CHAPTER V

PROXIMAL FUNCTION SPACES

5.1 Introduction

In this chapter we introduce the concept of proximal function space.

Section 2 of the chapter deals with the notion of a proximal evaluation map and some results.

In section 3, it is shown that the collection of p-continuous maps f from X to Y, denoted by Y^X, is a separated proximity space. In addition, to the p-continuity of evaluation map, several results are proved for proximal function space (Y^X, Π).

An application of identification proximity is described in section 4. The concepts of proximal homotopy is dealt with in section 5.

5.2 Proximal Associates

The present section contains definitions and results. We denote by Y^X the set of all p-continuous maps from X to Y.

DEFINITION 5.2.1: Let X and Y be proximity spaces. Let [A, U] denote the collection of p-continuous maps $f \in Y^X$ such that $f(A) \ll U$, $A \in P(X)$, $U \in P(Y)$. Thus $[A, U] = \{f \in Y^X : f(A) \ll U\}$.

DEFINITION 5.2.2: Let X and Y be two proximity spaces. The map $e : Y^X \times X \to Y$, defined by $e(f, x) = f(x)$, $x \in X$ is called the proximal evaluation map.

DEFINITION 5.2.3: Let X, Y and Z be proximity spaces. Let $\alpha : X \times Y \to Z$ be p-continuous in y for each fixed x; then

$$[\hat{\alpha}(x)](y) = \alpha(x, y) \qquad (*)$$

defines, a map from Y to Z and so $x \to \hat{\alpha}(x)$ is a map from X to Z^Y. Conversely, given a map $\hat{\alpha} : X \to Z^Y$, the formula (*) defines a map $\alpha : X \times Y \to Z$ p-continuous in y for each fixed x.

Two maps $\alpha : X \times Y \to Z$ and $\hat{\alpha} : X \to Z^Y$ related by the above formula are called proximal associates.

5.3 Proximal Function Space

In this section, a separated proximity Π is defined on the set Y^X. Several definitions have been given and p-continuity of maps have been proved.

THEOREM 5.3.1: *A binary relation* Π *on* $P(Y^X)$ *defined by: for* $\mathcal{A}, \mathcal{B} \in P(Y^X)$; $\mathcal{A} \not\Pi \mathcal{B}$ *if and only if there exist* $[A, U]$, $[B, V] \in P(Y^X)$, $A, B \in P(X)$, $U, V \in P(Y)$ *such that* $\mathcal{A} \subset [A, U]$, $\mathcal{B} \subset [B, V]$ *and* $U \not\delta V$ *is a separated proximity on* Y^X.

PROOF: We shall prove 2.1.1 (P2) and strong axiom 2.1.1(P4) only.

(ii) Suppose $(\mathcal{A} \cup \mathcal{B}) \not H \mathcal{C}$. To show that $\mathcal{A} \not H \mathcal{C}$ and $\mathcal{B} \not H \mathcal{C}$. Since $(\mathcal{A} \cup \mathcal{B}) \not H \mathcal{C}$ there exist [A, U], [B, V] such that $(\mathcal{A} \cup \mathcal{B}) \subset [A, U]$, $\mathcal{C} \subset [B, V]$ and $U \not\delta V$ i.e. $\mathcal{A} \subset [A, U]$, $\mathcal{B} \subset [A, U]$ and $\mathcal{C} \subset [B, V]$ such that $U \not\delta V$. It follows that $\mathcal{A} \not H \mathcal{C}$ and $\mathcal{B} \not H \mathcal{C}$. Conversely, suppose that $\mathcal{A} \not H \mathcal{C}$ and $\mathcal{B} \not H \mathcal{C}$. Then there exist [A, U], [B, V], [C, W] and [D, W_0], where A, B, C, D $\in$ P(X) and U, V, W, $W_0 \in$ P(Y) satisfying

$\mathcal{A} \subset [A, U]$, $\mathcal{C} \subset [B, V]$ and $U \not\delta V$.

Also, $\mathcal{B} \subset [C, W]$, $\mathcal{C} \subset [D, W_0]$ and $W \not\delta W_0$.

Since $U \not\delta V$ and $W \not\delta W_0$, we get $U \cup W \not\delta V \cap W_0$. Also, $\mathcal{A} \subset [A, U]$, $\mathcal{B} \subset [C, W]$ this imply $\mathcal{A} \cup \mathcal{B} \subset [A, U] \cup [C, W]$ or $\mathcal{A} \cup \mathcal{B} \subset [A \cup C, U \cup W]$ and $\mathcal{C} \subset [B, V] \cap [D, W_0]$ or $\mathcal{C} \subset [B \cap D, V \cap W_0]$. It follows that $(\mathcal{A} \cup \mathcal{B}) \not H \mathcal{C}$.

