ell^i = \eta_i \eta^i = 0 \; ,$$

(2.15) $$\ell_i \eta^i = 1 \; ,$$

the null vector ℓ_i, be a double null eigenvector of T_{ik}. The Vaidya spacetime of radiating star reads for

(2.16) $$\rho = p = 0.$$

For ,

(2.17) $$\mu = 0 \; ,$$

T_{ik} reduces to degenerate Type I fluid and it describes string dust i.e.

(2.18) $$\mu = p = 0 \; .$$

Hence, we get

(2.19) $$T_{ik} = \rho(\ell_i \eta_k + \ell_k \eta_i)$$

For such a distribution, the energy condition reads

(a) The Strong and weak energy conditions

(2.20) $$p \geq 0 \quad ,$$

(2.21) $$\rho \geq 0 \quad ,$$

(2.22) $$\mu > 0 \quad .$$

(b) The dominant energy conditions

(2.23) $$\rho \geq p \geq 0 \quad ,$$

(2.24) $$\mu > 0 \quad .$$

(c) The weak energy condition

(2.25) $$\mu = 0 \quad ,$$

(2.26) $$\rho + p \geq 0 \quad ,$$

(2.27) $$\rho \geq 0 .$$

(d) The strong energy condition

(2.28) $$\mu = 0 ,$$

(2.29) $$\rho + p \geq 0 ,$$

(2.30) $$p \geq 0 .$$

(e) The dominant energy condition

(2.31) $$\rho \geq 0 ,$$

(2.32) $$-\rho \leq p \leq \rho .$$

It is to be noted that

(2.33) $$T_{ik} \, l^i l^k = 0 ,$$

and

(2.34) $$T_{ik} \, n^i n^k = \mu .$$

For the metric (2.1), one may put

$$\ell_i = g_i^0 \quad , \tag{2.35}$$

$$\eta_i = g_i^1 + \frac{1}{2}\left(1 - \frac{2m}{(n-1)r^{n-1}}\right) g_i^0 . \tag{2.36}$$

Let us now solve the Einstein equation

$$G_{ik} = -8\pi \, T_{ik} \quad . \tag{2.37}$$

In view of eqs. (2.31) - (2.33) i.e. these equatiions satisfy the conditions given by eq. (2.13). Now let us substitute eqs. (2.9) to (2.12) in equations (2.35) - (2.36), one obtains

$$8\pi p = -\frac{m''}{(n-1)r^{n-1}} \quad , \tag{2.38}$$

$$8\pi \rho = \frac{nm'}{(n-1)\, r^n} \quad , \tag{2.39}$$

$$8\pi \mu = -\frac{n\dot{m}}{(n-1)\, r^n} \quad . \tag{2.40}$$

Let us now examine the equation (2.13) i.e. Type II fluid. The part $\mu\, l_i\, l_k$ of T_{ik} be the component of matter field which moves along the null hypersurface u = constant. For $\rho = p = 0$, one obtains the Vaidya solution in higher dimensions. Hence, the distributioin describes Vaidya radiating star in Type II fluid in higher dimensions. Here, it is to be noted that for $\mu = 0$, we do not recover the energy momentum tensor for the perfect fluid distribution. Rather, it describes an imperfect fluid distribution. Hence, by proper choice of the mass functiion, the energy conditions may be satisfied. Without loss of generality one may put the mass functioin m (u,r) as

$$(2.41) \qquad m\,(u,r) = \sum_{+\infty}^{-\infty} a_i\,(u)\, r^i ,$$

where a_i (u) are arbitrary functions of the retarted time u. Hence, we obtain

$$(2.42) \qquad 8\pi\, p = -\frac{1}{(n-1)} \sum_{+\infty}^{-\infty} i\,(i-1)\, a_i\, r^{i-n-1} ,$$

$$(2.43) \qquad 8\pi\rho = \frac{n}{(n-1)} \sum_{+\infty}^{-\infty} i\, a_i\, r^{i-n-1} ,$$

$$(2.44) \qquad 8\pi\mu = -\frac{n}{n-1}\sum_{+\infty}^{-\infty} \dot{a}_i(u)\, r^{i-n} \ .$$

For four-dimensional spacetime for n = 2 the above solutions go overto Wang family. Hence, our solutions are the higher dimensional generalisation. Again for $\rho = \mu = p = 0$, one obtains Schwarzschild solution in higher dimensions.

