CHAPTER FIVE
Intergenerational Altruism, Steady State Expectations and the Possibility of Divergence

The most disturbing result to come out of our analyses in Chapters Three and Four has been the possibility, in the case of iso-elastic instantaneous utility functions, that the distribution of inheritances of wealth might become more unequal over equilibrium growth paths. Moreover, such a possibility may be closely associated with the possible presence of conflicts between long-run economic growth and distributive equity with respect to inheritances of material wealth. In this chapter and the next, therefore, we consider, for the case of iso-elastic instantaneous utility functions, whether such possibilities can exist even with some alternative assumptions about the nature of expectations, about the length of time after death over which altruistic feelings extend and about the nature of individual utilities.

In this chapter we assume that individuals are not endowed with perfect foresight but have steady state expectations about the aggregate economic variables i.e. they expect the future values per efficiency unit of labour of aggregate economic variables to be stationary at current levels. However, we continue to assume that instantaneous utility across generations is the same iso-elastic function of the absolute rate of individual consumption and that individual objective functions incorporate altruistic feelings about all future generations of descendants.

Therefore in this chapter all assumptions made in section 3.1 and at the beginning of section 3.2 in Chapter Three are maintained, except that individuals are no longer assumed to perfectly foresee the time path of aggregate economic variables. Instead, all individuals living at any instant \( \tau \geq 0 \) expect that
\[
\forall \tau > \tau : k_\tau = k_\tau \land c_\tau = c_\tau \land w_\tau = w_\tau \land r_\tau = r_\tau .
\]
An individual belonging to the \( i^{th} \) wealth group \( (i = 1,2,\ldots,q) \) at point in time \( \tau \geq 0 \) therefore solves the following problem to determine his/her current rate of consumption.
Chapter Five

Max

\[
\int_{t}^{\infty} c_{it}^{\frac{\sigma-1}{\sigma}} m_{it}^{\frac{\tau-1}{\tau}} \exp \left( \frac{\mu(1-\frac{1}{\sigma})(t-\tau)}{1-\frac{1}{\sigma}} \right) \exp(-\theta(t-\tau)) dt, \text{ when } \sigma \neq 1,
\]

\[
\int_{t}^{\infty} (\ln(c_{it}) \exp(-\theta(t-\tau)) + \ln(m_{it}) \exp(-\theta(t-\tau))) dt, \text{ when } \sigma = 1 \quad (1d)
\]

subject to

\[
\frac{da_{it}}{dt} = w_{i} + (r_{i} - n - \mu) a_{it} - c_{it}, \quad \text{for all } t \geq \tau \quad (2d)
\]

\[
\lim_{t \to \infty} a_{it} \exp(-(r_{i} - n - \mu)(t - \tau)) \geq 0 \quad (3d)
\]

\[
c_{it} \geq 0, \quad \text{for all } t \geq \tau, \text{ when } \sigma \neq 1
\]

\[
c_{it} > 0, \quad \text{for all } t \geq \tau, \text{ when } \sigma = 1 \quad (4d)
\]

\[
a_{it} \text{ given} \quad (5d)
\]

where, \( c_{it} \) is a piecewise continuous function of \( t \) on \([\tau, \infty)\) and \( a_{it} \) is a continuous function of \( t \) with piecewise continuous derivatives on \([\tau, \infty)\).

Accordingly all definitions outlined in section 3.1, Chapter Three, are adhered to in this chapter except that of an equilibrium growth path, which we now redefine as follows.

Given an initial amount, \( k_{0} (\geq 0) \), of capital per efficiency unit of labour and an initial distribution of inherited wealth \( (a_{10}, a_{20}, \ldots, a_{q0}) \) in the economy, a programme \( \{\bar{c}_{i}, \bar{k}_{i}, \bar{c}_{i}, \bar{a}_{i}, \bar{a}_{i}, \ldots, \bar{c}_{i}, \bar{a}_{i}, \bar{a}_{i}\}_{i=0}^{\infty} \) is an equilibrium growth path for the economy if and only if

I. \( \bar{k}_{0} = k_{0} \land \forall i \in \{1, 2, \ldots, q\}: \bar{a}_{i0} = a_{i0} \land \forall t \geq 0: \bar{c}_{i} = \sum_{i=1}^{q} l_{i} \bar{c}_{i} \land \bar{k}_{i} = \sum_{i=1}^{q} l_{i} \bar{a}_{i} \geq 0 \)

II. \( \forall i \in \{1, 2, \ldots, q\} \land \forall t \geq 0: \frac{da_{it}}{dt} = w_{i} + (r_{i} - n - \mu) a_{it} - c_{it}, \quad \text{where } \forall t \geq 0: w_{i} = f(\bar{k}_{i}) - f(\bar{k}_{i}), f'(\bar{k}_{i}) \land r_{i} = f'(\bar{k}_{i}) \)
III. \( \forall \tau \geq 0 \land \forall i \in \{1, 2, \ldots, q\} \exists \) a solution\(^1\) \( \left\{ (c_{it}^{\prime}, a_{it}^{\prime})\right\}_{i=1}^{\tau} \) to the problem (1d) - (5d) when \( a_{it} = a_{it}^0 \land w_t = f(k_t) - k_t f'(k_t) \land r_t = f'(k_t), \) such that \( c_{it}^{\prime} = \tilde{c}_{it}. \)