(iv) Suppose $\mathcal{A} \not H \mathcal{B}$. By definition, there exist A, B $\in$ P(X) and U, V $\in$ P(Y) such that $\mathcal{A} \subset [A, U]$, $\mathcal{B} \subset [B, V]$ and $U \not\delta V$. Then there exist W, $W_0 \in$ P(Y) such that $U \not\delta W$, $V \not\delta W_0$ and $Y\text{-}W \not\delta Y\text{-}W_0$ i.e. $U \ll Y\text{-}W \ll W_0 \ll Y\text{-}V$. Hence $\mathcal{A} \not H [A, W]$.

Similarly, we can show that $\mathcal{B} \not H [A, Y\text{-}W]$.

Then choose $\mathcal{C} \equiv [A, W]$ a subset of Y^X such that $\mathcal{A} \not H \mathcal{C}$ and $(Y^X - \mathcal{C}) \not H \mathcal{B}$.

THEOREM 5.3.2: *If* Y *is Hausdorff (separated), then* Y^X *is also Hausdorff.*

PROOF: Let f, g $\in Y^X$ with $f \neq g$. Then there exists $p \in X$ such that $f(p) \neq g(p)$. Since Y is separated, $f(p) \,\bar{\delta}\, g(p)$, so obtain subsets G and H in Y such that $f(p) \ll G$, $g(p) \ll H$ and $G \,\bar{\delta}\, H$. This gives $f \in (p, G)$, $g \in (p, H)$ and $(p, G) \cap (p,H) = \emptyset$. Hence the assertion.

5.3.3: Let X, Y and Z be three proximity spaces. For $f \in Y^X$ and $g \in Z^Y$, the composition $g \circ f \in Z^X$ so that $T(f, g) = g \circ f$ defines a map from $Y^X \times Z^Y$ to Z^X.

THEOREM 5.3.4: *The map* T *defined as above is always p-continuous in each argument separately, i.e.*

(a) $g \to g \circ f_1$ *is p-continuous map* $F : Z^Y \to Z^X$ *for each fixed* $f_1 \in Y^X$.

(b) $f \to g_1 \circ f$ *is a p-continuous map* $G : Y^X \to Z^X$ *for each fixed* $g_1 \in Z^Y$.

PROOF: (a) Suppose $\mathcal{C}, \mathcal{D} \in P(Z^X)$ such that $\mathcal{C} \,\bar{\Pi}\, \mathcal{D}$, then for some C, D in P(X) such that $\mathcal{C} \subset [C, U]$, $\mathcal{D} \subset [D, V]$ and $U \,\bar{\delta}\, V$, where $U, V \in P(Z)$. To show $F^{-1}(\mathcal{C}) \,\bar{\Pi}\, F^{-1}(\mathcal{D})$. Now, let $g \in F^{-1}(\mathcal{C})$, then $F(g) \in \mathcal{C}$ i.e. $(g \circ f_1) \in \mathcal{C} \subset [C, U]$ for each fixed f_1.This gives $g \in [f_1(C), U]$. Then $F^{-1}(\mathcal{C}) \subset [f_1(C),U]$. Similarly, $F^{-1}(\mathcal{D}) \subset [f_1(D), V]$. Since $U \,\bar{\delta}\, V$, $f_1(C), f_1(D) \in P(Y)$ it follows that $F^{-1}(\mathcal{C}) \,\bar{\Pi}\, F^{-1}(\mathcal{D})$. Hence the map is p-continuous.

(b) This is proved similarly.

THEOREM 5.3.5: *Let* X, Y *and* Z *be three proximity spaces. Then the map* $T : Y^X \times Z^Y \to Z^X$ *is p-continuous.*

PROOF: Follows by the definition of function space proximity Π.

THEOREM 5.3.6: *Let* X *and* Y *be two proximity space. Then the evaluation map* $e : Z^Y \times Y \to Z$ *is p-continuous.*

PROOF: Follows from above, by taking X a singleton.

THEOREM 5.3.7: *If* $\alpha : X \times Y \to Z$ *is p-continuous, then* $\hat{\alpha} : X \to Z^Y$ *is also p-continuous.*

PROOF: Let A, B $\in$ P (X) such that A δ B. For the p-continuity of $\hat{\alpha}$, it is to be shown that $\hat{\alpha}(A)\ \Pi\ \hat{\alpha}(B)$ in Z^Y. To the contrary, suppose $\hat{\alpha}(A)\ \not\Pi\ \hat{\alpha}(B)$ in Z^Y. Then, by definition of Π, there exist [W, U] $[W_0, V] \subseteq Z^Y$ such that $\hat{\alpha}(A) \subseteq [W, U]$, $\hat{\alpha}(B) \subseteq [W_0, V]$ and U $\not\delta$ V. This implies, $\alpha\ (A \times W) \subseteq U$, $\alpha\ (B \times W_0) \subseteq V$. Since U $\not\delta$ V and α is p-continuous, it follows that $\alpha^{-1}(\alpha\ (A \times W))\ \not\delta\ \alpha^{-1}(\alpha(B \times W_0))$ and hence A $\not\delta$ B and W $\not\delta$ W_0, which is a contradiction. Consequently, $\hat{\alpha}$ is p-continuous.

Converse of the above theorem also holds.

