CHAPTER-VI

THREADS AND ULTRAFILTERS IN R-PROXIMITY SPACES

6.1 Introduction

In the present chapter, by a proximity space we shall mean a pair (X, δ) where X is a non-empty set and δ is a symmetric relation between subsets of X satisfying 6.2.1((PI)-P(IV)) condition: In addition, if δ satisfies P(V), δ is called an R-proximity on X; the pair (X, δ) is known as R-proximity space.

Note that the topology induced by the R-proximity is regular. Using the concept of R-proximity spaces, the theory of proximal regular covers has been developed and the notion of proximal threads have been introduced.

In section 2, it has been shown that X is an R-proximity space if and only if the family RC(X) of all p-regular covers of a proximity space is refined. This characterization of R-proximity space has further been used to prove a characterization for x_p-family ξ to be a x_p-ultrafilter in terms of threads.

6.2 Proximal Regular Cover-Some Definitions

DEFINITION 6.2.1: A proximity on X is a symmetric relation δ between subsets of X satisfying the following conditions:

P(I) $\phi \not\delta A$ for every $A \subseteq X$ ($\not\delta$ means "not-δ");

P(II) $A \delta A$ for every $A \neq \phi$;

P(III) $A \delta (B \cup C)$ if and only if $A \delta B$ or $A \delta C$;

P(IV) If x and y are distinct points of X, then $\{x\} \not\delta \{y\}$.

In addition, if

P(V) $x \ll A$, then there is $B \subseteq X$ such that $x \ll B \ll A$, holds, δ is called an R-proximity on X, where by $B \ll A$, we mean that A is a proximal neighbourhood of B and $B \ll A$ if and only if $B \not\delta X\text{-}A$. The condition P(V) is called the axiom of regularity. These axioms lead to an operator c on 2^X defined by $c(A) = \{x : \{x\} \delta A\}$, called the closure operator. It induces the topology whose closed sets are precisely the sets A such that c (A) = A.

Note that the topology induced by an R-proximity δ is regular. In fact, we have

"A topology is regular if and only if it is the topology induced by a R-proximity".

DEFINITION 6.2.2: Let (X, δ) be a R-proximity space and A be a closed set in X. Then A is called proximal regular or p-regular subset of X if it is the closure of an open set in X, i.e. A is a p-regular subset of X if

$$A = \overline{\text{Int} A},$$

where closure and interior are taken with respect to the induced topology from δ.

DEFINITION 6.2.3: A finite closed cover α of X is called a proximal regular cover or a p-regular cover if all the elements of α are p-regular closed sets and the interiors of distinct elements of α do not intersect i.e. if for A, B $\in \alpha$

$\text{Int } A \cap \text{Int } B = \phi$.

If α is an infinite closed cover with these properties, then they must be locally finite in X.

6.3 x_p-Ultrafilter And Threads

In the present section, the notion of x_p-family, distinguished x_p-family, x_p-ultrafilter and threads have been introduced. Some results and characterizations are also proved.

DEFINITION 6.3.1: Let (X, δ) be a R-proximity space. Denote by R (X) the collection of p-regular closed sets of X. Let $\xi \subseteq R(X)$. Then ξ is called x_p-centralized if

$\cap$ {Int A : A $\in$ ξ'} $\neq \phi$, for every finite sub family $\xi' \subseteq \xi$.

If ξ is x_p-centralized family, then it is called a x_p-family. If x_p-family ξ is not contained in any other x_p-family, then it is called a x_p-ultrafilter.

DEFINITION 6.3.2: Let (X, δ) be a R-proximity space and ξ be a x_p-family. Then the x_p-family ξ is called distinguished if

$$\cap \{A : A \in \xi\} \neq \phi \ .$$

It is said to be distinguished at the point x, if

$$x \in \cap \{A : A \in \xi\} \ .$$

STAR OF A POINT 6.3.3: Let $\alpha(x)$, where $x \in X$ be the star of x w.r.t. α is defined as

$$\alpha(x) = \cup \{A \in \alpha : x \, \delta \, A\}.$$

We shall denote by RC(X) the family of p-regular covers of X.

