Chapter – IV

A NEW CLASS OF LRS BIANCHI TYPE-I COSMOLOGICAL MODELS WITH VARIABLE G AND $^\wedge$
Chapter 4

A NEW CLASS OF LRS BIANCHI TYPE-I COSMOLOGICAL MODELS WITH VARIABLE G AND \( \Lambda \)

4.1 Introduction

Einstein’s theory of gravitation contains two parameters: Newton’s gravitational constant \( G \) and the cosmological constant \( \Lambda \) \[1\]. A possible time variable \( G \) has been suggested by Dirac \[2\] and extensively discussed in the literature \[3,4\]. Since its introduction, its significance has been studied from time to time by various workers \[5-7\]. In modern cosmological theories, the cosmological constant remains a focal point of interest. A wide range of observations now compellingly suggests that the universe possesses a non-zero cosmological constant \[8\]. In context of quantum field theory a cosmological term corresponds to the energy density of vacuum. It was suggested that the universe might have been born from an excited vacuum fluctuation which triggered off inflationary expansion, followed by super-cooling and subsequent re-heating with the release of locked up vacuum energy. The cosmological term, which is measure of the energy of empty space, provided a repulsive force opposing the gravitational pull between the galaxies. If the cosmological term exits, the energy it represents counts as mass because, as Einstein showed, mass and energy are equivalent. If the cosmological term is
large enough, its energy plus the matter in the universe could add up to the number that inflation predicts. Unlike the case of standard inflation, a universe with a cosmological term would expand faster with time because of the push from the cosmological term [9]. But recent research suggests that the cosmological term corresponds to a very small value of the order of $10^{-58} \text{cm}^{-2}$ [10].

It has been suggested by Linde [6] that $\Lambda$ is a function of temperature and related to the spontaneous symmetry breaking process. Therefore, it could be a function of time in a spatially homogeneous expanding universe [11]. The last measurements of the Hubble parameters [12,13] point to an intrinsic fragility of the standard (photon conserving) cosmology, in such a way that the models without a cosmological constant seem to be effectively ruled out (Refs.14,15, and references therein). Any model of the universe should yield a lifetime greater than that of the oldest objects in it. The age of the oldest stars in globular clusters, the oldest known objects in the universe, are of the order of 16 billion years [16]. Even allowing for the uncertainty in the determination of Hubble's constant ($H_0$), it is difficult for the standard models without a cosmological constant to lead to an age of the universe greater than that of these stars [15-17].

Even since Dirac first considered the possibility of a variable $G$ [2], there have been numerous modifications of general relativity to allow for a variable $G$ [18]. Recently a modification linking the variation of $G$ with that of $\Lambda$ has been considered within the framework of general relativity by many researchers [19-24]. However cosmological models with time-dependent $G$ and $\Lambda$ and the solutions $\Lambda \sim R^{-2} \sim t^{-2}$ were first obtained by Bertolami [25,26]. The modification mentioned above is appearing since it leaves the form of Einstein's equations formally unchanged by allowing a variation of $G$ to be accompanied by a change in $\Lambda$.

In the present work, we consider Einstein's field equations with time-varying
A and G and take the energy-momentum tensor of a perfect fluid. We assume that the conservation law of matter holds. Here we will adopt an approach which is more compact than others, and which not allows us to confirm earlier solutions but also leads to new ones.

4.2 Field Equations

We consider LRS Bianchi type I space-time

\[ ds^2 = dt^2 - A^2 dx^2 - B^2 (dy^2 + dz^2) \]  

(4.1)

where \( A = A(x, t), B = B(x, t) \).