THEOREM 5.3.8: *If* $\hat{\alpha} : X \to Z^Y$ *is p-continuous, then* $\alpha : X \times Y \to Z$ *is also p-continuous.*

PROOF: Consider the composite map $X \times Y \xrightarrow{\hat{\alpha} \times I} Z^Y \times Y \xrightarrow{e} Z$, where $(\hat{\alpha} \times I)$ is defined by $(\hat{\alpha} \times I)(x, y) = (\hat{\alpha}(x), y)$ for $x, y \in X \times Y$ and e is the evaluation map. Since, $\hat{\alpha}$ is p-continuous, so $\hat{\alpha} \times I$ is also p-continuous. In view of theorem 5.3.6, being the composite map of two p-continuous maps, the p-continuity of the map α follows.

5.4 Applications

In this section, we provide several applications of this function space proximity defined in the above section. First of these, namely, the map $p \times I : X \times Y \longrightarrow R \times Y$ is an p-identification, if $p : X \longrightarrow R$ is so, without assuming the local compactness of the space Y.

Second application, provides a connection between the function space proximity and the proximal homotopy between two p-continuous maps of X into Y.

DEFINITION 5.4.1: Let X be a proximity space and G be a partition of X. Then the set X/G, whose points are the members of the given partition G, when endowed with the quotient proximity is called the quotient proximity space and the map $p : X \longrightarrow X/G$ mapping a point x to the unique set of G that contains x, is called the quotient proximity map.

REMARK 5.4.2: Let X and Y be proximity spaces and f : X $\longrightarrow$ Y be a proximally continuous onto map. Then $G_f = \{f^{-1}(y) : y \in Y\}$ is a partition of X. Note that if $x \in X$, $p(x) = [x] = f^{-1}[f(x)]$. Let us define a function $\phi_f : X/G_f \longrightarrow Y$ by

$$\Phi_f[(x)] = f(x).$$

Then ϕ is well defined and $\phi \circ p = f$.

DEFINITION 5.4.3 [35]: Let X and Y be proximity spaces and f : X $\longrightarrow$ Y be a proximally continuous onto map. Then f is called a p-identification map whenever the proximity on Y is the quotient proximity.

Note that not every p-continuous onto map is a p-identification map. However, the proximal quotient map p : X $\longrightarrow$ X/G is a p-identification map.

RESULT 5.4.4 [35]: Let X, Y and Z be proximity spaces and let f : X $\longrightarrow$ Y be a p- continuous onto map. Then f is a p-identification map iff for each proximity space Z and each map g : Y $\longrightarrow$ Z. The p-continuity of g o f implies that of g.

RESULT 5.4.5 [35]: Let f : X $\longrightarrow$ Y be a p-identification map and h : X $\longrightarrow$ Z is proximally continuous. Suppose that $h \circ f^{-1}$ is single valued i.e., h is constant on each fibre $f^{-1}(y)$. Then

(i) $h \circ f^{-1} : Y \longrightarrow Z$ is proximally continuous;

(ii) h o f^{-1} is a p-open map iff h is a p-open map whenever every set X is f-saturated, i.e., whenever for every $A \subseteq X$, $A = f^{-1}(f(A))$.

THEOREM 5.4.6: *Let* $p : X \longrightarrow R$ *be an p-identification. Then the map* $p \times I : X \times Y \longrightarrow R \times Y$ *is an p-identification.*

PROOF: Let Z be proximity space and $g : R \times Y \longrightarrow Z$ be any p-map such that $g \circ (p \times I) : X \times Y \longrightarrow Z$ is p-continuous; in view of the result 5.4.4, the theorem will be proved if we can show that g is p-continuous. Since $\alpha = g \circ (p \times I)$ is p-continuous, then the associated map $\hat{\alpha} : X \rightarrow Z^Y$ is also p-continuous. Moreover, $\hat{\alpha} p^{-1} : R \rightarrow Z^Y$ is single-valued, in fact, $\hat{\alpha} p^{-1} = \hat{g}$, since

$$[\hat{\alpha} p^{-1}(r)](y) = (p^{-1}(r), y) = g(pp^{-1}(r), y) = g(r, y) = [\hat{g}(r)](y).$$

By 5.4.5, we conclude that $\hat{\alpha} p^{-1} = g : R \longrightarrow Z^Y$ is p-continuous and also $g : R \times Y \longrightarrow Z$ is p-continuous.

REMARK 5.4.7: From the above result, it follows that the result holds good in classical (topological) context.

5.5 Proximal Function Space And Homotopy

DEFINITION 5.5.1: Let f, g be two continuous maps of Y into Z. Then f and g are said to be proximally homotopic if there is a p-continuous map, $h : Y \times I \longrightarrow Z$, where $I = [0, 1]$, such that h(y, 0) = f(y) and h(y, 1) = g(y) for every $y \in Y$ and h is called p-homotopy between f and g.

5.5.2 In view of theorem 5.3.7 and theorem 5.3.8, we see that a p-homotopy $h : Y \times I \longrightarrow Z$ between p-continuous maps f and g corresponds to a p-path in Z^Y from the point f to the point g in Z^Y.

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