Proposition 5.1: Given \( \theta, n, \mu \) and \( \sigma \) such that \( \mu < \theta + \frac{\mu}{\sigma} < \frac{\theta}{1 - \frac{1}{\sigma}}, \) if \( \sigma > 1, \)

\[ \frac{\theta}{1 - \frac{1}{\sigma}} < \mu < \theta + \frac{\mu}{\sigma}, \]
if \( \sigma < 1, \) and \( \theta > 0, \) if \( \sigma = 1, \) there exist values of the initial ratio of capital to effective labour, \( k_0, \) and initial distributions of inherited wealth in the economy for which equilibrium growth paths exist with the property that the degree of inequality in the distribution of inherited wealth increases, remains unchanged or decreases over the growth path according as the long-run rate of growth of output per capita \( \mu \) is greater than, equal to, or less than \( \mu_0, \) where \( f'(k_0) = \theta + n + \frac{\mu_0}{\sigma}. \)

Proof: - Let \( \theta, n, \mu \) and \( \sigma \) be such that \( \mu < \theta + \frac{\mu}{\sigma} < \frac{\theta}{1 - \frac{1}{\sigma}}, \) if \( \sigma > 1, \)

\[ \frac{\theta}{1 - \frac{1}{\sigma}} < \mu < \theta + \frac{\mu}{\sigma}, \]
if \( \sigma < 1, \) and \( \theta > 0, \) if \( \sigma = 1. \) Let the initial amount of capital per efficiency unit of labour, \( k_0, \) be such that \( f'(k_0) = \theta + n + \frac{\mu_0}{\sigma} > n + \mu. \) Also, let

\[ f'(k_0) = \theta + n + \frac{\mu_0}{\sigma} < n + \frac{\theta}{1 - \frac{1}{\sigma}}, \]
when \( \sigma > 1, \) and \( f'(k_0) = \theta + n + \frac{\mu_0}{\sigma} > n + \frac{\theta}{1 - \frac{1}{\sigma}}, \)
when \( \sigma < 1. \)

Since \( f'(k_0) > n + \mu \) and it can be shown that \( \forall k > 0: \frac{f(k)}{k} > f'(k), \) we can consider any initial distribution of inheritances \( \left( a_{i0}, a_{i1}, \ldots, a_{iq} \right), \) such that

\(^1\) The definition of a solution is the same as that given in section 3.1, Chapter Three.
\[ \forall i \in \{1, 2, \ldots, q\}: \left[ f(k_0) - k_0 f'(k_0) + \{ f'(k_0) - (n + \mu) \} a_{i, 0} > 0 \right] \]
\[ \land \quad f'(k_0) < f'(k^*) = \theta + n + \frac{\mu}{\sigma} \quad \left( 1 - \frac{a_{i, 0}}{k_0} \right) \leq \frac{\left\{ f(k^*) - (n + \mu) \right\} k^*}{f'(k^*) - (n + \mu)} \]
\[ \quad \vdots \quad (5.1.1) \]

Given \( f'(k_0) > n + \mu \) and \( \theta > \mu \left( 1 - \frac{1}{\sigma} \right) \) we can define a growth path

\[ \left\{ (\bar{c}_i, \bar{k}_i), (\bar{a}_u, \bar{a}_{i, u}), (\bar{a}_{i, u}, \bar{a}_{i, u}), \ldots, (\bar{c}_{i, q}, \bar{a}_{i, q}) \right\}_{i=0}^{\infty} \]

such that

\[ \forall t \geq 0 : f'(\bar{k}_t) > n + \mu \]
\[ \bar{k}_0 = k_0 \land \quad \forall t \geq 0: \quad \frac{\overline{d\bar{k}_t}}{dt} = \frac{\sigma \left\{ f'(\bar{k}_t) - \left( \theta + n + \frac{\mu}{\sigma} \right) \right\} f'(\bar{k}_t) - (n + \mu) \bar{k}_t}{f'(\bar{k}_t) - (n + \mu) \bar{k}_t} \]
\[ \quad \vdots \quad (5.1.2) \]
\[ \forall t \geq 0 \land \forall i \in \{1, 2, \ldots, q\}: \quad \bar{a}_{i, 0} = a_{i, 0} \land \quad \frac{\overline{d\bar{a}_{i, u}}}{dt} = \frac{\sigma \left\{ r_i - \left( \theta + n + \frac{\mu}{\sigma} \right) \right\} r_i - (n + \mu) \bar{a}_{i, u}}{r_i - (n + \mu) \bar{a}_{i, u}} \]
\[ \quad \vdots \quad (5.1.3) \]

where, \( \forall t \geq 0 : w_i = f'(\bar{k}_t) - \bar{k}_t f'(\bar{k}_t) \land r_i = f'(\bar{k}_t) \)

To see how the above conditions can define a growth path, first of all note that

if \( f'(k_0) = \theta + n + \frac{\mu}{\sigma} > n + \mu \) then \( \frac{\overline{d\bar{k}_t}}{dt} \bigg|_{t=0} = 0 \) according to the above conditions.