Einstein field equations with time-dependent cosmological and gravitational "constant" [27]

\[ R_{ij} - \frac{1}{2} g_{ij} R + \Lambda(t) g_{ij} = -8\pi G(t) T_{ij} \]  

(4.2)

and the perfect-fluid energy-momentum tensor

\[ T_{ij} = (p + \rho) u_i u_j - p g_{ij} \]  

(4.3)

together with comoving coordinates \( u_i u_i = 1 \), yield the following four independent equations

\[ \frac{2\dot{B}}{B} + \frac{\dot{B}^2}{B^2} - \frac{B''}{A^2 B^2} = -8\pi G\rho + \Lambda \]  

(4.4)

\[ \dot{B} - \frac{B'\dot{A}}{A} = 0 \]  

(4.5)

\[ \frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} - \frac{B''}{A^2 B} + \frac{A'\dot{B}}{A^3 B} = -8\pi G\rho + \Lambda \]  

(4.6)

\[ \frac{2B''}{A^2 B} - \frac{2A'\dot{B}}{A^3 B} + \frac{B''}{A^2 B^2} - \frac{2\dot{A}\dot{B}}{A B} - \frac{\dot{B}^2}{B^2} = -8\pi G\rho - \Lambda \]  

(4.7)

On the other hand, the vanishing of the covariant divergence of the Einstein tensor in equation (2) and the usual energy-momentum conservation relation \( T_{ij}^{ij} \)

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If we suppose the energy conservation law to hold, then eq.(8) reduces to
\[
\dot{\rho} + (p + \rho)\left(\frac{\dot{A}}{A} + \frac{2\dot{B}}{B}\right) = 0
\] (4.9)
\[
\dot{\lambda} = -8\pi G \rho
\] (4.10)

Here and in what follows a prime and a dot indicate partial differentiation with respect to \(x\) and \(t\) respectively. To solve the system of equations we confine ourselves to assume an equation of state
\[
p = \gamma \rho, \quad 0 \leq \gamma \leq 1
\] (4.11)

### 4.3 Solutions of the field equations

Equation (5), after integration, yields
\[
A = \frac{B'}{t},
\] (4.12)
where \(l\) is an arbitrary function of \(x\).

Equations (4) and (6), with the use of equation (12), reduce to
\[
\frac{B}{B'} \frac{d}{dx} \left(\frac{\dot{B}}{B}\right) + \frac{\dot{B}}{B'} \frac{d}{dt} \left(\frac{B'}{B}\right) + \frac{l^2}{B^2} \left(1 - \frac{B'}{B\dot{l}}\right) = 0
\] (4.13)

Here \(A\) and \(B\) are separable in \(x\) and \(t\), let us assume \(\frac{B'}{B}\) as functions of \(x\) alone.

Now equation (13) gives after integration
\[
B = l S(t),
\] (4.14)
where \(S(t)\) is an arbitrary function of \(t\).

With the help of equation (14), equation (12) becomes
\[
A = \frac{\dot{l}}{l} S
\] (4.15)
Now the metric (1) takes the form

\[ ds^2 = dt^2 - S^2(t)[dX^2 + e^{2X}(dy^2 + dz^2)], \quad (4.16) \]

where \( X = \ln l \).

With the use of equations (14) and (15), equations (4) and (7) give

\[ 8\pi G \rho - \Lambda = \frac{1}{S^2} - 2\frac{\dot{S}}{S} - \frac{\ddot{S}}{S^2} \]

\[ 8\pi G \rho + \Lambda = 3\frac{\ddot{S}}{S^2} - \frac{3}{S^2} \]

(4.17)  (4.18)

Using equation (11) and eliminating \( \rho(t) \) from equations (17) and (18), we have

\[ \frac{2\dot{S}}{S} + (1 + 3\gamma)\frac{\dot{S}^2}{S^2} - (1 + 3\gamma)\frac{1}{S^2} - (1 + \gamma)\Lambda = 0 \]

(4.19)

Now the expressions for the energy density and the pressure are given by

\[ 4\pi G p = 4\pi G \gamma \rho = \frac{\gamma}{(1 + \gamma)} \left[ \frac{\dot{S}^2}{S^2} - \frac{\ddot{S}}{2S} - \frac{1}{S^2} \right] \]

(4.20)

The function \( S(t) \) remains undetermined. To obtain its explicit dependence on \( t \), one may have to introduce additional assumptions. In the following we assume the deceleration parameter to be constant to achieve this objective i.e.

\[ q = -\frac{S\ddot{S}}{S^2} = -\left( \frac{\dot{H} + H^2}{H^2} \right) = b \text{ (constant)}, \]

(4.21)

where \( H = \dot{S}/S \) is the Hubble parameter. The above equation may be rewritten as

\[ \frac{\ddot{S}}{S} + b\frac{\dot{S}^2}{S^2} = 0 \]

(4.22)

This equation (22) is integrated to obtain

\[ S(t) = \begin{cases} (d + ct)^{\frac{1}{1+b}} & \text{when } b \neq -1 \\ S_0 \ e^{H_0 t} & \text{when } b = -1 \end{cases} \]

(4.23)
where \( c, d, S_0 \) and \( H_0 \) are constants of integration.