DIRECTED FAMILY 6.3.4: The family RC(X) is said to be directed if for every pair of p-regular covers $\alpha_1, \alpha_2 \in$ RC(X), there exist a p-regular cover $\alpha_3 \in$ RC(X) such that $\alpha_3 > \alpha_1$, $\alpha_3 > \alpha_2$.

DEFINITION 6.3.5: A cover λ is a refinement of a cover γ or that λ refines γ if for each $V \in \lambda$ there is a set $U \in \gamma$ such that $V \subseteq U$.

6.3.6: For $\alpha, \beta \in RC(X)$, we have

$\gamma = \alpha \wedge \beta = \{\overline{\text{Int } A \cap \text{Int} B}: A \in \alpha, B \in \beta$ and $\text{Int } A \cap \text{Int } B \neq \phi\}$.

Denote by $RC_0(X)$ a refined and directed family of p-regular covers of X.

DEFINITION 6.3.7: Consider the Tychonoff Product T_0 of the discrete proximity space $\hat{\alpha}, \alpha \in RC_0(X)$

i.e. $T_0 = \prod\{\hat{\alpha}, \alpha \in RC_0(X)\}$.

Note that a point $\xi \in T_0$ is a family of the form

$\xi = \{A^{\alpha} : \alpha \in RC_0(X)\}$

and is called a thread, if $A^{\alpha_2} \subseteq A^{\alpha_1}$ whenever

$\alpha_2 > \alpha_1, \alpha_1, \alpha_2 \in RC_0(X)$.

DEFINITION 6.3.8: Denote by $T_0(X)$, the collection of all threads in $RC_0(X)$. A thread $\xi \in T_0$ (X) is said to be distinguished if

$$\cap \{A : A \in \xi\} \neq \phi.$$

This is called distinguished at the point x if

$$x \in \cap \{A : A \in \xi\}.$$

THEOREM 6.3.9: *Let* (X, δ) *be a R-proximity space,* α *a p-regular cover of* X *and* $x_0 \in X$. *Then* $x_0 \in \text{Int}\ \alpha(x_0)$, *where* $\alpha(x_0) = \cup \{A \in \alpha : x_0\ \delta\ A\}$.

PROOF: Consider $E_\alpha(x_0) = \cup \{A \in \alpha : x_0\ \bar{\delta}\ A\}$. Since each $A \in \alpha$ is closed in X and α is a p-regular finite cover (or locally finite cover) it follows that $E_\alpha(x_0)$ is closed in X. Hence X - $E_\alpha(x_0)$ is open, i.e. $x_0 \in X - E_\alpha(x_0)$ implies $x_0 \ll X - E_\alpha(x_0) \subseteq \alpha(x_0)$. This gives $x_0 \ll \alpha(x_0)$ and hence $x_0 \ll \text{Int}\ \alpha(x_0)$. Consequently, $x_0 \in \text{Int}\ \alpha\ (x_0)$.

THEOREM 6.3.10: *Let* (X, δ) *be an* R*-proximity space. Suppose that* α_1 *and* α_2 *are two p-regular covers of* X *and that* $\alpha_2 > \alpha_1$, *i.e.* α_2 *is refinement of* α_1. *Then*

(a) *each element of* α_2 *is contained in a unique element of* α_1;

(b) *each element of* α_1 *is the union of all elements of* α_2, *contained in the given element of* α_1.

PROOF: We prove only (a). To the contrary assume that for an element $A_i^{\alpha_2} \in \alpha_2$, there exists $A_{j_1}^{\alpha_1}, A_{j_2}^{\alpha_1} \in \alpha_1$ such that

$$A_i^{\alpha_2} \subset A_{j_1}^{\alpha_1}\ ,\ A_i^{\alpha_2} \subset A_{j_2}^{\alpha_1}.$$

Then

$$\text{Int}\ A_i^{\alpha_2} \subset \text{Int}\ A_{j_1}^{\alpha_1}\ ,\ \text{Int}\ A_i^{\alpha_2} \subset \text{Int}\ A_{j_2}^{\alpha_1}.$$

Consequently, $\text{Int}\, A_{j_1}^{\alpha_1} \cap \text{Int}\, A_{j_2}^{\alpha_1} \neq \phi$, a contradiction to the fact that α_1 is a p-regular cover (def. 6.2.3).

