CHAPTER-IV

RETRACTIONS IN THE CATEGORY PROXB

4.1 Introduction

Fibrewise general topology or General topology of continuous map is concerned most of all in extending the main notions and results concerning spaces to continuous maps.

In the present chapter the extension of the main notions and results concerning proximity spaces to p-continuous maps has been dealt with.

The category **Prox**, of separated proximity spaces as objects and p-continuous maps as morphisms, is a subcategory of the category **Top** of topological spaces as objects and continuous maps as morphisms.

We shall denote by **ProxB**, the category with objects as proximity maps or p-maps into the proximity space B and fibrewise p-maps as morphisn

This is treated as generalization of the category **Prox,** since **Prox** is isomorphic to the particular case of **ProxB** in which the space B consists of a single point.

4.2 Proximal Retraction

We shall denote by (X, f) the fibrewise proximity space, where $f : X \rightarrow B$ is the projection. The morphism $\eta : X \rightarrow Y$ is called the fibrewise proximity map, where (X, f) , (Y, g) are fibrewise proximity spaces, when $g \circ \eta = f$.

Let X be a proximity space and A be a proximal subspace of X. Then A is called a proximal retract (p-retract) of X if the identity map $I_d : A \rightarrow A$ is extendable to a p-continuous map $r : X \rightarrow A$. The map r is called proximal retraction.

DEFINITION 4.2.1: Let X be a proximity space and A be a proximal subspace of X. Let (X, f) and (A, g) be two fibrewise proximity spaces. Then

(i) a map $r : X \rightarrow A$ is said to be a fibrewise proximal retraction of f to g if it is a proximal retraction and a fibrewise p-map i.e. $f = g \circ r$. We say that g is a proximal retract of f.

(ii) if there is a proximal neighbourhood Y of A in X and a fibrewise proximal retraction $r : Y \rightarrow A$, then g is said to be a proximal neighbourhood retract of f.

DEFINITION 4.2.2: Let X be a fibrewise proximity space over B. A fibrewise p-map $e : X \rightarrow X$ is said to be fibrewise p-idempotent if $e \circ e = e$.

THEOREM 4.2.3: *Let* X *be a fibrewise proximity space over* B, A *is a proximal subspace of* X *and* $r : X \rightarrow A$ *be a* *fibrewise proximal retraction. Let* i : A → X, *be the inclusion map, then the composition* i o r = e : X → X *is fibrewise p-idempotent.*

PROOF: Note that e is a fibrewise p-map, being the composition of two fibrewise p-maps i and r . By hypothesis r : X → A is fibrewise proximal retraction, so r o i = I_dA, the identity map on A. These imply

e o e = (i o r) o (i o r) = i o (r o i) o r = i o r = e.

Converse of the above theorem also holds:

THEOREM 4.2.4: *Let* (X, f) *and* (A, g) *be FPS,* e : X → X *a fibrewise p-idempotent,* e(X) = A *and* r : X → A *is given by* r(x) = e(x), x ∈ X. *Then* r *is a fibrewise proximal retraction of* f *to* f/A.

PROOF: By hypothesis e = e o e and r : X → A is a fibrewise p-map, so g o r = f. Let a ∈ A. Since A = e(X) there exist x ∈ X such that a = e(x) and hence

r(a) = e(a) (by hypothesis)

= e (e(x)) (since a = e(x))

= (e o e)(x)

= e(x) (since e o e = e)

= a.

Consequently, r is a fibrewise proximal retraction of f to f/A.

DEFINITION 4.2.5: Let (X, f), (A, g) and (Z, h) be fibrewise proximity spaces, $A \subseteq X$ and $g = f/A$. Let $\psi : X \to Z$ and $\phi : A \to Z$ be fibrewise proximity maps or fibrewise p-maps. Then

(i) ψ is called a fibrewise proximal extension of ϕ if $\psi/A = \phi$.

(ii) ψ is called a fibrewise proximal neighbourhood extension of ϕ if there is a proximal neighbourhood U of A in X such that $\psi/U = \phi$.

THEOREM 4.2.6: *Let* (X, f) *and* (A, g) *be fibrewise proximity spaces and* $A \subseteq X$, $g = f/A$. *Then* g *is a proximal retraction of* f *if and only if for every fibrewise proximity space* (Z, h) *and every fibrewise p-map* $\phi : A \to Z$, ϕ *has a fibrewise proximal extension of* X *to* Z.