Using eq. (21) in eqs. (19) and (20) lead to

\[
\Lambda = \frac{1}{(1 + \gamma)} \left[ (1 + 3\gamma - 2b)H^2 - (1 + 3\gamma)\frac{1}{S^2} \right]
\]

(4.24)

\[
4\pi G\rho = \frac{1}{(1 + \gamma)} \left[ (1 + b)H^2 - \frac{1}{S^2} \right]
\]

(4.25)

With the use of equations (11), (14) and (15), eq.(9) reduces to

\[
\dot{\rho} + 3H(1 + \gamma)\rho = 0
\]

(4.26)

which after integration yields

\[
\rho = kS^{-3(1+\gamma)},
\]

(4.27)

where \( k \) is the constant of integration.

Using equations (24) and (27) in eq. (10) leads to

\[
G = -\frac{1}{4\pi(1 + \gamma)k} \int \dot{S}S^{3\gamma}[1 + 3\gamma + (1 + 3\gamma - 2b)(S \dot{S} - \dot{S}^2)]dt + k_1,
\]

(4.28)

where \( k_1 \) is the constant of integration.

### 4.3.1 Non-flat Models

Case (i) : \( b \neq -1 \). For singular model, since \( S_0 = 0 \), equation (23) leads to

\[
S = S_0 \ t^{(\frac{3\gamma+1}{1+b})}
\]

(4.29)

Using equation (29) in equations (24), (28) and (25), we have

\[
\Lambda = \frac{1}{(1 + \gamma)t^2} \left[ \frac{(1 + 3\gamma - 2b)}{(1 + b)^2} - \frac{(1 + 3\gamma)}{S_0^2} t^{\frac{3\gamma+1}{1+b}} \right]
\]

(4.30)

\[
G = \frac{S_0^{(3\gamma+1)} t^{(\frac{3\gamma-3b+1}{1+b})}}{4\pi k(1 + \gamma)} \left[ \frac{S_0^2}{(1 + b)} - t^{\frac{3b}{1+b}} \right] + k_2
\]

(4.31)
where \( k_2 \) is the constant of integration. It is observed from equation (32) that \( \rho \geq 0 \) provided \( t(\tau+\gamma) \leq S^2_0 \left( \frac{1+b}{1+b'} \right) \). From equation (30), it is observed that the cosmological parameter \( \Lambda \) decreases with time. From equation (31), it can be seen that \( G \geq 0 \) provided that \( t(\tau+\gamma) \leq S^2_0 \left( \frac{1+b}{1+b'} \right) \). Hence we also observe that \( G \) is an increasing function of time. This equation was recently invoked \[20, 31, 32\] in support of an argument for an increasing \( G \). The model holds only for \( t(\tau+\gamma) \leq S^2_0 \left( \frac{1+b}{1+b'} \right) \).

For \( b = 0 \), equations (30) - (32) reduce to

\[
\Lambda = \frac{1}{(1+\gamma)t^2} \left[ (1+3\gamma) - \frac{(1+3\gamma)}{S^2_0} \right] \tag{4.33}
\]

\[
G = \frac{S^3_0(3\gamma+1)t(3\gamma+1)}{4\pi k(1+\gamma)} \left[ S^2_0 - 1 \right] + k_2 \tag{4.34}
\]

\[
\rho = \frac{k}{S^3_0(\gamma+1)t(\gamma+1) + \frac{k_2 S^2_0 t^2}{(S^2_0 - 1)}} \tag{4.35}
\]

From equation (33), it is observed that the cosmological parameter \( \Lambda \) varies as the inverse square of time which matches its natural units. From equation (34), it can be seen that \( G \geq 0 \) provided \( S^2 \geq 1 \) and \( k_2 \geq 0 \). From equations (11) and (35), it is observed that the energy conditions given by Ellis \[14\]

(i) \( (\rho + p) > 0 \)

(ii) \( (\rho + 3p) > 0 \)

(iii) \( \rho > 0 \)

and the dominant energy conditions given by Hawking and Ellis \[15\]

(i) \( (\rho - p) \geq 0 \)

(ii) \( (\rho + p) \geq 0 \)
are satisfied provided \( S_0^2 > 1, k > 0 \) and \( k_3 > 0 \).