The following can easily be proved.

THEOREM 6.3.11: *Let* (X, δ) *be a R-proximity space and* α, β *be two p-regular covers of* X. *Then* $\gamma = \alpha \wedge \beta$ *is a p-regular cover of* X, *refining both* α *and* β.

6.4 Characterization of R-Proximity Spaces

THEOREM 6.4.1: *The family* RC(X) *of all p-regular covers of a proximity space* X *is refined if and only if* X *is an R-proximity space.*

PROOF: Suppose that X is an R-proximity space. Let $x_0 \in$ X and an open set G_{x_0} in $\tau(\delta)$ containing x_0 then $x_0 << G_{x_0}$. Using axiom of regularity 6.2.1 [P(V)], there exist an (open) set G'_{x_0} in X such that $x_0 << G'_{x_0} \subseteq \overline{G'_{x_0}} << G_{x_0}$. Hence $x_0 \in G'_{x_0}$ *and* $\overline{G'_{x_0}} \subseteq G_{x_0}$.

Let $\alpha_0 = \left\{\overline{G'_{x_0}}, \overline{X - \overline{G'_{x_0}}}\right\}$ be a p-regular cover. Obviously, $\alpha_0(x_0) = \overline{G'_{x_0}} \subseteq G_{x_0}$. Consequently, the family RC(X) is refined.

Conversely, suppose that RC(X) is refined. To prove the theorem, it suffices to show that [P(V)] is satisfied. Let $x_0 \in$

X and there is an open set G_{x_0} containing x_0 then $x_0 << G_{x_0}$. Since RC(X) is refined, therefore there exist $\alpha_0 \in RC(X)$ such that $\alpha_0 (x_0) \subseteq G_{x_0}$.

By the theorem 6.3.9, $x_0 \in Int\ \alpha_0 (x_0)$. This gives $x_0 << Int\ \alpha_0 (x_0)$.

Put $Int\ \alpha_0 (x_0) = G'_{x_0}$. Then

$$x_0 << G'_{x_0} \subseteq \overline{G'_{x_0}} = \overline{Int\ (\alpha_0(x_0))} = \alpha_0 (x_0) \subseteq G_{x_0}.$$

Hence X is an R-proximity space.

THEOREM 6.4.2: *Let* X *be an* R*-proximity space and* RC(X) *the family of all p-regular covers of* X. *If* ξ *is a distinguished thread in* RC(X), *then* $\cap$ {A: A $\in \xi$} *consists of exactly one point.*

PROOF: Let ξ be a distinguished thread in RC(X). We have to show that $\cap$\{A: A $\in \xi$\} consists of exactly one point. To the contrary assume that for $x_1, x_2 \in X$ such that $x_1 \neq x_2$, $x_1 \in \cap$\{A: A $\in \xi$\} and $x_2 \in \cap$ \{A: A $\in \xi$\}.

Since X is an R-proximity space, $x_1 \neq x_2$ gives $x_1 << X - x_2$ so there exist an open set G_{x1} in $\tau(\delta)$ containing x_1 such that

$$x_1 << G_{x_1} \subseteq \overline{G_{x_1}} << X - x_2 \text{ which gives } x_2 \notin \overline{G_{x_1}}.$$

Consider $\alpha_0 = \left\{\overline{G_{x_1}}, \overline{X\text{-}\overline{G_{x_1}}}\right\}$ the p-regular cover. Since ξ is a thread, either $\overline{G_{x_1}} \in \xi$ or $\overline{X\text{-}\overline{G_{x_1}}} \in \xi$. Suppose $\overline{G_{x_1}} \in \xi$. In this case, $x_2 \notin \overline{G_{x_1}}$, and hence it follows that $x_2 \notin \cap\{A: A \in \xi\}$ which is a contradiction.