Then there must exist an interval \( (0, t^0) \), \( t^0 > 0 \), on which \( \bar{k}_t \) is defined with

\( f'(\bar{k}_t) > n + \mu \). Therefore, by (5.1.2), \( \bar{k}_t \) is differentiable on that interval. Further, \( \bar{k}_t \)

must be constant on that interval with \( f'(\bar{k}_t) = \theta + n + \frac{\mu}{\sigma} \). (5.1.2) then implies that
\[ \bar{k}_o = \lim_{t \to \infty} \bar{k}_i, \text{ so that } f'(\bar{k}_o) = \theta + n + \frac{\mu}{\sigma} > n + \mu. \] Therefore, once again we can define an interval \((t^0, t^1)\), \(t^1 > t^0\), in which \(\bar{k}_i\) is defined and \(f'(\bar{k}_i) > n + \mu\). Proceeding in this way it is possible to see that if \(f'(k_0) = \theta + n + \frac{\mu}{\sigma}\) then the above conditions define a growth path with \(\bar{k}_i = k_0, \forall t \geq 0\).

Now suppose that \(f'(k_0) > \theta + n + \frac{\mu}{\sigma} > n + \mu\). Then, by condition (5.1.2),
\[
\left. \frac{d\bar{k}_i}{dt} \right|_{t=0} > 0 \text{ so that there exists an interval } (0, t^0), t^0 > 0, \text{ in which } \bar{k}_i \text{ is defined and } f'(\bar{k}_i) > \theta + n + \frac{\mu}{\sigma} > n + \mu. \] It follows, by (5.1.2), that \(\frac{d\bar{k}_i}{dt} > 0\) on this interval. Therefore, \(\bar{k}_i\) is monotonic and bounded on \((0, t^0)\), so that \(\lim_{t \to \infty} \bar{k}_i\) exists and \(f'(\lim_{t \to \infty} \bar{k}_i) \geq \theta + n + \frac{\mu}{\sigma}\). If (5.1.2) is to be satisfied then we must have \(\bar{k}_o = \lim_{t \to \infty} \bar{k}_i\), so that \(f'(\bar{k}_o) \geq \theta + n + \frac{\mu}{\sigma} > n + \mu\). Therefore, we can define an interval \((t^0, t^1)\), \(t^1 > t^0\), on which \(\bar{k}_i\) is defined with \(f'(\bar{k}_i) \geq \theta + n + \frac{\mu}{\sigma} > n + \mu\). Proceeding in this manner, it is possible to show that if \(f'(k_0) > \theta + n + \frac{\mu}{\sigma}\) then the above conditions (5.1.2) and (5.1.3) define a growth path with \(k^* \geq \bar{k}_i > k_0, \forall t > 0\), where \(f'(k^*) = \theta + n + \frac{\mu}{\sigma}\). In a similar fashion we can show that if \(\theta + n + \frac{\mu}{\sigma} > f'(k_0) > n + \mu\) then the above conditions define a growth path with \(k_0 > \bar{k}_i \geq k^*, \forall t > 0\).

Hence, since \(f'(k_0) > n + \mu\) and \(\theta > \mu\left(1 - \frac{1}{\sigma}\right)\), therefore from (5.1.2) we get,
\[
f'(k^*) = \theta + n + \frac{\mu}{\sigma} \wedge f'(k_g) = n + \mu \rightarrow \lim_{t \to \infty} \bar{k}_i = k^* \wedge \]
\[k_0 < k^* \rightarrow \forall t > 0: 0 < k_0 < \bar{k}_i \leq k^* < k_g] \wedge [k_0 = k^* \rightarrow \forall t > 0: 0 < k_0 = \bar{k}_i = k^* < k_g].\]
\begin{align*}
&\land \left[ k_0 > k^* \rightarrow \forall t > 0 : k_g > k_0 > \bar{k}_t \geq k^* > 0 \right] \quad \ldots (5.1.4) \\

From (5.1.2) – (5.1.4) we can show that \\
\bar{k}_0 = k_0 \land \left[ \forall i \in \{1,2,\ldots,q\} : \bar{a}_i = a_{i0} \right] \land \forall t \geq 0 : \bar{c}_t = \sum_{i=1}^{q} l_i \bar{c}_i \land \bar{k}_t = \sum_{i=1}^{q} l_i \bar{a}_i \geq 0 \quad \ldots (5.1.5)
\end{align*}

From (5.1.2) and (5.1.3) we get \( \forall t \geq 0 \land \forall i \in \{1,2,\ldots,q\} : \\
d\left( \frac{\bar{a}_i}{k_t} \right) \over dt = \frac{\sigma \left[ r_t - \left( \theta + n + \frac{\mu}{\sigma} \right) \right]}{r_t - (n + \mu) k_t} \left( 1 - \frac{\bar{a}_i}{k_t} \right) \quad \ldots (5.1.6)