In this case, the Ricci Scalar becomes

\[
R = \frac{1}{S_0^2} \left( \frac{(1 - b)t}{t} \right) - (1 - b) t \frac{(1 + b)}{(1 + b)} \tag{4.36}
\]

It is observed from eq.(36) that when \( t \to 0 \); (i) \( R \to \infty \) if \( b = 0 \), (ii) \( R \to \infty \) if \( b \geq 1 \) and (iii) \( R \to \infty \) if \( b \leq -2 \). The equation (36) also suggest that when \( t \to \infty \); (i) \( R \to 0 \) if \( b > 0 \) and (ii) \( R \to \infty \) if \( b \leq -2 \).

The scalars of expansion and shear are given by

\[
\theta = \frac{3}{(1 + b) t}, \quad \sigma = 0 \tag{4.37}
\]

The model has singularity at \( t = 0 \). At \( t \to \infty \), the expansion ceases. Here \( \xi = 0 \), which confirms the isotropic nature of the space-time.

**Case (ii) :** \( b = -1 \). In this case, equation (22) becomes

\[
\dot{H} = 0 \quad \text{and} \quad H = H_0 = \text{constant} \tag{4.38}
\]

Using eq.(38) in eqs.(24), (28) and (25), we have

\[
\Lambda = \frac{1}{(1 + \gamma)} \left[ 3(1 + \gamma)H_0^2 - \frac{(1 + 3\gamma)}{S_0^2 e^{2H_0 t}} \right] \tag{4.39}
\]

\[
G = \frac{S_0^{(1+3\gamma)} e^{(1+3\gamma)H_0 t}}{4\pi (1 + \gamma) k} + k_1 \tag{4.40}
\]

\[
\rho = \frac{k}{S_0^{(1+3\gamma)} e^{3(1+\gamma)H_0 t} + k_3 e^{2H_0 t}} \tag{4.41}
\]

where \( k_3 \) is the constant of integration. From equations (11)and (41), we observe that the energy conditions given by Ellis [14] and the dominant energy conditions given by Hawking and Ellis [15] are satisfied provided \( k \geq 0 \) and \( k_3 \geq 0 \). When \( t \to 0 \), then \( \rho \to \frac{k}{S_0^{3(1+\gamma)+k_3}} \) which is a finite quantity. From equations (39) and (40), we observe that as \( t \) increases, \( \Lambda \) decreases while \( G \) increases as \( t \) increases. So in this case also, \( G \) is an increasing function of time. Here \( G \geq 0 \) provided \( k > 0 \) and \( k_1 \geq 0 \). For \( H_0 = 0 \), we get a static universe with constant density with
It is observed that $G$ is decreasing and $\Lambda$ is increasing with time if $H_0 > 0$ and vice versa. This inflationary model starts from a non-singular orgine.

In this case, the Ricci scalar becomes

$$ R = \frac{2}{S_0^2} e^{2H_0 t} + 12H_0^2 $$  \hspace{1cm} (4.42) $$

It is observed from equation (42) that (i) when $t \rightarrow 0$, $R \rightarrow \left(\frac{2}{S_0^2} + 12H_0^2\right)$ and (ii) when $t \rightarrow \infty$, then $R \rightarrow 12H_0^2$ when $H_0 > 0$ and $R \rightarrow \infty$ when $H_0 < 0$. The expansion and sheer scalars are given by

$$ \theta = 3H_0, \quad \sigma = 0 $$ \hspace{1cm} (4.43) $$

The model represents a uniform expansion as can be seen from equation (43). The flow of the fluid is geodetic as the acceleration vector $f_i = (0,0,0,0)$.