PROOF: Suppose $g : A \to B$ is a proximal retract of f . Then there exist a fibrewise proximal retraction $r : X \to A$ with $g \circ r = f$. Consequently, $\phi : A \to Z$ has a fibrewise extension $\phi \circ r : X \to Z$, since $\phi \circ r/A = \phi$.

Conversely, suppose that for every fibrewise proximity space (Z, h) and every fibrewise p-map $\phi : A \to Z$, ϕ has a fibrewise proximal extension of X to Z, then taking (Z, h) as (A, g), the fibrewise p-map I_dA has a fibrewise proximal extension r.

4.3 Proximal Adjuction Space

DEFINITION 4.3.1: Let $(X, \ll_1)$ and $(Y, \ll_2)$ be two proximity spaces. Denote by $X \oplus Y$ the disjoint union of X and Y. The binary relation '$\ll$' on $P(X \oplus Y)$ defined by "for $\mathcal{A}, \mathcal{B} \in P(X \oplus Y)$, $\mathcal{A} \ll \mathcal{B}$ if and only if $\mathcal{A} \cap X \ll_1 \mathcal{B} \cap X$ and $\mathcal{A} \cap Y \ll_2 \mathcal{B} \cap Y$" is a separated proximity on $X \oplus Y$.

We shall prove 2.1.9 (N3) and (N4) only.

(N3) $\mathcal{A} \ll (\mathcal{B} \cap \mathcal{C})$ implies $\mathcal{A} \cap X \ll_1 (\mathcal{B} \cap \mathcal{C}) \cap X$ and $\mathcal{A} \cap Y \ll_2 (\mathcal{B} \cap \mathcal{C}) \cap Y$; therefore, $\mathcal{A} \cap X \ll_1 \mathcal{B} \cap X$, $\mathcal{A} \cap X \ll_1 \mathcal{C} \cap X$ and $\mathcal{A} \cap Y \ll_2 \mathcal{B} \cap Y$, $\mathcal{A} \cap Y \ll_2 \mathcal{C} \cap Y$ i.e. $\mathcal{A} \ll \mathcal{B}$ and $\mathcal{A} \ll \mathcal{C}$.

Conversely, suppose $\mathcal{A} \ll \mathcal{B}$ and $\mathcal{A} \ll \mathcal{C}$. Then

$\mathcal{A} \cap X \ll_1 \mathcal{B} \cap X$, $\mathcal{A} \cap Y \ll_2 \mathcal{B} \cap Y$ and $\mathcal{A} \cap X \ll_1 \mathcal{C} \cap X$, $\mathcal{A} \cap Y \ll_2 \mathcal{C} \cap Y$; so $\mathcal{A} \cap X \ll_1 (\mathcal{B} \cap \mathcal{C}) \cap X$, $\mathcal{A} \cap Y \ll_2 (\mathcal{B} \cap \mathcal{C}) \cap Y$ i.e. $\mathcal{A} \ll (\mathcal{B} \cap \mathcal{C})$.

(N4) Suppose $\mathcal{A} \ll \mathcal{B}$. This implies $\mathcal{A} \cap X \ll_1 \mathcal{B} \cap X$ and $\mathcal{A} \cap Y \ll_2 \mathcal{B} \cap Y$. We obtain subsets G and H of X and Y respectively such that $\mathcal{A} \cap X \ll_1 G \ll_1 \mathcal{B} \cap X$ and $\mathcal{A} \cap Y \ll_2 H \ll_2 \mathcal{B} \cap Y$. Accordingly, $\mathcal{A} \cap X \ll_1 (G \cup H) \cap X \ll_1 \mathcal{B} \cap X$, $\mathcal{A} \cap Y \ll_2 (H \cup G) \cap Y \ll_2 \mathcal{B} \cap Y$ and so $\mathcal{A} \ll \mathcal{C} \ll \mathcal{B}$, where $\mathcal{C} = G \cup H$. Thus (N4) also holds.

The pair $(X \oplus Y, \ll)$ with '$\ll$' as above, is called the *disjoint proximal sum* of X and Y.

The subspace proximities on X and Y induced by (X ⊕ Y, ≪) are the same as the given proximities on X and Y respectively.