We can prove that \( \forall k > 0 : \frac{f(k)}{k} > f'(k) \). Therefore, from (5.1.4) and (5.1.6) it can be shown that \( \forall i \in \{1,2,\ldots,q\} : \\
a_{i0} \geq k_0 \rightarrow \forall t \geq 0 : \bar{a}_i \geq \bar{k}_t \wedge w_t + (r_t - n - \mu) \bar{a}_i \geq f(\bar{k}_t) - (n + \mu) \bar{k}_t > 0 \quad \ldots (5.1.7)

By our choice of the initial distribution of inherited wealth \( (a_{i0}, a_{20}, \ldots, a_{q0}) \) we note that \( \forall i \in \{1,2,\ldots,q\} : w_o + (r_o - n - \mu) a_{i0} > 0 \quad \ldots (5.1.8) \)

Now, we can show from (5.1.2) and (5.1.3) that \( \forall t \geq 0 \land \forall i \in \{1,2,\ldots,q\} : \\
d\left( w_t + (r_t - n - \mu) \bar{a}_i \right) \over dt = -\bar{k}_t f'(\bar{k}_t) \left( 1 - \frac{\bar{a}_i}{k_t} \right) \frac{\sigma \left[ r_t - \left( \theta + n + \frac{\mu}{\sigma} \right) \right]}{r_t - (n + \mu) k_t} \left[ f(\bar{k}_t) - (n + \mu) \bar{k}_t \right] + \sigma \left[ r_t - \left( \theta + n + \frac{\mu}{\sigma} \right) \right] \left( w_t + (r_t - n - \mu) \bar{a}_i \right) 

\begin{align*}
\text{Solving the above differential equation we get, } \forall t \geq 0 \land \forall i \in \{1,2,\ldots,q\} : \\
&\left( w_t + (r_t - n - \mu) \bar{a}_i \right) \exp \left( -\int_{0}^{t} \sigma \left[ r_v - \left( \theta + n + \frac{\mu}{\sigma} \right) \right] dv \right) = \left( w_o + (r_o - n - \mu) \bar{a}_{i0} \right) - \\
\end{align*}
From (5.1.6) it follows that,

$$\forall i \in \{1,2,\ldots,q\}: a_{i0} < k_0 \rightarrow \forall t \geq 0: \bar{a}_t \leq \bar{k}_t$$  \hspace{1cm} (5.1.10)

From (5.1.3), (5.1.4), (5.1.8) – (5.1.10) we can show that

$$\forall i \in \{1,2,\ldots,q\} \land \forall t \geq 0: \left[ f'(k_0) \geq \theta + n + \frac{\mu}{\sigma} \land a_{i0} < k_0 \rightarrow w_i + (r_i - n - \mu)\bar{a}_t > 0 \right]$$ \hspace{1cm} (5.1.11)

From (5.1.3) we get

$$\forall t \geq 0 \land \forall i \in \{1,2,\ldots,q\}: \bar{a}_t = a_{i0} \exp\left( \int_0^t \left( r_v - \left( \theta + n + \frac{\mu}{\sigma} \right) \right) dv \right) +$$

$$\int_0^t \frac{\sigma \left( r_z - \left( \theta + n + \frac{\mu}{\sigma} \right) \right)}{r_z - (n + \mu)} w_z \exp\left( \int_0^z \left( r_w - \left( \theta + n + \frac{\mu}{\sigma} \right) \right) dw \right) dz$$ \hspace{1cm} (5.1.12)

From (5.1.5) and (5.1.12) it therefore follows that

$$\forall t \geq 0 \land \forall i \in \{1,2,\ldots,q\}: \bar{a}_t = \bar{k}_t + (a_{i0} - k_0) \exp\left( \int_0^t \left( r_v - \left( \theta + n + \frac{\mu}{\sigma} \right) \right) dv \right)$$

Therefore,\ \forall t \geq 0 \land \forall i \in \{1,2,\ldots,q\}: w_i + (r_i - n - \mu)\bar{a}_t =

$$w_i + (r_i - n - \mu)\bar{k}_t + (a_{i0} - k_0) \exp\left( \int_0^t \left( r_v - \left( \theta + n + \frac{\mu}{\sigma} \right) \right) dv \right) (r_i - n - \mu)$$
If \( f'(k_0) < \theta + n + \frac{\mu}{\sigma} \), then, since \( \bar{k}_t \) is a continuous function of \( t \) on \([0, \infty)\), given (5.1.4), it follows from above that

\[
\forall t \geq 0 \land \forall i \in \{1, 2, \ldots, q\}:
[a_{i0} < k_0 \rightarrow w_t + (r_t - n - \mu)\bar{a}_u > w_t + (r_t - n - \mu)\bar{k}_t + (a_{i0} - k_0)(r_t - n - \mu)
= \left[ \frac{f(\bar{k}_t) - (n + \mu)\bar{k}_t}{r_t - (n + \mu)} - (k_0 - a_{i0}) \right](r_t - n - \mu) \quad \ldots (5.1.13)
\]