### 4.3.2 Flat Models

The condition for flat model is obtained as

$$ \frac{1}{S^2} = (1 - b)H^2 $$  \hspace{1cm} (4.44) $$

Using equation (44) in equations (24), (28) and (25), we find the expressions for $\Lambda$, $G$ and $\rho$ as given by

$$ \Lambda = \frac{(3\gamma - 1)bH^2}{(1 + \gamma)} $$  \hspace{1cm} (4.45) $$

$$ G = \frac{b(b - 3\gamma)}{2\pi(1 + \gamma)k(b - 1)} \int \hat{S}S^{3\gamma} dt + k_1 $$  \hspace{1cm} (4.46) $$

$$ 2\pi G\rho = \frac{bH^2}{(1 + \gamma)} $$  \hspace{1cm} (4.47) $$

**Case (i) :** $b \neq -1$. Using equation (29) in equations (45)-(47) yield

$$ \Lambda = \frac{(3\gamma - 1)b}{(1 + \gamma)(1 + b)^2t^2} $$  \hspace{1cm} (4.48) $$
\[ G = \frac{b(b - 3\gamma)S_0^{(3\gamma+1)}}{4\pi(1 + \gamma)k(b - 1)(3\gamma + 1)} e^{\frac{(3\gamma+1)}{3\gamma+1}} + k_1 \]  
\[ \rho = \frac{2k(b - 1)(3\gamma + 1)}{t^2 \left[(b - 3\gamma)S_0^{(3\gamma+1)}(1 + b)\right]^{(2+\gamma)} + k_2} \]

From equation (50), it is observed that \( \rho > 0 \) provided \( k > 0, k_2 > 0 \) and \( b > 3\gamma \).

**Case (ii) :** \( b = -1 \). Using eq.(38) in eqs. (45)-(47), we have

\[ \Lambda = -\frac{(1 + 3\gamma)}{(1 + \gamma)} H_o^2 = \text{constant} \]
\[ G = \frac{S_0^{(1+3\gamma)}e^{(1+3\gamma)Ho t}}{4\pi(1 + \gamma)k + k_1} \]
\[ \rho = \frac{2kH_o^2}{S_0^{(1+3\gamma)}e^{(1+3\gamma)Ho t} + k_4} \]

where \( k_4 \) is an arbitrary constant. From equations (11) and (53), it is observed that the energy conditions are satisfied provided \( k > 0 \) and \( k_4 > 0 \). Here we also observe that \( \rho \rightarrow \frac{2kH_o^2}{S_1^{(1+3\gamma)} + k_4} \) when \( t \rightarrow 0 \) and \( \rho \rightarrow 0 \) when \( t \rightarrow \infty \). From equation (52), we also observe that \( G \) increases as \( t \) increases which supports an argument of increasing \( G \) [20, 31, 32].

### 4.4 Concluding Remarks

We have investigated a new class of LRS Bianch-type I cosmological models in which the cosmological and gravitational constants vary with time. The nature of the gravitational constant \( G \), cosmological constant \( \Lambda \) and the energy density \( \rho \) have been examined for both the power law and exponential expansion of both the flat universe and non-flat universe. The gravitational constant \( G \) is an increasing function of time in all cases. In most variable \( G \) cosmologies [27,30] \( G \) is a decreasing function of time. But the possibility of an increasing \( G \) has also been suggested [31]. Recently Abdel-Rahman [20] and Singh et al [32] supports the views in favour of the gravitational constant to be an increasing function of
time. The cosmological constant $\Lambda$ reduces gradually as the universe expands. The cosmological constant, on the other hand, is depleted as the universe expands. For both flat and non-flat models in case (i) when $b \neq -1$, it is also observed that the cosmological constant $\Lambda$ varies as the inverse square of time when $b = 0$, which matches its natural units. This supports the views in favour of the dependence $\Lambda \sim t^{-2}$ first expressed by Bertolami [25, 26].

We have found some new ones with interesting properties. However we have concentrated on the solutions and their detailed physical implications will require further investigation and this is under way.
Bibliography