Now, in view of eqs (2.38) - (2.40), we observe that

$$(2.45) \qquad \rho = 0$$

implies

$$(2.46) \qquad m' = 0$$

or

$$(2.47) \qquad m'' = 0 \ ,$$

which implies

$$(2.48) \qquad p = 0 \ .$$

Hence, Vaidya solution simply follows from $\rho = 0$. Again

$$\mu = 0 \tag{2.49}$$

implies

$$\dot{m} = 0 . \tag{2.50}$$

2.3 Some Particular Solutions:

Let us now consider some particular cases

case (i) Let us consider the function $a_i(u)$ as

$$a_i(u) = \begin{cases} a/2 , & i = 1 \\ 0 & i \neq 1 \end{cases} , \tag{2.51}$$

where a be an arbitrary constant. Hence

$$m(u,r) = \frac{a}{2} r. \tag{2.52}$$

In view of eqs. (2.42) - (2.44) , one obtains

(2.53) $$p = 0 ,$$

(2.54) $$8\pi\rho = \frac{na}{2(n-1)r^n} ,$$

(2.55) $$\mu = 0 ,$$

showing that the matter field is of Type I and the metric reads

(2.56) $$ds^2 = 2\,dudr + (1 - \frac{a}{r^{n-2}})\,du^2 ,$$

which may be identified as higher dimensional representation of field of a Schwarzschild particle with global monopole or particle in string dust. Again for n = 2, we recover the monopole solution.

<u>Case (ii)</u>

Let us consider the function a_i (u) such that

(2.57) $$a_i = \begin{cases} a_o , & i = (n-1) \\ 0 , & i \neq (n-1) \end{cases} ,$$

where a_o be a constant. For this case, one obtains

(2.58) $$8\pi p = - a_o (n-2)/r^2 \quad ,$$

(2.59) $$8\pi \rho = na_o/r^2 \quad ,$$

(2.60) $$\mu = 0 \quad .$$

Again it gives Type I distribution and the metric reads

(2.61) $$ds^2 = 2\, dudr + (1-2a_o/n-1)\, du^2 \quad ,$$

which also describes the global monopole spacetime for n = 2.

<u>Case (iii)</u> Let us define a_i (u) as

(2.62) $$a_i = \begin{cases} \dfrac{\Lambda(n-1)}{n(n-+1)} \quad , & i = n+1 \\ 0 \quad , & i \neq n+1 \end{cases} \quad .$$

For this case, we obtain

$$(2.63) \qquad m(u,r) = \frac{\Lambda (n-1)}{n(n+1)} \; r^{n+1} \quad .$$

Others parameters read

$$(2.64) \qquad 8\pi p = - 8\pi \rho = \Lambda \quad ,$$

$$(2.65) \qquad \mu = 0 \quad ,$$

where Λ as the cosmological constant and the metric reads

$$(2.66) \qquad ds^2 = 2\,du\,dr + \left(1 - \frac{2\Lambda}{2(n+1)} r^2 \right) du^2 \quad .$$

This describes de Sitter and anti de Sitter higher dimensional spacetime for

$$(2.67) \qquad \Lambda g \gtrless 0 \quad .$$

For n = 2, it reduces to de Sitter spacetime.

Case (iv) Let us select a_i (u) as

$$(2.68) \quad a_i(u) = \begin{cases} f(u) & \text{for } i = 0 \\ -\dfrac{4\pi e^2(u)}{n} & \text{for } i = \ell - n \\ 0 & \text{for } i \neq 0,\ \ell - n \end{cases},$$

where two arbitrary function f(u) and e(u) describe mass and electric charge at the retarded time u. The corresponding physical parameters read

$$(2.69) \quad m = f(u) - \frac{4\pi e^2(u)}{n\, r^{n-2}},$$

$$(2.70) \quad 8\pi \rho = 8\pi p = \frac{4\pi e^2(u)}{nr^2},$$

$$(2.71) \quad 8\pi \mu = -\frac{1}{(n-1)r^n}\left(n\dot{f} - \frac{8\pi e \dot{e}}{r^{n-1}}\right),$$

and the metric in this case reads

$$(2.72)\qquad ds^2 = 2\,du\,dr + [1 - \frac{2f(u)}{(n-1)r^{n-1}} + \frac{8\pi e^2}{n(n-1)r^{2(n-1)}}]du^2,$$

which is identified as the Bonnor-Vaidya solution in higher dimensions. The electromagnetic field is

$$(2.73)\qquad F_{ik} = \frac{e(u)}{r^n}\left(g'_i\, g^o_k - g^o_i\, g'_k \right),$$

with the four-current vector

$$(2.74)\qquad 4\pi J^i = -\frac{\dot{e}(u)}{r^n}\, g^i_1 .$$

<u>Case (v)</u> Let us select the $a_i(u)$ such that

$$(2.75)\qquad a_i(u) = \begin{cases} f(u) & i = 0 \\ -\dfrac{g(u)}{nk-1} & i = 1-nk \quad (k \neq 1/n) \\ 0 & i \neq 0,\ 1-nk \end{cases},$$

where f(u) and g(u) are arbitrary constants an k be the positive constant and

(2.76) $$k < 1 .$$

In this case the physical parameters read

(2.77) $$m(u,r) = f(u) - \frac{g(u)}{(kn-1)\, r^{kn-1}} ,$$

(2.78) $$p = k \rho$$

(2.79) $$8\pi\rho = \frac{ng(u)}{(n-1)r^{(nk-1)}} .$$

For k = 1, the above solution describes Bonnor-Vaidya solutioin and the metric read as

