CHAPTER-III

FIBREWISE PROXIMITY SPACES

3.1 Introduction

The present chapter deals with the concept of fibrewise proximity spaces (FPS). The theory developed here is an analogue of the theory of fibrewise topological spaces [19]. In fibrewise topology we work over a topological base space B. When B is a point space, the theory reduces to that of ordinary topological space. Since a p-continuous map is always continuous with respect to the induced topologies, every fibrewise proximity space is a fibrewise topological space.

In section 2, a theory of fibrewise proximity spaces is developed. Several definitions and results have been given.

In section 3, separation axioms in fibrewise proximity space have been introduced and several results are proved.

3.2 Fibrewise Proximity Space

Recall that a fibrewise set over a set B, called the base set, consists of a set X together with a map $p : X \rightarrow B$, called the projection. For $b \in B$, we shall denote by X_b its inverse image $p^{-1}(b)$. Also for $W \subseteq B$, $p^{-1}(W)$ is denoted by X_w.

DEFINITION 3.2.1: A fibrewise proximity on a fibrewise set X over B, where B is a proximity space, is any proximity on X which makes the projection p-continuous.

DEFINITION 3.2.2: Let B be a proximity space. A fibrewise proximity space (FPS) is a fibrewise set over B with a fibrewise proximity.

Note that every proximity space may be treated as a fibrewise proximity space over itself with the identity map as projection.

RESULT 3.2.3 [19]: Let $\phi : X \to Y$ be a p-open and p-closed fibrewise onto map, where X and Y are fibrewise proximity spaces over B. Let $\alpha : X \to R$ be a p-map, which is bounded above in the sense that α is bounded above on each fibre of X. Then $\beta : Y \to R$ is a p-continuous map, where $\beta(\eta) = \sup_{\xi \in \phi^{-1}(\eta)} \alpha(\xi)$.

DEFINITION 3.2.4: Let X and Y be fibrewise proximity spaces over B with projections p and q. A map $\phi : X \to Y$ is said to be fibrewise p-map if $q \circ \phi = p$ i.e. if $\phi(X_b) \subseteq Y_b$ for each point $b \in B$.

Note that by restriction for each proximal subspace B' of B, a fibrewise p-map $\phi : X \rightarrow Y$ determines a fibrewise p-map $\phi_{B'} : X_{B'} \rightarrow Y_{B'}$ over B'.

The restriction $\phi_b : X_b \rightarrow Y_b$ of ϕ to the fibre over the point $b \in B$ is an ordinary p-map.

DEFINITION 3.2.5: A map $\phi : X \rightarrow Y$ is said to fibrewise p-continuous if for $A \subseteq X_W$, where $W \subseteq B$ and for any proximal neighbourhood K of $\phi(A)$, its inverse image, $\phi^{-1}(K)$ is a proximal neighbourhood of A i.e.

$$\phi(A) << K \Rightarrow A << \phi^{-1}(K).$$

When $A = \{x\}$, the above condition reduces to

$$\phi(x) << K \Rightarrow x << \phi^{-1}(K).$$

Note that a fibrewise p-continuous map is a fibrewise continuous map with respect to the induced topologies.

3.3 Separation Axioms

In the present section separation axioms in fibrewise proximity spaces are defined and some results have been proved.

DEFINITION 3.3.1: Let X be a FPS. Then X is called a fibrewise T_0-proximity space if each fibre X_b, $b \in B$ is T_0. In other words, X is T_0, if whenever x_1, $x_2 \in X_b$, $b \in B$ and $x_1 \neq x_2$ there exists $x_1 << G_{x_1}$, $x_2 \notin G_{x_1}$ or $x_2 \, \delta \, X\text{-}G_{x_1}$.

THEOREM 3.3.2: T_0*-ness is hereditary in FPS.*

PROOF: Let (Y, δ_Y) be a subspace of a FPS X. Let $y_1, y_2 \in Y_b$, $b \in B$ such that $y_1 \neq y_2$. Since X is fibrewise T_0 proximity space, there exists a proximal neighbourhood say H of y_1 in X such that $y_2 \notin H$, i.e. $y_1 << H$ and $y_2 \notin H$ or $y_2\ \delta$ X-H. This gives $y_1 << H \cap Y$ and $y_2 \notin H \cap Y$. Thus there exists a proximal neighbourhood $H \cap Y$ of y_1 in Y such that $y_2 \notin H \cap Y$. Hence Y is fibrewise T_0 proximity space.

