CHAPTER-II

PREREQUISITES

2.1 Proximity Spaces

DEFINITION 2.1.1 [cf. 29]: A binary relation δ on the power set of X is called an Efremovič proximity on X if and only if δ satisfies the following axioms:

P1. $A \delta B$ implies $B \delta A$;

P2. $(A \cup B) \delta C$ if and only if $A \delta C$ or $B \delta C$;

P3. $A \delta B$ implies $A \neq \phi$, $B \neq \phi$;

P4. $A \not\delta B$ implies there exists a subset E such that $A \not\delta E$ and $X-E \not\delta B$;

P5. $A \cap B \neq \phi$ implies $A \delta B$.

The pair (X, δ) is called a proximity space. In addition, if δ also satisfies:

P6. $\{x\} \delta \{y\}$ implies $x = y$,

then the pair (X, δ) is called a separated proximity space. The condition (P4), described above, is referred to as "Strong Axiom".

Let (X, δ) be a proximity space. For a subset A of X define cl A $= \{x \in X : x \delta A\}$. Then 'cl' is a *Kuratowski closure operator* on X. The topology obtained by this operator is called the *topology induced* by δ, and is denoted by $\tau(\delta)$.

DEFINITION 2.1.2 [cf. 29]: If on a set X, there is a topology τ and a proximity δ such that $\tau = \tau(\delta)$, then τ and δ are said to be compatible.

Note that for subsets A and B of a proximity space (X, δ), $A \delta B$ if and only if $clA \delta clB$, where closure is taken with respect to $\tau(\delta)$.

DEFINITION 2.1.3 [cf. 29]: Let X be a set. For $A, B \in P(X)$, $A \delta_1 B$ if and only if $A \cap B \neq \phi$, defines a proximity and is called the *discrete proximity* on X.

If $A \delta B$ for every pair of nonempty subsets A and B of X, then we obtain the *indiscrete proximity* on X.

DEFINITION 2.1.4 [cf. 29, 14]: Let (X, τ) be a completely regular space. Subsets A and B of X are said to be distinguishable if and only if there is a continuous function $f : X \rightarrow [0, 1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.

REMARK 2.1.5 [cf. 29]: If (X, τ) is a completely regular space, then the proximity δ defined by: for subsets A, B of X.

2.1.5.1 A $\not\delta$ B if and only if A and B are functionally distinguishable, is compatible with τ.

DEFINITION 2.1.6 [cf. 29]: If δ_1 and δ_2 are two proximities on a set X, then δ_2 said to be finer than δ_1, expressed by $\delta_1 < \delta_2$ if and only if A δ_2 B implies A δ_1 B.

The following theorem shows that a finer proximity structure induces a finer topology:

THEOREM 2.1.7 [cf. 29]: (a) *Let* δ_1 *and* δ_2 *be two proximities defined on a set* X, *then* $\delta_1 < \delta_2$ *implies* $\tau(\delta_1) \subseteq \tau(\delta_2)$.

(b) *Let* τ_1 *and* τ_2 *be two completely regular* T_2 *topologies on* X, δ_1 *be any separated proximity on* X *compatible with* τ_1, δ_2 *be defined by* (2.1.5.1) *with respect to* τ_2. *Then* $\tau_1 \subseteq \tau_2$ *implies* $\delta_1 < \delta_2$.

DEFINITION 2.1.8 [cf. 29, 43]: A subset B of a proximity space (X, δ) is called a δ-neighbourhood or proximal neighbourhood of A, denoted by, A << B, if and only if A $\not\delta$ X-B.

REMARK 2.1.9: (i) In terms of proximal neighbouhoods, the 'Strong Axiom P4' is equivalent to

2.1.9.1 A $\not\delta$ B implies there exist subsets C and D such that A << C, B << D and C $\not\delta$ D.

Consequently, the topology $\tau(\delta)$ of a separated proximity space (X, δ) is Hausdorff.

(ii) Given a proximity space (X, δ), the relation '$\ll$' satisfies the following properties:

N1. $\phi \ll X$;

N2. if $A \ll B$, then $A \subseteq B$;

N3. $A \ll B \cap C$ if and only if $A \ll B$ and $A \ll C$;

N4. if $A \ll B$, then there exists a subset C such that

$A \ll C \ll B$.

If δ is a separated proximity, then

N5. if $a \neq b$, then $a \ll X - \{b\}$.