From (5.1.2) and (5.1.4) we know that \( \forall t \geq 0: f'(k_0) - (n + \mu) > 0 \), and that if

\[
f'(k_0) < \theta + n + \frac{\mu}{\sigma} \text{ then } \forall t \geq 0: \frac{d\bar{k}_t}{dt} \leq 0 \land \bar{k}_t \leq k^* \text{, where } r^* = f'(k^*) = \theta + n + \frac{\mu}{\sigma}. \text{ It then follows that}
\]

\[
f'(k_0) < \theta + n + \frac{\mu}{\sigma} \rightarrow \forall t \geq 0: \frac{f(\bar{k}_t) - (n + \mu)\bar{k}_t}{r_t - (n + \mu)} \geq \frac{f(k^*) - (n + \mu)k^*}{r^* - (n + \mu)} \quad \ldots (5.1.14)
\]

(5.1.1), (5.1.13) and (5.1.14) therefore imply that

\[
\forall i \in \{1, 2, \ldots, q\} \land \forall t \geq 0: \left[ f'(k_0) < \theta + n + \frac{\mu}{\sigma} \land a_{i0} < k_0 \rightarrow w_t + (r_t - n - \mu)\bar{a}_u > 0 \right]
\]

\ldots (5.1.15)

Hence, from (5.1.7), (5.1.11) and (5.1.15) we get

\[
\forall i \in \{1, 2, \ldots, q\} \land \forall t \geq 0: w_t + (r_t - n - \mu)\bar{a}_u > 0 \quad \ldots (5.1.16)
\]

Consider any \( i \in \{1, 2, \ldots, q\} \) and any \( \tau \geq 0 \). Let us define the programme \( \{(\bar{a}_u, \bar{a}_u)\}_\tau \) as follows.

\[
\bar{a}_u = \bar{a}_u \land \forall t \geq \tau:
\bar{c}_u = -\frac{(\sigma - 1)(r_t - n) - \theta\sigma}{r_t - n - \mu} \left\{ w_t + (r_t - n - \mu)\bar{a}_u \right\} \exp \left\{ \sigma \left\{ r_t - \left( \theta + n + \frac{\mu}{\sigma} \right) \right\}(t - \tau) \right\}
\wedge \frac{d\bar{a}_u}{dt} = w_t + (r_t - n - \mu)\bar{a}_u - \bar{c}_u \quad \ldots (5.1.17)
\]
Chapter Five

By assumption, \( f'(k_0) < n + \frac{\theta}{1 - \frac{1}{\sigma}} \) and \( \theta + n + \frac{\mu}{\sigma} < n + \frac{\theta}{1 - \frac{1}{\sigma}} \), when \( \sigma > 1 \).

By assumption, \( f'(k_0) > n + \frac{\theta}{1 - \frac{1}{\sigma}} \) and \( \theta + n + \frac{\mu}{\sigma} > n + \frac{\theta}{1 - \frac{1}{\sigma}} \), when \( \sigma < 1 \).

Moreover, \( \theta > 0 \), when \( \sigma = 1 \). Therefore, from (5.1.4) we can show that

\[
\forall \tau \geq 0: \frac{(\sigma - 1)(r_i - n) - \theta \sigma}{r_i - n - \mu} < 0
\]  

(5.1.18)

From (5.1.16) – (5.1.18) it follows that

\[
\forall \tau \geq \tau_0: \tilde{a}_{it} > 0
\]  

(5.1.19)

From (5.1.17) we can show that \( \forall \tau \geq \tau_0 \):

\[
\tilde{a}_{it} \exp(- (r_i - n - \mu)(t - \tau)) = \tilde{a}_{it} + w_i \int_{t}^{\tau} \exp(- (r_i - n - \mu)(v - \tau)) dv + \frac{(\sigma - 1)(r_i - n) - \theta \sigma}{r_i - n - \mu} \{w_i + (r_i - n - \mu)\tilde{a}_{it}\} \int_{t}^{\tau} \exp(\{\sigma - 1\}(r_i - n) - \theta \sigma)(v - \tau)) dv
\]

Given (5.1.4) and (5.1.18), it can be shown from above that

\[
\lim_{t \to \infty} \tilde{a}_{it} \exp(- (r_i - n - \mu)(t - \tau)) = 0
\]  

(5.1.20)

(5.1.17), (5.1.19) and (5.1.20) imply that

\[
\forall i \in \{1, 2, ..., q\} \land \forall \tau \geq 0: \{(\tilde{c}_{it}, \tilde{a}_{it})\}_{t=\tau}^{\infty} \text{ is an admissible programme for the problem (1d) – (5d) when } w_i = f(\bar{k}_i) - \bar{k}_i f(\bar{k}_i), \quad r_i = f'(\bar{k}_i) \text{ and } a_{it} = \tilde{a}_{it}
\]  

(5.1.21)

Again consider any \( i \in \{1, 2, ..., q\} \) and any \( \tau \geq 0 \). Consider any programme \( \{(c'_{it}, a'_{it})\}_{t=\tau}^{\infty} \), distinct from \( \{(\tilde{c}_{it}, \tilde{a}_{it})\}_{t=\tau}^{\infty} \), which satisfies (2d) – (5d) and is an admissible programme for the problem (1d) – (5d).