Similarly, the case when $\overline{X\text{-}\overline{G_{x_1}}} \in \xi$ leads to a contradiction.

Hence $\cap \{A: A \in \xi\}$ consists of exactly one point.

THEOREM 6.4.3: Let X *be an* R*-proximity space and* R(X) *is the family of all regular closed sets in* X. *Then every* x_p*-family* η *(distinguished at the point* $x_0 \in X$*) is contained in some* x_p*-ultrafilter.*

PROOF: Let η be a x_p–family (distinguished at the point x_0) i.e. $x_0 \in \cap\{A: A \in \eta\}$. Denote by

$$A(\eta) = \{\eta_0 : \eta_0 \supset \eta, \text{ where } \eta_0 \text{ is an } x_p\text{-family}\}.$$

Let $\eta_1, \eta_2 \in A(\eta)$ and assume that $\eta_2 > \eta_1$, if $\eta_1 \subset \eta_2$. Then $A(\eta)$ is partially ordered. Suppose that $R \subset A(\eta)$ is a nest in $A(\eta)$. Then R has a maximal element in $A(\eta)$, namely $\eta^* = \cup\{\eta : \eta \in R\}$ which is x_p-family (distinguished at $x_0 \in X$) containing η.

Consequently, $A(\eta)$ has a maximal element.

THEOREM 6.4.4: *Let* X *be a R-proximity space and* R(X) *is the family of all regular closed sets in* X. *Then*

(a) *the* x_p*-family* η *is a* x_p*-ultrafilter if and only if the following conditions are satisfied:*

(i) *if* $A_1, A_2 \in \eta$, *then* $\overline{\text{Int } A_1 \cap \text{Int} A_2} \in \eta$;

(ii) *if* $A_1 \in \eta$ *and* $A_1 \subseteq A_2$, $A_2 \in R(X)$, *then* $A_2 \in \eta$;

(iii) *if* $B \in R(X)$ *and* $\text{Int } B \cap \text{Int } A \neq \phi$ *for every* $A \in \eta$, *then* $B \in \eta$.

(b) *if* η *is a* x_p*-ultrafilter and* $A \in R(X)$, *then either* $A \in \eta$ *or* $\overline{X - A} \in \eta$.

PROOF (a): (i) and (ii) are obvious. For the proof of (iii), let $B \in R(X)$ such that $\text{Int } B \cap \text{Int } A \neq \phi$ for every $A \in \eta$.

If we put $\eta^* = \{B\} \cup \eta \cup \{\overline{\text{Int B} \cap \text{Int} A} : A \in \eta\}$, then η^* is a x_p-family. Hence by the maximality of the x_p-family η, we obtain $\eta = \eta^*$ and $B \in \eta$.

Thus, each x_p- ultrafilter satisfies conditions (i), (ii) and (iii).

Conversely, suppose that the x_p-family η satisfies these three conditions. We have to show that η is a x_p-ultrafilter.

Assume that $\eta \subseteq \eta^*$. Then, in each case we have $\text{Int B} \cap \text{Int A} \neq \phi$ if $B \in \eta^*$ and $A \in \eta \subseteq \eta^*$.

But η satisfies condition (iii), hence $B \in \eta$, so that $\eta^* = \eta$ i.e. η is a x_p-ultrafilter.

(b) This is easy to prove.

THEOREM 6.4.5: *Let* X *be a R-proximity space. A* x_p-*family* ξ *is a* x_p-*ultrafilter in* R(X) *(resp. distinguished* x_p-*ultrafilter) if and only if* ξ *is a thread (resp. a distinguished thread) in the family* RC(X) *of all p-regular covers of* X.

PROOF: Let us first suppose that the x_p-family ξ is a x_p-ultrafilter. We now show that ξ is a thread.