DEFINITION 3.3.3: The fibrewise proximity space X over B is fibrewise R_0-proximity space if for each point $x \in X_b$, $b \in B$, and for each proximal neighbourhood $V >> x$ in X, there exists a proximal neighbourhood $W >> b$ in B such that $X_w \cap (\bar{x}) << V$.

THEOREM 3.3.4: *Let* $\phi : X \to Y$ *be a fibrewise proximal embedding, where* X *and* Y *are fibrewise proximity spaces over the same base* B. *If* Y *is fibrewise* R_0*-proximity space, then so is* X.

PROOF: Let $x \in X_b$, $b \in B$ and let $V >> x$ in X. Then $V = \phi^{-1}V'$, where $V' >> y = \phi(x)$ in Y. Since Y is fibrewise R_0, there exists a proximal neighbourhood $W >> b$ in B such that $Y_w \cap (\bar{y}) << V'$. By the p-continuity of ϕ, $\phi^{-1} (Y_w \cap (\bar{y})) << \phi^{-1}V' = V$.

Thus, for any $x \in X_b$, $b \in B$ and $V >> x$ in X, there exists a proximal neighbourhood $W >> b$ in B such that $X_w \cap (\overline{x}) << V$. So, X is fibrewise R_0-proximity space.

THEOREM 3.3.5: *Let* $\phi : X \to Y$ *be a p-closed and p-continuous fibrewise onto map, where* X *and* Y *are fibrewise proximity spaces over* B. *If* X *is fibrewise* R_0*-proximitiy space, then so is* Y.

PROOF: Let $y \in Y_b$, $b \in B$ and let $V >> y$ in Y. As ϕ is onto, let $\phi(x) = y$, for some $x \in X_b$. Since ϕ is p-continuous map, therefore, $x << \phi^{-1}(V)$ in X. By hypothesis, X is fibrewise R_0-proximity space, there exists a proximal neighbourhood $W >>$ b in B such that $X_w \cap (\overline{x}) << \phi^{-1}(V)$. This gives $Y_w \cap \phi(\overline{x}) << \phi(\phi^{-1}(V)) = V$ (since ϕ is onto). Hence $Y_w \cap \{\overline{\phi(x)}\} << V$ (since ϕ is p-closed). Consequently, Y is fibrewise R_0-proximity space.

DEFINITION 3.3.6: Let X be a FPS. Then X is a fibrewise Hausdorff (separated) proximity space, if whenever x_1, $x_2 \in X_b$, $b \in B$ such that $x_1 \neq x_2$, then there exists disjoint proximal neighbourhoods V_1 and V_2 of x_1 and x_2 in X.

THEOREM 3.3.7: *Let* $\phi : X \to Y$ *be p-continuous, fibrewise injective map, where* X *and* Y *are fibrewise proximity* *spaces over* B. *If* Y *is fibrewise separated proximity space, then so in* X.

PROOF: Let x_1, x_2 ∈ X_b, b ∈ B and $x_1 \neq x_2$. Since ϕ is injective, therefore $\phi(x_1), \phi(x_2) \in Y_b$ and $\phi(x_1) \neq \phi(x_2)$. As Y is fibrewise separated proximity space, there exists disjoint proximal neighbourhoods A, B of $\phi(x_1)$, $\phi(x_2)$ respectively in Y; i.e. $\phi(x_1) << A$, $\phi(x_2) << B$ and A $\bar{\delta}$ B. Since ϕ is p-continuous, it follows that $\phi^{-1}(\phi(x_1)) << \phi^{-1}(A)$, $\phi^{-1}(\phi(x_2)) << \phi^{-1}(B)$ and $\phi^{-1}(A)$ $\bar{\delta}$ $\phi^{-1}(B)$ or $x_1 << \phi^{-1}(A)$, $x_2 << \phi^{-1}(B)$ and $\phi^{-1}(A)$ $\bar{\delta}$ $\phi^{-1}(B)$. Hence, X is fibrewise separated proximity space.

REMARK 3.3.8: A fibrewise separated proximity space is a separated proximity space.

DEFINITION 3.3.9: The FPS X over B is fibrewise functionally separated proximity space (FFSPS), if whenever x_1, x_2 ∈ X_b, b ∈ B and $x_1 \neq x_2$ there exists a proximal neighbourhood $W >> b$ and a p-continuous map $\alpha : X_W \to I$ such that $\alpha(x_1) = 0$ and $\alpha(x_2) = 1$.

