Chapter 2

On The Hypersurface of a Finsler Space with the Special Metric $\alpha + \frac{\beta^{n+1}}{(\alpha-\beta)^n}$

We consider an $n$-dimensional Finsler space $F^n = (M^n, L(x, y))$, i.e., a pair consisting of an $n$-dimensional differential manifold $M^n$ equipped with a fundamental function $L(x, y)$. The concept of the $(\alpha, \beta)$-metric $L(\alpha, \beta)$ was introduced by M. Matsumoto [38] and has been studied by many authors [16], [24], [66]. A Finsler metric $L(x, y)$ is called an $(\alpha, \beta)$-metric $L(\alpha, \beta)$ if $L$ is a positively homogeneous function of $\alpha$ and $\beta$ of degree one, where $\alpha^2 = a_{ij}(x)y^iy^j$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on $M^n$.

A hypersurface $M^{n-1}$ of the $M^n$ may be represented para-
metrically by the equation \( x^i = x^i(u^\alpha), \alpha = 1, \ldots, n - 1 \), where \( u^\alpha \)
are Gaussian coordinates on \( M^{n-1} \). The following notations are also employed [26]:
\[ B^i_{\alpha \beta} = \frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta}, \quad B^i_{0\beta} = \nu^\alpha B^i_{\alpha \beta}, \quad B^{ij\ldots}_{\alpha \beta \ldots} = B^i_{\alpha} B^j_{\beta} \ldots. \]
If the supporting element \( y^i \) at a point \( (u^\alpha) \) of \( M^{n-1} \) is assumed to be tangential to \( M^{n-1} \), we may then write \( y^i = B^i_{\alpha}(u) \nu^\alpha \), so that \( \nu^\alpha \) is thought of as the supporting element of \( M^{n-1} \) at the point \( (u^\alpha) \). Since the function \( L(u, v) = L(x(u), y(u, v)) \) gives rise to a Finsler metric of \( M^{n-1} \), we get an \((n-1)\)-dimensional Finsler space \( F^{n-1} = (M^{n-1}, L(u, v)) \).

In the present chapter, we consider an \( n \)-dimensional Finsler space \( F^n = (M^n, L) \) with \((\alpha, \beta)\)-metric \( L(\alpha, \beta) = \alpha + \frac{\beta^{n+1}}{(\alpha - \beta)^n} \) and the hypersurface of \( F^n \) with \( b_i(x) = \partial_i b \) being the gradient of a scalar function \( b(x) \). We prove the conditions for this hypersurface to be a hyperplane of 1st kind, 2nd kind and we also prove that this hypersurface is not a hyperplane of 3rd kind.

### 2.1 Preliminaries

Let \( F^n = (M^n, L) \) be a special Finsler space with the metric

\[
L(\alpha, \beta) = \alpha + \frac{\beta^{n+1}}{(\alpha - \beta)^n}. \tag{2.1.1}
\]
The derivatives of the (2.1.1) with respect to $\alpha$ and $\beta$ are given by

$$
L_\alpha = \frac{(\alpha - \beta)^{n+1} - n\beta^{n+1}}{(\alpha - \beta)^{n+1}}, \quad L_\beta = \frac{(n + 1)(\alpha - \beta)\beta^n}{(\alpha - \beta)^{n+1}},
$$

$$
L_{\alpha\alpha} = \frac{n(n + 1)\beta^{n+1}}{(\alpha - \beta)^{n+2}}, \quad L_{\beta\beta} = \frac{n(n + 1)\alpha^2\beta^{n-1}}{(\alpha - \beta)^{n+2}},
$$

$$
L_{\alpha\beta} = \frac{-n(n + 1)\alpha\beta^n}{(\alpha - \beta)^{n+2}},
$$

where $L_\alpha = \frac{\partial L}{\partial \alpha}, L_\beta = \frac{\partial L}{\partial \beta}, L_{\alpha\alpha} = \frac{\partial L_\alpha}{\partial \alpha}, L_{\beta\beta} = \frac{\partial L_\beta}{\partial \beta}$ and $L_{\alpha\beta} = \frac{\partial L_\alpha}{\partial \beta}$.