The following theorem provides a way to obtain a proximity δ on a set X with the help of a binary relation $\ll$ on P(X):

THEOREM 2.1.10 [cf. 43, 29]: *If* $\ll$ *is a binary relation on* P(X) *satisfying* N1-N4 *and* δ *is defined by* $A \not\delta B$ *if and only if* $A \ll X-B$, *then* δ *is a proximity on* X *and* B *is a proximal neighbourhood of* A *if and only if* $A \ll B$. *Moreover, if* $\ll$ *also satisfies* N5, *then* δ *is separated.*

The intersection of all proximal neighbourhood of a subset A of X is the closure of A, i.e. $clA = \cap \{B : B \gg A\}$.

DEFINITION 2.1.11 [cf. 29]: Let (X, δ_1) and (Y, δ_2) be proximity spaces. A map $f : X \rightarrow Y$ is said to be a proximity map or a p-map or a p-continuous map if and only if $A\ \delta_1\ B$ implies $f(A)\ \delta_2\ f(B)$ if and only if $C\ \not\delta_2\ D$ implies $f^{-1}(C)\ \not\delta_1\ f^{-1}(D)$, if and only if $C \ll_2 D$ implies $f^{-1}(C) \ll_1 f^{-1}(D)$.

Note that the composition of p-continuous maps is a p-continuous map. If $A \subseteq X$, then the inclusion map $i : A \rightarrow X$ is a p-continuous map.

DEFINITION 2.1.12 [cf. 29]: Two proximity spaces (X, δ_1) and (Y, δ_2) are called proximally isomorphic or p-homeomorphic if there exist a one-one map f from X onto Y such that both f and f^{-1} are p-continuous maps. Such a map $f : X \rightarrow Y$ is called a p-homeomorphism between X and Y. If $f : X \rightarrow Y$ defines a p-homeomorphism between X and f (X), then f is called a p-embedding.

DEFINITION 2.1.13 [cf. 29]: Let (X, δ) be a proximity space and Y be a subset of X. For subsets A, B of Y, the relation "$A\ \delta_Y\ B$ if and only if $A\ \delta\ B$" defines a proximity on Y; δ_Y is called the subspace proximity or induced proximity on Y and $\tau(\delta_Y)$ is the subspace topology induced by $\tau(\delta)$.

DEFINITION 2.1.14: Consider the family $\{(X_\alpha, \delta_\alpha) : \alpha \in \Lambda\}$ of (separated) proximity spaces. Let $X = \Pi\{(X_\alpha : \alpha \in \Lambda\}$ denote the Cartesian product of these spaces. A relation δ on P (X) given by

"$A \delta B$ if and only if for each pair of finite covers $\{ A_i : i = 1, 2, \ldots, m\}$ and $\{B_j : j = 1, 2, \ldots, n\}$ of A and B respectively, there exists an A_i and B_j such that $P_\alpha(A_i)\ \delta_\alpha\ P_\alpha(B_j)$ for each $\alpha \in \Lambda$, where P_α denotes the projection of X onto X_α",

defines a (separated) proximity on X, and is called the product proximity on X.

Note that a map f from a proximity space (Y, δ_1) to $X = \Pi\{(X_\alpha : \alpha \in \Lambda\}$ is a proximity mapping if and only if the composition $P_\alpha \circ f : Y \to X_\alpha$ is a proximity mapping for each projection P_α.

DEFINITION 2.1.15: A map $f : (X, \delta_1) \to (Y, \delta_2)$ is called p-open if for subsets $A, B \in P(X)$, $A \ll_1 B$ implies $f(A) \ll_2 f(B)$.

DEFINITION 2.1.16: Let (X, δ_1) and (Y, δ_2) be proximity spaces. A map f from X to Y is called p-closed if for subset A, B of X, $A\ \bar{\delta}_1\ B$ implies $f(A)\ \bar{\delta}_2\ f(B)$.

LEMMA 2.1.17: Let (X, δ) be a proximity space and let clA $(\overline{A})$ and int A (A^0), denote respectively, the closure and interior of A in $\tau(\delta)$ .Then

(i) $A \ll B$ implies $\overline{A} \ll B$, and

(ii) $A \ll B$ implies $A \ll B^0$.

Therefore $A \subseteq B^0$, showing that a δ-neighbourhood is a topological neighbourhood.

DEFINITION 2.1.18: Let γ and δ be two proximities on a non-empty set X. Then γ is said to be finer than δ if for subsets A, B of X,

$A \gamma B \Rightarrow A \delta B$ and is written $\delta < \gamma$.

Equivalently, γ is finer than δ if $A \ll\delta B \Rightarrow A \ll\gamma B$.