Let \( \forall T > \tau: Z_T = \int_{t}^{T} u_i((c'_{it}, m_i) \exp(- \theta(t - \tau)) dt + \int_{t}^{\tau} u_i((c'_{it}, m_i) \exp(- \theta(t - \tau)) dt ,
\]
where \( u_i() \) is defined at the beginning of section 3.3, Chapter Three, for all \( t \geq 0 \).

From (5.1.19) it follows that we can define \( \forall t \geq \tau \):

\[
\widetilde{\lambda}_i = \left[ \frac{\partial u_i(c_{it}, m_t) \exp(-\theta(t - \tau))}{\partial c_{it}} \right]_{c_{it} = \widetilde{c}_{it}} = \frac{1}{\sigma} \int_{\tau}^{1} m_{\tau}^{1-\sigma} \exp\left(\left(\mu\left(1 - \frac{1}{\sigma}\right) - \theta\right)(t - \tau)\right) > 0
\]

... (5.1.22)

Moreover, let us define \( \forall t \geq \tau \):

\[
\tilde{H}_i(c_{it}, a_{it}, t) = H(c_{it}, a_{it}, \tilde{\lambda}_i, t) = u_i(c_{it}, m_t) \exp(-\theta(t - \tau)) + \tilde{\lambda}_i \{ w_t + (r_t - n - \mu) a_t - c_t \}
\]

Then, since both \( \{ (\tilde{c}_{it}, \tilde{a}_{it}) \}_{t=\tau}^\infty \) and \( \{ (c_{it}', a_{it}') \}_{t=\tau}^\infty \) satisfy (2d), we can write

\[
\forall T > \tau : Z_T = \int_{\tau}^{T} \tilde{H}_i(\tilde{c}_{it}, \tilde{a}_{it}, t) - \tilde{\lambda}_i \frac{d\tilde{a}_{it}}{dt} - \tilde{H}_i(c_{it}', a_{it}', t) + \tilde{\lambda}_i \frac{da_{it}'}{dt} \] 

i.e. \( \forall T > \tau : Z_T = \int_{\tau}^{T} \left[ \frac{d\tilde{a}_{it}}{dt} \tilde{a}_{it}' - \tilde{\lambda}_i \right] \] 

\[
= \int_{\tau}^{T} \left[ \frac{d\tilde{a}_{it}}{dt} \tilde{a}_{it}' - \tilde{\lambda}_i \right] - \int_{T}^{\tau} \left[ \frac{d\tilde{a}_{it}}{dt} \tilde{a}_{it}' - \tilde{\lambda}_i \right] \]

i.e. \( \forall T > \tau : Z_T = \int_{\tau}^{T} \left[ \frac{d\tilde{a}_{it}}{dt} \tilde{a}_{it}' - \tilde{\lambda}_i \right] + \tilde{\lambda}_i \] 

(since \( \tilde{\lambda}_i = \tilde{a}_{it} = a_{it}' \))

\[
\forall T > \tau : Z_T = \int_{\tau}^{T} \left[ \frac{d\tilde{a}_{it}}{dt} \tilde{a}_{it}' - \tilde{\lambda}_i \right] + \tilde{\lambda}_i \] 

... (5.1.23)
From (5.1.22) we obtain \( \forall t \geq \tau : \frac{1}{\lambda_{it}} \frac{d\lambda_{it}}{dt} = -\frac{1}{\sigma c_{it}} \frac{dc_{it}}{dt} - \left( \theta - \mu \left( 1 - \frac{1}{\sigma} \right) \right) \), and from (5.1.17) and (5.1.19) we obtain \( \forall t \geq \tau : \frac{1}{\sigma c_{it}} \frac{dc_{it}}{dt} = r_{t} - \left( \theta + n + \frac{\mu}{\sigma} \right) \). It therefore follows that \( \forall t \geq \tau : \frac{d\lambda_{it}}{dt} = -\lambda_{it} \left( r_{t} - n - \mu \right) \) ... (5.1.24)

Given (5.1.19) and (5.1.22), we can show that

\[ \forall t \geq \tau : \frac{\partial u_{i}(c_{it})}{\partial c_{it}} \bigg|_{c_{it} = \tilde{c}_{it}} m_{i}^{-\frac{1}{\sigma}} \exp \left( \left( \mu \left( 1 - \frac{1}{\sigma} \right) - \theta \right)(t - \tau) \right) \] ... (5.1.25)

From (5.1.23) – (5.1.25) we get

\[
\forall T > \tau : Z_{T} = \int_{\tau}^{T} \left[ \frac{u_{i}(\tilde{c}_{it}) - u_{i}(c'_{it}) - \frac{\partial u_{i}(c_{it})}{\partial c_{it}} (\tilde{c}_{it} - c'_{it})}{m_{i}^{-\frac{1}{\sigma}}} \right] \exp \left( - \left( \theta - \mu \left( 1 - \frac{1}{\sigma} \right) \right)(t - \tau) \right) dt \\
+ \tilde{\lambda}_{it} (a'_{it} - \tilde{a}_{it})
\]

Since \( \{(c_{it}, \tilde{a}_{it})\}_{t=r}^{\infty} \) and \( \{(c'_{it}, a'_{it})\}_{t=r}^{\infty} \) are distinct programmes satisfying (2d), and \( \tilde{a}_{it} = a'_{it} \), therefore \( \exists t* \in [r, \infty) : \tilde{c}_{it} = c'_{it} \).