In the special Finsler space $F^n = (M^n, L)$ the normalized element of support $l_i = \partial_i L$ and the angular metric tensor $h_{ij}$ are given by [66]:

$$
l_i = \alpha^{-1} L_\alpha Y_i + L_\beta b_i,
$$

$$
h_{ij} = p a_{ij} + q_0 b_i b_j + q_1 (b_i Y_j + b_j Y_i) + q_2 Y_i Y_j,
$$

where

$$
Y_i = a_{ij} y^j,
$$

$$
p = LL_\alpha \alpha^{-1} = \frac{\alpha (\alpha - \beta)^n + \beta^{n+1}}{(\alpha - \beta)^{2n+1}} \frac{((\alpha - \beta)^{n+1} - n\beta^{n+1})}{\alpha (\alpha - \beta)^{2n+1}},
$$

$$
q_0 = LL_\beta \beta = \frac{n(n + 1) (\alpha (\alpha - \beta)^n + \beta^{n+1})}{(\alpha - \beta)^{2n+2}} \frac{\alpha^2 \beta^{n-1}}{(\alpha - \beta)^{2n+2}},
$$

$$
q_1 = LL_{\alpha\beta} \alpha^{-1} = \frac{-n(n + 1) (\alpha (\alpha - \beta)^n + \beta^{n+1})}{(\alpha - \beta)^{2(n+1)}} \beta^n,
$$

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\( g_2 = L \alpha^{-2}(L_{\alpha\alpha} - L_{\alpha})^{-1} \)

\[
= \frac{\alpha(\alpha - \beta)^n + \beta^{n+1}}{\alpha^3(\alpha - \beta)^{2(n+1)}} \left[ n\beta^{n+1} \{(n + 2)\alpha - \beta\} - (\alpha - \beta)^{n+2} \right].
\]

The fundamental tensor \( g_{ij} = \frac{1}{2} \partial_i \partial_j L^2 \) and its reciprocal tensor \( g^{ij} \) is given by [66]

\[
g_{ij} = p a_{ij} + p_0 b_i b_j + p_1 (b_i Y_j + b_j Y_i) + p_2 Y_i Y_j,
\]

where

\[
p_0 = q_0 + L^2\beta,
\]

\[
= \frac{1}{(\alpha - \beta)^{2n+2}} \cdot (n + 1) \left\{ n\alpha^3(\alpha - \beta)^n \beta^{n-1} + (n + 1)\alpha^2\beta^{2n} + n\alpha^2\beta^{2n} - 2\alpha\beta^{2n+1} \right\} + \beta^{2n+2},
\]

\[
p_1 = q_1 + L^{-1}pL\beta,
\]

\[
= \frac{1}{\alpha(\alpha - \beta)^{2n+2}} \cdot \left\{ (1 - n)\alpha - \beta \right\} (n + 1)\alpha\beta^n(\alpha - \beta)^n - 2n(n + 1)\alpha\beta^{2n+1} + n\beta^{2(n+1)} - \beta^{n+1}(\alpha - \beta)^{n+1},
\]

\[
\text{(2.1.3)}
\]

\[
p_2 = q_2 + p^2L^{-2},
\]

\[
= \frac{1}{\alpha^3(\alpha - \beta)^{2(n+1)}} \left\{ \alpha(\alpha - \beta)^n + \beta^{n+1} \right\} \left\{ n\beta^{n+1} \{(n + 2)\alpha - \beta\} - (\alpha - \beta)^{n+2} \right\} + \left\{ (\alpha - \beta)^{2n+2} + n^2\beta^{2n+2} - 2n(\alpha - \beta)^{n+1}\beta^{n+1} \right\}.
\]

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\[ g_{ij} = p^{-1} a_{ij} - S_0 b_i b_j - S_1 (b_i y^i + b_j y^j) - S_2 y^i y^j, \] (2.1.4)

where

\[ b^i = a_{ij} b_j, \quad S_0 = \frac{(pp_0 + (p_0 p_2 - p_1^2) \alpha^2)}{\zeta p}, \]
\[ S_1 = \frac{(pp_1 + (p_0 p_2 - p_1^2) \beta)}{\zeta p}, \] (2.1.5)
\[ S_2 = \frac{(pp_2 + (p_0 p_2 - p_1^2) b^2)}{\zeta p}, \quad b^2 = a_{ij} b^i b^j, \]

\[ \zeta = p^2 (p + p_0 b^2 + p_1 \beta) + (p_0 p_2 - p_1^2) (\alpha^2 b^2 - \beta^2). \]