THEOREM 3.3.10: *Subspaces of fibrewise functionally separated proximity space (FFSPS) are fibrewise functionally separated proximity space.*

PROOF: Let X be a FFSPS over B and Y be a fibrewise proximal subspace of X. Then Y is also a FPS over B. Let $y_1, y_2 \in Y_b, b \in B$ such that $y_1 \neq y_2$. As X is FFSPS, there exists a proximal neighbourhood $W >> b$ and a p-continuous map $\alpha : X_w \rightarrow I$ such that $\alpha(y_1) = 0$ and $\alpha(y_2) = 1$, where $X_w \subseteq X$. Obviously, $\beta : X_w \cap Y \rightarrow I$ is a map in Y such that $\beta(y_1) = 0$ and $\beta(y_2) = 1$. Hence subspace of fibrewise functionally separated proximity space are fibrewise functionally separated proximity space.

THEOREM 3.3.11: *Let* $\phi : X \rightarrow Y$ *be p-continuous, fibrewise injective map, where* X *and* Y *are fibrewise proximity spaces over* B. *If* Y *is fibrewise functionally separated proximity space, then so is* X.

PROOF: Let x_1, $x_2 \in X_b$, $b \in B$ such that $x_1 \neq x_2$. Then $\phi(x_1), \phi(x_2) \in Y_b$ and $\phi(x_1) \neq \phi(x_2)$ i.e. $\phi(x_1) \not\delta \phi(x_2)$. Since Y is fibrewise functionally separated proximity space, there exists a proximal neighbourhood $W >> b$ and a p-continuous map $\alpha : Y_w \rightarrow I$ such that $\alpha(\phi(x_1)) = 0$ and $\alpha(\phi(x_2)) = 1$ or $(\alpha o \phi)(x_1) = 0$ and $(\alpha o \phi)(x_2) = 1$ (since α and ϕ both are p-continuous). Hence $\beta(x_1) = 0$ and $\beta(x_2) = 1$, where $\beta : X_w \rightarrow I$ is p-continuous. Thus, X is fibrewise functionally separated proximity space.

DEFINITION 3.3.12: The fibrewise proximity space X over B is fibrewise p-regular if for each point $x \in X_b$, $b \in B$ and for each proximal neighbourhood $V >> x$ in X, there exists a proximal neighbourhood $W >> b$ in B and a proximal neighbourhood U of x in X_W such that $X_W \cap \overline{U} << V$.

THEOREM 3.3.13: *Let* $\phi : X \to X'$ *be a fibrewise proximal embedding, where* X *and* X' *are fibrewise proximity space over* B. *If* X' *is fibrewise p-regular, then so is* X.

PROOF: Let $x \in X_b$, $b \in B$ and let $V >> x$ in X. Then $V = \phi^{-1}V'$, where $V' >> x' = \phi(x)$ in X'. Since X' is fibrewise p-regular, there exists a proximal neighbourhood $W >> b$ and a proximal neighbourhood $U' >> x'$ in X'_W such that $X'_W \cap \overline{U'} << V'$. Then $U = \phi^{-1}U' >> x$ in X_W such that $X_W \cap \overline{U} << V$. Thus X is fibrewise p-regular.

THEOREM 3.3.14: *Let* $\phi : X \to Y$ *be a p-open, p-closed and p-continuous fibrewise onto map, where* X *and* Y *are fibrewise proximity space over* B. *If* X *is fibrewise p-regular, then so is* Y.

PROOF: Let $y \in Y_b, b \in B$ and let $V >> y$. Since ϕ is onto, $y = \phi(x)$, for some $x \in X_b$. Let $U = \phi^{-1}(V) >> x$.

By hypothesis, X is fibrewise p-regular so there exists a proximal neighbourhood $W >> b$ and a proximal neighbourhood

$U' >> x$ such that $X_w \cap \overline{U'} << U$. Since ϕ is p-open, therefore $\phi(X_w \cap \overline{U'}) << \phi(U)$. This gives $Y_w \cap \phi(\overline{U'}) << V$. Hence $Y_w \cap \{\overline{\phi U'}\} << V$ (since $\phi\overline{U'} = \{\overline{\phi U'}\}$). Consequently, Y is fibrewise p-regular.