(2.80) $$ds^2 = 2dudr + \left[1 - \frac{2f(u)}{(n-1)r^{n-1}} + 2g(u)\ (n-1) \right.$$

$$\left. (kn-1)r^{(n\&h)-2} \right] du^2 .$$

This is identified as Husain (1996) solution and is asymptotically flat for

(2.81) $k < 1/n$,

representing bounded source and for

(2.82) $k > 1/n$,

be the cosmological.

But for

(2.83) $kn = 1$

(2.84) $m(u,r) = f(u) + a(u) \ln r$,

energy conditions are violated and so it is ruled out.

Hence let us discuss two sub-cases, one for bounded source and other for cosmological model.

Sub-Case (a) Let us consider the sub-case (a) as

(2.85) $k > 1/n$

For simplicity let us put $k = 1$. Then, one may obtain

(2.86) $$2f = A(1- \tan hu) ,$$

(2.87) $$2g = 1 + B \tan hu ,$$

where A and B are constants such that

(2.88) $$A \geq 0 ,$$

(2.89) $$0 \leq B \leq 1 .$$

Under above situation the metric reads

(2.90) $$ds^2 = 2\,du\,dr + \left(\frac{1 - A(1-\tan hu)}{(n-1)\,r^{n-1}} + \frac{1+B \tan hu}{(n-2)^2 r^{2(n-1)}} \right) du^2 .$$

It has a naked singularity at r = 0 in the limit $u \longrightarrow \infty$. But for $u \longrightarrow -\infty$, it may have horizons depending upon the relative values of A and B, and horizons are

$$(2.91) \qquad (n-1)\, r^{n-1} = A \pm \sqrt{A^2+B-1} \quad .$$

<u>**Sub-Case (b)**</u> Let us consider sub-case (b) as

$$(2.92) \qquad k < 1/n \quad .$$

One obtains

$$(2.93) \qquad 2f = C + A\,(1- \tan hu) \quad ,$$

$$(2.94) \qquad 2g = B\,(1- \tan hu) \quad ,$$

and the metric reads

$$(2.95) \qquad ds^2 = 2du\, dr + \left[1+ \frac{A(1-\tan hu)}{(n-1)\, r^{n-1}} + \frac{B(1-\tan hu)}{(n-1)\,(kn-1)\, r^{n(k+1)-2}} \right] du^2$$

where

$$(2.96) \qquad A,\, B \geq 0.$$

Again in the limit $u \longrightarrow \infty$, the metric will represent either naked singularity at $r = 0$ for

(2.97) $$C \neq 0$$

or it would be flat for

(2.98) $$C = 0 \quad .$$

For $u \longrightarrow -\infty$ with $C = 0$, apparent horizons are

(2.99) $$\frac{n(k+1)-2}{R^{n-1}} - \frac{2A}{n-1} R^{\frac{nk-1}{n-1}} + \frac{2B}{(n-1)(nk-1)} = 0,$$

(2.100) $$R = r^{n-1} \quad .$$

But for

(2.101) $$n = 2 \quad \text{and} \quad k = 1/3 \quad ,$$

one obtains

(2.102) $$(r-2A)^3 = (3B)^3\, r \quad .$$

2.4 Concluding Remarks:

We have obtained a large family of inhomogeneous non-static spherically symmetric solutions of the Einstein equation for null fluid in higher dimensions. It encompasses higher dimensional versions of many previously known solutions, such as, Vaidya, charged Vaidya and Husain solutions and also some new solutions describing global monopole or string dust. In this way we have presented the general version of the 4-dimensional spherically symmetric solutions describing Type II fluid to (n+2)-dimensional spherically symmetric solutions and essentially retaining their physical behaviour. In particular higher dimensional version of Husain solution that describes gravitational collapse leading to asymptotically flat black hole solutions for $k > 1/n$. The general metric depends upon the parameter k and two arbitrary functions of retarded coordinate u, which are constrained by the energy conditioins. Also the long retarded time limit of the asymptotically flat solutiions would fall between Schwarzschild and Reissner-Nordstrom solutions. However, it is possible to obtain more exact solutions of the similar kinds by imposing the equation of state $p = k\rho$. The linear combinations of all the cases presented above would also be a solution.

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