(i) Let $\alpha \in RC(X)$. It is sufficient to show that there is an element $A \in \alpha$ such that $A \in \xi$.

Suppose that to the contrary, no such A exists in α. Then by the theorem 6.4.4 (b), we have

$\overline{X - A} \in \xi$ for all $A \in \alpha$.

Using the theorem 6.4.4 (a), we have

$\overline{\cap \left\{\text{Int}\, \overline{X - A} : A \in \alpha \right\}} \in \xi$.

But $\text{Int}\, \overline{X - A} = X - A$, therefore, $\cap\{(X - A) : A \in \alpha\} = X - \cup \{A : A \in \alpha\} = \phi$, a contradiction. Thus $A \in \xi$.

Now, let $A_1, A_2 \in \alpha$ and $A_1, A_2 \in \xi$. Then $A_2 \subset \overline{X - A_1}$, follows from the fact that α is regular cover and $\text{Int}\, A_1 \cap \text{Int}\, A_2 = \phi$. Consequently, $\overline{X - A_1} \in \xi$, which is a contradiction, by theorem 6.4.4 (b). Hence $\xi \cap \alpha$ is non-empty and consists of exactly one element for each $\alpha \in RC(X)$.

(ii) Let $\alpha_1, \alpha_2 \in RC(X)$ and $\alpha_2 > \alpha_1$. Let $\{A_1\} = \xi \cap \alpha_1$, $\{A_2\} = \xi \cap \alpha_2$ i.e. A_1 (resp. A_2) is a unique element of α_1 (resp. α_2) belonging to ξ. It is sufficient to show that $A_2 \subset A_1$. This result follows from theorem (6.3.10) and above theorem.

Conversely, suppose that the x_p-family ξ is a thread. We now show that ξ is a x_p-ultrafilter.

Let $A_1, A_2 \in \xi$. By the definition of thread, we have $\{A_1\} = \xi \cap \alpha_1$, $\{A_2\} = \xi \cap \alpha_2$, where $\alpha_1, \alpha_2 \in RC(X)$.

Now, let us consider another p-regular cover $\alpha_3 = \alpha_2 \wedge \alpha_1$, $\alpha_3 > \alpha_2$ and $\alpha_3 > \alpha_1$ (by using the theorem 6.3.11) consider an element $A_3 \in \xi \cap \alpha_3$. Then by the definition of a thread, $A_3 \subset A_1$, $A_3 \subset A_2$.

But $\alpha_3 = \alpha_1 \wedge \alpha_2$, so that

$$A_3 = \left\{\overline{\text{Int } A' \cap \text{Int } A''} : A' \in \alpha_1,\ A'' \in \alpha_2\right\}$$

Clearly, $A_1 = A'$, $A_2 = A''$. Then

$$A_3 = \left\{\overline{\text{Int } A_1 \cap \text{Int } A_2} \in \xi\right\}.$$

Hence, it follows that ξ is x_p-centralized, i.e., ξ is a x_p-family. Let us assume that ξ is not a maximal x_p-family, i.e. ξ is not a x_p-ultrafilter. Then ξ is contained in some x_p-ultrafilter $\bar{\xi}$.

From (i) we find that $\bar{\xi} \cap \alpha$ is non-empty and consists of exactly one element for each $\alpha \in RC(X)$. To the contrary, this property is already possessed by a smaller subfamily of ξ. Hence ξ is a x_p-ultrafilter. Consequently, if ξ is a distinguished thread, then ξ is also a distinguished x_p-ultrafilter.

REMARK 6.4.6: Since the topology induced by an R-proximity space is regular, the results holds good for regular topological spaces as well.

The following result holds.

THEOREM 6.4.7: *Let* X *be a* R-*proximity space and* $x_0 \in X$. *Let* $A_0 \in R(X)$ *such that* $x_0 \in A_0$. *Then there is a distinguished thread* ξ *in* RC(X) *such that*

$$\{x_0\} = \cap \{A : A \in \xi\} \text{ and } A_0 \in \xi.$$

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