THEOREM 3.3.15: *If* X *is fibrewise p-regular and fibrewise* T_0 *proximity space over* B, *then* X *is fibrewise Hausdorff proximity space.*

PROOF: Let $x_1, x_2 \in X_b, b \in B$ and $x_1 \neq x_2$. Since X is fibrewise T_0 proximity space, there exists a proximal neighbourhood say G of x_1, which does not contain x_2 i.e. $x_1 << G, x_2 \notin G$ or $x_2 \, \delta \, X - G$.

Also, since X is fibrewise p-regular, there exists a proximal neighbourhood $W >> b$ in B and a proximal neighbourhood $U >> x_1$ in X_w such that $X_w \cap \overline{U} << G$. That $X_w - (X_w \cap \overline{U})$ is a proximal neighbourhood of x_2 follows from the fact that $x_2 \notin G$ and $X_w \cap \overline{U} << G$. As U and $X_w - (X_w \cap \overline{U})$ are disjoint proximal neighbourhood of x_1 and x_2 the result follows.

DEFINITION 3.3.16: Let X be a fibrewise proximity space over B. Then X is said to be a fibrewise completely regular proximity space if for each $x \in X_b, b \in B$ and a proximal neighbourhood V of x, there exists a proximal neighbourhood $W >> b$ and a p-continuous map $p_o : X_w \to I$ such that $p_o(x) = 1$ and $p_o(X - V) = 0$.

THEOREM 3.3.17: *Let* $\phi : X \to X'$ *be a fibrewise proximal embedding, where* X *and* X' *are fibrewise proximity spaces over* B. *If* X' *is fibrewise completely regular proximity space, then so is* X.

PROOF: Let $x \in X_b, b \in B$ and $V >> x$ in X. Choose $V = \phi^{-1}V'$, where $V' >> x' = \phi(x)$ in X'. Since X' is fibrewise completely regular proximity space, there exists a proximal neighbourhood $W >> b$ in B and a p-continuous map $p'_o : X'_w \to I$ such that $p'_o(x') = 1$ and $p'_o(X' - V') = 0$. Obviously, $p'_o \circ \phi / X_w \equiv p_o : X_w \to I$ is the required map. Hence X is fibrewise completely regular proximity space.

THEOREM 3.3.18: *Let* $\phi : X \to Y$ *be a p-open, p-closed and p-continuous fibrewise onto map, where* X *and* Y *are fibrewise proximity space over* B. *If* X *is fibrewise completely regular proximity space, then so is* Y.

PROOF: Let $y \in Y_b$, $b \in B$ and let $V >> y$ in Y. As ϕ is onto. Let $\phi(x)=y$, for some $x \in X_b$. Since ϕ is p-continuous map, therefore, $x << \phi^{-1}(V) = U$ in X. By hypothesis, there exists a proximal neighbourhood $W >> b$ in B and a p-continuous map

$p_o : X_w \to I$ such that $p_o(x) = 1$ and $p_o(X - U) = 0$.

Using 3.2.3, we obtain a p-map $\beta : Y_w \to I$ such that

$$\beta(y) = \sup_{x \in \phi^{-1}(y)} \alpha(x)$$

$$= \sup\left\{p_o(x) : x \in \phi^{-1}(y)\right\} = 1$$

$$\beta(Y - V) = \sup\{p_o(x) : x \in \phi^{-1}(Y - V)\}$$

$$= \sup\{p_o(x) : x \in X - U\} = 0.$$

Hence Y is fibrewise completely regular proximity space.

THEOREM 3.3.19: *If* X *is a fibrewise completely regular proximity space and fibrewise* T_0 *proximity space over* B, *then* X *is fibrewise functionally Hausdorff proximity space.*

PROOF: Let $x_1, x_2 \in X_b, b \in B$ and $x_1 \neq x_2$. Since X is fibrewise T_0 proximity space, there exists a proximal neighbourhood say G of x_1 which does not contain x_2 i.e. $x_1 << G, x_2 \notin G$ or $x_2 \; \delta \; X - G$.

Also, since X is fibrewise completely regular proximity space, there exists a proximal neighbourhood W of b in B ($W >> b$) and a p-continuous map $p_o : X_w \to I$ such that $p_o(x_1) = 1$ and $p_o(X - G) = 0$. Since $x_2 \in X - G$, it follows that $p_o(x_2) = 0$. Thus, X is fibrewise functionally Hausdorff proximity space.