Moreover, from (5.1.17) it follows that \( \tilde{c}_{it} \) is a continuous function of \( t \) on \([r, \infty)\), and by definition, \( c'_{it} \) is a piecewise continuous function of \( t \) on \([r, \infty)\). Therefore, it follows that

\[ \exists t^{0} : \left[ t \leq t^{0} \land \forall t \in [t^{0}, t^{*}] : \tilde{c}_{it} = c'_{it} \right] \lor \exists t^{1} : \left[ t^{*} < t^{1} \land \forall t \in [t^{*}, t^{1}] : \tilde{c}_{it} = c'_{it} \right] \]

From (5.1.19), given our definition of \( u_{i}(\cdot) \), \( \forall t \geq 0 \), it can be shown using the Mean Value Theorem\(^2\) that

\(^2\) See Binmore (1982), pp.103-104.
Therefore, it follows that
\[ \exists T^* > \tau : [\forall T \geq T^* : Z_T - \bar{\lambda}_{it} (a'_{it} - \bar{a}_{it}) > 0 \text{ and } Z_T - \bar{\lambda}_{it} (a'_{it} - \bar{a}_{it}) \text{ is a non-decreasing function of } T] \]

From (5.1.24) we get \( \forall T > \tau : \bar{\lambda}_{it} = \lambda_{it} \exp(- (r_T - n - \mu)(T - \tau)) \)

Therefore, it follows that
\[ \exists T^* > \tau : [\forall T \geq T^* : Z_T - \bar{\lambda}_{it} (a'_{it} - \bar{a}_{it}) \exp(- (r_T - n - \mu)(T - \tau)) > 0 \text{ and } Z_T - \bar{\lambda}_{it} (a'_{it} - \bar{a}_{it}) \exp(- (r_T - n - \mu)(T - \tau)) \text{ is a non-decreasing function of } T] \]

... (5.1.26)

From (5.1.22) we know that \( \bar{\lambda}_{it} > 0 \) ... (5.1.27)

Since, by assumption, \( \{c', a'\}_t \) satisfies (3d), from (5.1.20) we get,
\[ \lim_{T \to \infty} (a'_{it} - \bar{a}_{it}) \exp(- (r_T - n - \mu)(T - \tau)) \geq 0 \] ... (5.1.28)

Therefore, from (5.1.26) – (5.1.28) it follows that,
\[ \exists T' > \tau : [\forall T > \tau : T \geq T' \to Z_T > 0] \] ... (5.1.29)

Hence, given our suppositions, from (5.1.3), (5.1.17), (5.1.21) and (5.1.29) it follows that
\[ \forall i \in \{1, 2, ..., q\} \land \forall \tau \geq 0 : \{\bar{c}_{it}, \bar{a}_{it}\}_t \text{ is a unique solution to the problem (1d) – (5d) } \]
when \( w_t = f(k_t) - k_t f'(k_t), r_t = f'(k_t) \) and \( a_{it} = \bar{a}_{it} \), such that \( \bar{c}_{it} = \bar{c}_{it} \) ... (5.1.30)
Therefore, by definition of an equilibrium growth path, from (5.1.3), (5.1.5) and (5.1.30) we can conclude that \( \{ (c_{1t}, k_{1t}), (c_{2t}, a_{2t}), \ldots, (c_{q_t}, a_{qt}) \}_{t=0}^{\infty} \) is an equilibrium growth path for the economy.

Solving the differential equation in (5.1.6) we obtain \( \forall i \in \{1, 2, \ldots, q\} \wedge \forall t \geq 0 \):

\[
\left( \frac{\bar{a}_i}{k_i} - 1 \right) = \left( \frac{\bar{a}_{0i}}{k_0} - 1 \right) \exp \left( - \int_{0}^{t} \left[ \frac{\sigma \left( r_v - \left( \theta + n + \frac{\mu}{\sigma} \right) \right)}{r_v - (n + \mu)} \frac{w_v}{k_v} \right] dv \right)
\]

Therefore, \( \forall t \geq 0 \wedge \forall \text{ distinct } i, j \in \{1, 2, \ldots, q\} \):

\[
\left( \frac{\bar{a}_i - \bar{a}_j}{k_i} \right) = \left( \frac{a_{0i} - a_{0j}}{k_0} \right) \exp \left( - \int_{0}^{t} \left[ \frac{\sigma \left( r_v - \left( \theta + n + \frac{\mu}{\sigma} \right) \right)}{r_v - (n + \mu)} \frac{w_v}{k_v} \right] dv \right)
\] \quad \ldots (5.1.31)

