CHAPTER-I

INTRODUCTION

1.1 Proximity Spaces – General Background

Generalizations of the metric concept came into existence in various ways; important among them are uniformity and proximity etc. Given a metric space (X, d), we may define a relation δ on P(X) as follows: A δ B if and only if D(A,B) = 0, where D (A,B) = inf $\{d(a,b) : a \in A, b \in B\}$. This definition reflects a nearness between A and B. The concept of nearness was initiated by F. Riesz in the year 1908. The axiomatization of this notion by V. A. Efremovič and pioneering work of Efremovič [11, 12], Smirnov [32] and others [2, 3, 4] in this direction led to a theory of proximity spaces [cf. II, def.2.1.1]. Prior to this, a study was made by Krishna Murti and Wallace [42] concerning the use of separation of sets as the primitive concept and in each case similar but weaker axioms than those of Efremovič were used. Later on, the notion of a proximal neighbourhood was introduced by Efremovič [12] and an equivalent set of axioms was obtained to get a proximity space. By defining the concept of closure of a set A of X, he proved the following theorem, obtaining thereby a topology on the set:

If a subset A *of a proximity space* (X, δ) *is defined to be closed if and only if* x δ A *implies* x ∈ A, *then the collection of complements of all closed sets so defined yields a topology* τ =τ (δ) *on* X [cf. 29].

Thus, every proximity space is a topological space. In fact, the topology induced by δ is always completely regular. In addition, if δ is separated [cf. II, def. 2.1.1], then the induced topology is Tychonoff. Efremovič has further shown that:

If (X, τ) *is a completely regular space, then the proximity δ defined by*: *for* A, B ∈ P(X), "A δ B *if and only if* A *and* B *are functionally distinguishable" is compatible with* τ. *If* (X, τ) *is Tychonoff, then* δ *is separated.*

In a T_4- *space* (X, τ), *for* A, B ∈ P(X), A δ B *if and only if* clA ∩ clB ≠ ϕ *defines a compatible proximity.*

The pair (X, δ), where δ is compatible with the given topology [cf. II, def. 2.1.2] on X, is called an *associated proximity space.* A study in this direction was further carried out by Smirnov and he subsequently proved that every Hausdorff completely regular space has a maximal associated separated proximity space and it has a minimal associated separated proximity space if the space is locally compact.

The concept of a proximity mapping [cf. II, def.2.1.11] was taken into account by Efremovič [12] in order to study functions from one proximity space to another. This proximity map or proximally continuous map is a natural analogue of continuous maps. It can be verified that a proximity mapping between two proximity spaces is continuous with respect to the induced topologies. But the converse does not hold, i.e. a continuous mapping with respect to induced topologies need not be a proximally continuous mapping. However, we have the following result:

If (X, δ_1) *and* (Y, δ_2) *are proximity spaces and* X *is compact, then every continuous mapping* f *from* X *to* Y *is a proximity mapping* [cf. 29].

Note that a topological space may have different compatible proximities. For A, B $\in$ P(R), the two proximities δ_1 and δ_2 defined by A δ_1 B if and only if D(A,B) = 0 and A δ_2 B if and only if clA $\cap$ clB $\neq \phi$ respectively, are compatible with the usual topology on R. The set of all compatible proximities on a given set X is partially ordered by the relation '>' or 'finer than' [cf. II, def.2.1.6].

Every compact Hausdorff space X *has a unique compatible separated proximity* [cf. 29] δ *defined by* A δ B *if and only if* clA $\cap$ clB $\neq \phi$ *for* A, B $\in$ P(X).

1.2 Fibrewise Proximity Spaces

The history of fibrewise topology goes back to Riemann who introduced the concept. The modern development of the subject has its origin in the work of Hurewicz on fibre spaces and of Whitney of fibre bundles.

Fibrewise uniform spaces were studied by Niefield where both base space and the total space are required to the uniform and the projection is required to be uniformly continuous.

In fibrewise topology, we work over a topological base space B. When B is a point space, the theory reduces to that of ordinary topology. For results and definitions related to the theory of fibrewise topological spaces, we refer [19].

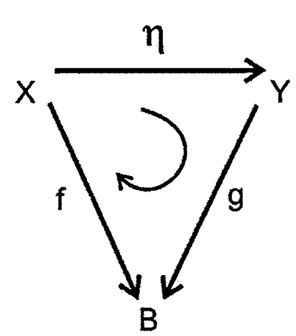
In chapter III of the present thesis, a theory of fibrewise proximity space has been developed. Every fibrewise proximity space is a fibrewise topological space. Introducing, the concept of fibrewise proximity spaces, the notion of separation axioms for these spaces are developed and several results are proved.

1.3 Retraction In the Category **ProxB**

The category theory was founded in the year 1950 by Eilenberg and Mclane. The developments in the theory of proximity spaces indicate that for Tychonoff spaces the category **Prox** of separated proximity spaces as objects and p-continuous maps as morphisms, is a more appropriate tool than the familiar category **Top** of all topological spaces as objects and continuous maps as morphisms.

In Chapter IV, the notion of the category **ProxB** is introduced; its objects are fibrewise proximity spaces or a proximity map into the proximity base space B from (X, δ) and morphisms are fibrewise p-maps.

For fibrewise proximity spaces (X, f) and (Y, g), the map $\eta : X \rightarrow Y$ is called a fibrewise p-map if the following diagram commute



Using the concept of proximal retract and proximal adjunction space the following characterization has been proved : For X, Y $\in$ **Prox** and a proximal subspace A of X, the map $f : A \rightarrow Y$ has a p- continuous extension if and only if Y is a proximal retract of the proximal adjunction space $X \cup_f Y$. Further the notion of fibrewise proximal adjunction space is introduced and some results have been proved.

1.4 Proximal Function Space

In the set Y^X, the uniform topology and the compact convergence topology made specific use of the metric d for the space Y. But both the topologies can not be extended where Y is a general topological space. However in the space C(X,Y) of continuous functions the compact open topology coincides with the compact convergence topology, when Y is a metric space, is introduced and studied by several [28].

In chapter V, we introduce the concept of proximal function space. It is shown that the set of p-continuous maps f from X to Y, is a separated proximity space.

1.5 Threads and Ultrafilters in R-Proximity Spaces

In chapter VI, the concept of R-proximity spaces is dealt with. After developing the theory of proximal regular covers, the notion of proximal threads has been introduced. It is shown that in the set R(X) of all proximal regular closed sets, a x_p-family ξ is a x_p-ultrafilter if and only if ξ is a thread in the set RC(X) of all proximal regular covers of X.

1.6 Absolute of R-Proximity Spaces

A space Y is called an extension of a space X if X is a dense subspace of Y. Compactifications and real compactification of Tychonoff spaces, and H-closed extensions of Hausdorff spaces are the important extensions. In 1958, A. Gleason showed that the projective objects in the category of compact spaces and continuous maps are the extremally disconnected space and given a compact space X, there is a space T(X) in this category mapping irreducibly onto X. In 1963, S. Elliadis generalized this result to the category of

Hausdorff spaces and perfect o-continuous maps and T(X) was termed as "Absolute" of X.

The importance of extremally disconnected spaces particularly in general topology, become transparent in the context of the theory of absolute topological spaces. Recall that A space X is called extremally disconnected if for every open set $U \subseteq X$ in X the set $\overline{U}$ is not only closed but also open in X, where $\overline{U}$ denotes the closure of U.

It is proved that the proximal space $\dot{X}$ of all distinguished ultrafilters, is an proximal absolute of a regular proximity space X.

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