DEFINITION 2.1.19 [cf. 13, 43]: Let (X, δ_1) be a proximity space, Y be a set and f be a map of X onto Y. Define a binary relation '$\ll_2$' on Y by "for subsets $C, D \in P(Y)$, $C \ll_2 D$ if and only if for each binary rational $s \in (0,1)$ there is some $C_S \subseteq Y$ such that $C_0 = C$, $C_1 = D$ and $s < t$ implies $f^{-1}(C_S) \ll_1 f^{-1}(Ct)$. The relation '$\ll_2$' satisfies N1-N5 and hence defines a separated proximity δ_2 given by $C \not\delta_2 D$ if and only if $C \ll_2 Y\text{-}D$, called the quotient proximity [25] induced by f. The quotient proximity is the finest proximity on Y which makes f a p-map. The space (Y, δ_2), where δ_2 is derived from $\ll_2$, is called the "quotient space" of X with respect to the map f.

2.2 Topological Spaces

The free union X+Y of disjoint spaces X, Y is the set $X \cup Y$ with topology: $U \subseteq X+Y$ is open if and only if $U \cap X$ is open in X and $U \cap Y$ is open in Y. Since $X \cap Y = \phi$, X and Y keep their own topologies and are disjoint open sets in X+Y.

DEFINITION 2.2.1: Let X and Y be two disjoint spaces, $A \subset X$ a closed subset and $f : A \to Y$ be continuous. In X+Y, generate an equivalence relation R_0 by $a \sim f(a)$, for each $a \in A$. The quotient space $(X+Y)/ R_0$ is said to be " X attached to Y by f". This quotient space is called an adjunction space and is denoted as $X \cup_f Y$.

DEFINITION 2.2.2: A category $\mathcal{C}$ consists of two classes, a class of ob$\mathcal{C}$, called the objects $\mathcal{C}$ and a class $\mathcal{M}$ called the morphisms of $\mathcal{C}$ together with the following axioms:

CT 1. the composition h o g o f of three morphisms is defined whenever the compositions h o g and g o f are defined;

CT 2. composition of morphisms is associative, i.e. (h o g) o f = h o (g o f) and both compositions are defined if either is defined;

CT 3. there is a bijection which assigns to each object X an identity morphism I_X and for each morphism f there are two identity morphisms I_X and I_Y such that $f = f \circ I_X$ and $f = I_Y \circ f$.

A category $\mathcal{S}$ is a *subcategory* of a category $\mathcal{C}$ if every object and morphism of $\mathcal{S}$ is also an object or morphism of $\mathcal{C}$.

DEFINITION 2.2.3 [10]: Let X and Y be two topological spaces. Define the map $e : X \times C(X, Y) \to Y$ by the equation

$e(x, f) = f(x)$, where $x \in X$; then the map e is called the evaluation map.

DEFINITION 2.2.4 [10]: Let X and Y be topological spaces. If A is a compact subspace of X and U is an open subset of Y, define

$$[A, U] = \{f \in C(X, Y) : f(A) \subseteq U\}.$$

The sets [A, U] form a subbasis for a topology on C(X, Y), i.e. called the compact open topology.

DEFINITION 2.2.5: A fibrewise set over a set B, called the base set, consists of a set a X together with a map $p : X \to B$, called the projection. For $b \in B$, we shall denote by X_b its inverse image $p^{-1}(b)$. Also for $W \subseteq B$, $p^{-1}(W)$ is denoted by X_W.

A fibrewise topology on a fibrewise set X over B, where B is a topological space, is any topology on X which makes the projection continuous.

DEFINITION 2.2.6: Let B be a topological space. A fibrewise topological space is a fibrewise set over B with a fibrewise topology.

DEFINITION 2.2.7: A continuous mapping $f : X \to Y$ is said to be compact if the subspace $f^{-1}(y)$ is compact for every $y \in Y$. A compact closed and continuous map is called a perfect mapping.

DEFINITION 2.2.8: A mapping f from a space X onto a space Y is irreducible if Y is not the image under f of a closed set F in X, other than X.

A topological space Y is an irreducible perfect pre image of a regular space X if there is an irreducible perfect mapping f of Y onto X.

DEFINITION 2.2.9: A space $\dot{X}$ is called a absolute of the space X if, $\dot{X}$ is an irreducible perfect pre image of X and every irreducible perfect pre image of $\dot{X}$ is homeomorphic to $\dot{X}$.

DEFINITION 2.2.10: Let f and g be two continuous functions from X to Y. Then f and g are said to be homotopic, written $f \approx g$, if there is a continuous map $H : X \times I \to Y$ (where $I = [0, 1]$) such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all $x \in X$. The map H is called a homotopy between f and g.

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