Since, \( \mu_0 \) is defined by \( f'(k_0) = \theta + n + \frac{\mu_0}{\sigma} \), \( \bar{k} \) is a continuous function of \( t \) on \( [0, \infty) \) and \( \forall k > 0 : f''(k) < 0 \), therefore, (5.1.4) implies that

\[
\forall t \geq 0 : \exp \left( - \int_{0}^{t} \left[ \frac{\sigma \left( r_v - \left( \theta + n + \frac{\mu}{\sigma} \right) \right)}{r_v - (n + \mu)} \frac{w_v}{k_v} \right] dv \right) > 1 \quad \ldots (5.1.32)
\]

From (5.1.31) and (5.1.32) it thereby follows that

\[
\forall t \geq 0 \wedge \forall \text{ distinct } i, j \in \{1, 2, \ldots, q\}:
\]

\[
[0 < I_{[0,1]}(i, j) < 1 \leftrightarrow \mu < \mu_0] \wedge [I_{[0,1]}(i, j) = 1 \leftrightarrow \mu = \mu_0] \wedge [I_{[0,1]}(i, j) > 1 \leftrightarrow \mu > \mu_0]
\]

Hence, under the assumed conditions, there exist values of the initial ratio of capital to labour in the economy and initial distributions of inherited wealth for which
Chapter Five

\( \{ (\bar{c}_t, \bar{k}_t), (\bar{c}_1, \alpha_1), (\bar{c}_2, \alpha_2), \ldots, (\bar{c}_q, \alpha_q) \} \) is an equilibrium growth path for the economy along which the degree of inequality in the distribution of inherited wealth increases, remains unchanged or decreases according as \( \mu >, =, < \mu_0 \).

We can also note from (5.1.2) and (5.1.4) that the long-run rate of growth of per capita output, consumption and capital (wealth) is equal to \( \mu \).

This concludes our proof of Proposition 5.1.

Summary and Conclusion

The analysis in this chapter was motivated by the need to consider whether the disquieting possibilities of divergence in the distribution of inherited wealth over equilibrium growth paths and of long-run trade-offs between the objectives of economic growth and distributive equity with respect to inherited wealth emerging from our analysis of the case of iso-elastic instantaneous utility functions in previous chapters could be confined to the specific analytical framework considered in these chapters.

An important determining variable of the dynamics of any market economy is the nature of expectations that individuals have about the future evolution of the economy. In this chapter we considered whether our assumptions in Chapters Three and Four regarding the nature of these expectations were crucial to the emergence of the above possibilities or whether the above possibilities could also emerge under different assumptions about the nature of individual expectations in market economies. Hence, our analysis in this chapter was based on a representation of the market economy, which though in all respects similar to that used for our analysis of the case of iso-elastic instantaneous utility functions in previous chapters, assumed that individuals were not endowed with perfect foresight but, in fact, had steady state expectations about the future path of macroeconomic variables.
The chapter contains only one proposition. Proposition 5.1 asserts that even if we assume that individuals have steady state expectations about macroeconomic variables, there exist initial values of the ratio of capital to effective labour in the economy and initial distributions of inherited wealth for which there are equilibrium growth paths with the property that the degree of inequality in the distribution of inherited wealth increases, remains unchanged or decreases over these equilibrium growth paths according as the rate of growth of the number of efficiency units per natural unit of labour in the economy (which is also equal to the long-run rate of growth of per capita output) is greater than, equal to or less than a value $\mu_0$. From our discussion of the proof of Proposition 5.1, given the initial ratio of capital to effective labour in the economy, we can interpret $\mu_0$ as that value of the rate of growth of the number of efficiency units per natural unit of labour for which the above variety of equilibrium growth path is also a steady state growth path for the economy.

Proposition 5.1 demonstrates that the assumption of perfect foresight is not necessary to derive the possibility of equilibrium growth paths over which the degree of inequality in the distribution of bequests in the economy increases. Moreover, such growth paths may also be associated with higher long-run rates of growth. Compared to the case of perfect foresight, the demonstration involved the assumption of more stringent conditions on the initial amount of capital per efficiency unit of labour in the economy and on the initial distribution of inherited wealth. However, unlike in the case of perfect foresight, it did not assume any specific form of the production function.

The results are better understood in the context of the discussion carried out at the end of section 4.2 in Chapter Four to elucidate the relationship between the rate of labour-augmenting technical progress and the dynamics of the distribution of inherited wealth. We can show from both Chapters Four and Five that in the case of the iso-elastic instantaneous utility function, if individuals solve an optimisation problem such as (1a) – (5a) (see Chapter Three), with or without perfect foresight about the time path of the economy’s capital stock, then the individual rate of consumption is proportional to the sum of the individual’s inherited wealth and the
discounted (at expected rates of interest) values of his/her current and his/her descendants' expected future wage incomes.

In such a case, as seen in Chapter Four, the greater the sum of these expected values, the greater the possibility of the distribution of inherited wealth becoming more unequal. We noted that a higher rate of growth in per capita endowments of effective labour had positive as well as negative effects on this sum. The negative effects flowed from the expected negative influence on the future ratios of capital to effective labour of a higher rate of growth of effective labour. In the case of perfect foresight, individuals took into account such influences, and demonstration of the possibility of increasing inequality involved restrictions on the production function. In the case of steady state expectations, however, individuals do not contemplate any change in the relevant ratio, so that additional restrictions on the nature of the production function are not required.