THEOREM 3.3.20: *Let* $\phi : X \to Y$ *be a p-open, p-closed and p-continuous fibrewise onto map, where* X *and* Y *are fibrewise proximity space over* B. *If* X *is fibrewise completely regular proximity space and fibrewise* T_0 *proximity space, then* Y *is fibrewise functionally Hausdorff proximity space.*

PROOF: Let $y_1, y_2 \in Y_b, b \in B$ such that $y_1 \neq y_2$. Since ϕ is onto map, then $\phi^{-1}(y_1), \phi^{-1}(y_2) \in X_b$ and $\phi^{-1}(y_1) \neq \phi^{-1}(y_2)$. Since X is fibrewise T_0 proximity space, there exists a proximal neighbourhood say G of $\phi^{-1}(y_1)$ which does not contain $\phi^{-1}(y_2)$ i.e. $\phi^{-1}(y_1) \ll G$, $\phi^{-1}(y_2) \not\ll G$ or $\phi^{-1}(y_2)\ \delta\ X - G$. Also, since X is fibrewise completely regular proximity space, there exists a proximal neighbourhood W of b in B and p-continuous map $p_o : X_w \to I$ such that $p_o\left(\phi^{-1}(y_1)\right) = 1$ and p_o (X-G) = 0. Since $\phi^{-1}(y_2) \in X - G$, it follows that $p_o\left(\phi^{-1}(y_2)\right) = 0$ i.e. $\beta(y_1) = 0$ and $\beta(y_2) = 1$, where $\beta : X_w \to I$ is a p-continuous map. Hence Y is fibrewise functionally Hausdorff proximity.

DEFINITION 3.3.21: Let X be a FPS. Then X is said to be a fibrewise normal proximity space (FNPS) if for each $b \in B$ and each pair of sets P and Q of X such that $P\ \bar{\delta}\ Q$, there exists a proximal neighbourhood $W >> b$ and disjoint proximal neighbourhoods C and D of $X_w \cap P$ and $X_w \cap Q$ respectively in X_w, i.e. $X_w \cap P << C$ and $X_w \cap Q << D$ and $C\ \bar{\delta}\ D$.

THEOREM 3.3.22: *Let* $\phi: X \to Y$ *be a p-closed and p-continuous fibrewise onto map, where* X *and* Y *are fibrewise proximity spaces over* B. *If* X *is fibrewise normal proximity space, then so is* Y.

PROOF: Let $b \in B$ and let U and V be the sets of Y such that $U \bar{\delta} V$. This implies $\phi^{-1}U \bar{\delta} \phi^{-1}V$ in X by the p-continuity of ϕ. Since X is fibrewise normal proximity space, there exists a proximal neighbourhood W of b and proximal neighbourhoods G and H of $X_w \cap \phi^{-1}U$ and $X_w \cap \phi^{-1}V$ respectively in X_w such that $G \bar{\delta} H$, i.e.

$X_w \cap \phi^{-1}U << G$, $X_w \cap \phi^{-1}V << H$ and $G \bar{\delta} H$.

or $X_w \cap \phi^{-1}U \bar{\delta} (X_w - G), X_w \cap \phi^{-1}V \bar{\delta} (X_w - H)$ and $G \bar{\delta} H$.

As ϕ is p-closed, the result follows.

THEOREM 3.3.23: Let $\phi: X \to X'$ *be a p-closed fibrewise proximal embedding, where* X *and* X' *are fibrewise proximity spaces over* B. *If* X' *is fibrewise normal proximity space, then so is* X.

PROOF: Let $b \in B$ and let $U \bar{\delta} V$ in X. Then $\phi U \bar{\delta} \phi V$ in X'. Since X' is fibrewise normal proximity space, therefore, there exists a proximal neighbourhood W of b and disjoint proximal neighbourhoods G and H of $X'_w \cap \phi U$ and $X'_w \cap \phi V$ respectively in X'_w, i.e. $X'_w \cap \phi U << G$, $X'_w \cap \phi V << H$ and

$G \not\delta H$. Since ϕ is p-continuous, therefore,

$$\phi^{-1}(X'_w \cap \phi U) << \phi^{-1}G, \phi^{-1}(X'_w \cap \phi V) << \phi^{-1}H \text{ and}$$

$\phi^{-1}G \not\delta \phi^{-1}H$. It follows that

$$X_w \cap U << \phi^{-1}G, \ X_w \cap V << \phi^{-1}H \text{ and } \phi^{-1}G \not\delta \phi^{-1}H \text{ in}$$

X_w. Thus X is fibrewise normal proximity space.

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