DEFINITION 4.3.2: Let A be a proximal subspace of (X, $\ll_1$) and f : A ⟶ Y be a p-continuous map. The quotient set of X ⊕ Y, obtained by identifying the point y of f(A) ⊆ Y with each x in f^{-1} (y), is endowed with the quotient proximity [cf. II, def. 2.1.20]; the resulting proximity space is denoted by X $\cup_f$ Y and is known as the *proximal adjunction space* with respect to the attaching p-map f.

THEOREM 4.3.3: *Let* X, Y ∈ **Prox** *and* A *be a proximal subspace of* X. *Then* f : A → Y *has a p-continuous extension* $\tilde{f}$ *to* X *iff* Y *is a proximal retract of the proximal adjunction space* X $\cup_f$ Y.

PROOF: If r : X $\cup_f$ Y → Y is a proximal retraction, then r o p_1 (Fig. 1) is a p-continuous extension of f to X where p_1 : X → X $\cup_f$ Y is q o i_X .

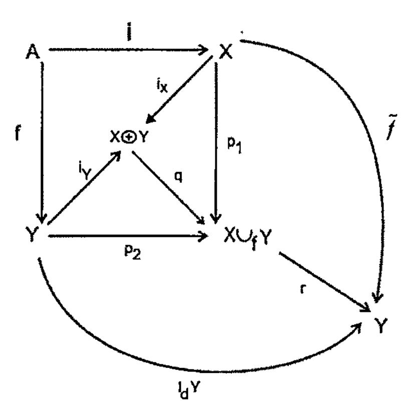

Fig. 1

Conversely, if $\tilde{f}$: X → Y is the p-continuous extension of f : A → Y, then the map ϕ is the required proximal retraction and Y is a proximal retract follow from Fig.2 below.

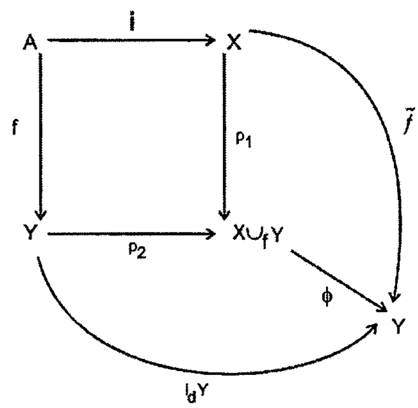

Fig. 2

4.4 Fibrewise Proximal Adjunction Spaces

In the present section a fibrewise version of proximal adjunction spaces is dealt with.

DEFINITION 4.4.1: Let (X, f) and (Y, g) be fibrewise proximity spaces, A be a proximal subspace of X, ϕ : A → Y be a fibrewise proximity map with g o ϕ = f/A and $X \cup_\phi Y$ be the proximal adjunction space.

The map h : $X \cup_\phi Y$ → B defined below is a fibrewise p-map.

$$h(x) = \begin{cases} f(x) : x \in X\text{-}A; \\ g(x) : x \in Y. \end{cases}$$

Then $(X \cup_{\phi} Y, h)$ is called a fibrewise proximal adjunction space determined by (X, f), (Y, g) and ϕ and h a proximal adjunction map determined by f, g and ϕ, and is denoted as $f \cup_{\phi} g$.

With the same notations as above, we have-

THEOREM 4.4.2: g *is a proximal retract of* h (i.e. g o r = h) *if and only if* ϕ *has a fibrewise proximal extension to a map of* X *to* Y.

PROOF: Suppose g is a proximal retract of h and $r : X \cup_{\phi} Y \to Y$ be a fibrewise proximal retraction. Let us define a map $\psi : X \to Y$ as $\psi(x) = r(p(x))$ for all $x \in X$. Note that for each $x \in A$, $\psi(x) = r(p(x)) = r(\phi(x)) = \phi(x)$. Thus, ψ is a fibrewise proximal extension of ϕ.

Conversely, suppose that $\psi : X \to Y$ is a fibrewise proximal extension of the given map ϕ. We now show that g is a proximal retract of h i.e. to show that there exist a map $r : X \cup_{\phi} Y \to Y$ such that $r/Y = I_d Y$, and g o r = h.

Let us define $r : X \cup_{\phi} Y \to Y$ as $r(z) = z$ if $z \in Y$. If $z \notin Y$, there exist unique $x \in X\text{-}A$ with $p(x) = z$. So define $r(z) = \psi(x)$. Now, the composite map $s = r \circ p : X \cup Y \to Y$ is sucsh that $s/X = \psi$ and $s/Y = I_d Y$ and is p-continuous. As, g o r = h, it follows that r is the desired fibrewise proximal retraction.

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