The \( hV \)-torsion tensor \( C_{ijk} = \frac{1}{2} \partial_k g_{ij} \) is given by [66]

\[ 2p C_{ijk} = p_1 (h_{ij} m_k + h_{jk} m_i + h_{ki} m_j) + \gamma_1 m_i m_j m_k, \]

where

\[ \gamma_1 = p \frac{\partial p_0}{\partial \beta} - 3 p_1 q_0, \quad m_i = b_i - \alpha^{-2} \beta Y_i. \] (2.1.6)

Here \( m_i \) is a non-vanishing covariant vector orthogonal to the element of support \( y^i \). Let \( \{ i \} \) be the components of Christoffel symbols of the associated Riemannian space \( R^n \) and \( \nabla_k \) be covariant differentiation with respect to \( x^k \) relative to this Christof-
fel symbols. We put

\[ 2E_{ij} = b_{ij} + b_{ji}, \quad 2F_{ij} = b_{ij} - b_{ji}, \]

where \( b_{ij} = \nabla_j b_i \).

Let \( C_T = (\Gamma^i_{jk}, \Gamma^i_{0k}, \Gamma^i_{jk}) \) be the Cartan connection of \( F^n \). The difference tensor \( D^i_{jk} = \Gamma^i_{jk} - \left\{ \begin{array}{c} i \\ jk \end{array} \right\} \) of the special Finsler space \( F^n \) is given by [32]

\[
D^i_{jk} = B^i E_{jk} + F^i_k B_j + F^i_j B_k + B^i_k b_{0k} + B^i_j b_{0j} \\
- b_{0m} g^{im} B_{jk} - C^i_{jm} A^m_k - C^i_{km} A^m_j + C_{jkm} A^m_s g^{is} \\
+ \chi^s (C^i_{jm} C^m_{sk} + C^i_{km} C^m_{sj} - C^m_{jk} C^i_{ms}),
\]

where

\[
B_k = p_0 b_k + p_1 Y_k, \quad B^i = g^{ij} B_j, \quad F^i_k = g^{ij} F_{ji} \\
B_{ij} = \left\{ p_1 (a_{ij} - \alpha^{-2} Y_i Y_j) + \frac{\partial p_0}{\partial \beta} m_i m_j \right\} / 2, \\
B^i_k = g^{ij} B_{ji}, \\
A^m_k = B^m_k E_{00} + B^m E_{k0} + B_k F^m_0 + B_0 F^m_k, \\
\lambda^m = B^m E_{00} + 2B_0 F^m_0, \quad B_0 = B_i y^i.
\]

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where '0' denote contraction with $y^i$ except for the quantities $p_0, q_0$ and $S_0$.

### 2.2 Induced Cartan connection

Let $F^{n-1}$ be a hypersurface of $F^n$ given by the equations $x^i = x^i(u^\alpha)$. The element of support $y^i$ of $F^n$ is to be taken tangential to $F^{n-1}$, that is

$$y^i = B^i_\alpha(u)v^\alpha.$$  \hspace{1cm} (2.2.1)

The metric tensor $g_{\alpha\beta}$ and $\nu$-torsion tensor $C_{\alpha\beta\gamma}$ of $F^{n-1}$ are given by

$$g_{\alpha\beta} = g_{ij}B_i^\alpha B_j^\beta, \quad C_{\alpha\beta\gamma} = C_{ijk}B_i^\alpha B_j^\beta B_k^\gamma.$$

At each point $u^\alpha$ of $F^{n-1}$, a unit normal vector $N^i(u, v)$ is defined by

$$g_{ij}(x(u, v), y(u, v))B_i^\alpha N^j = 0, \quad g_{ij}(x(u, v), y(u, v))N^iN^j = 1.$$

As for the angular metric tensor $h_{ij}$, we have

$$h_{\alpha\beta} = h_{ij}B_i^\alpha B_j^\beta, \quad h_{ij}B_i^\alpha N^j = 0, \quad h_{ij}N^iN^j = 1.$$  \hspace{1cm} (2.2.2)
If \((B^\alpha_i, N_i)\) denote the inverse of \((B^i_\alpha, N^i)\), then we have
\[
B^\alpha_i = g^{\alpha\beta} g_{ij} B^j_\beta, \quad B^\alpha_i B^\alpha_j = \delta^\alpha_\alpha, \\
B^\alpha_i N^i = 0, \quad B^i_\alpha N_i = 0, \quad N_i = g_{ij} N^j, \\
B^i_k = g^{k\ell} B_{\ell j}, \quad B^\alpha_i B^\alpha_j + N^i N_j = \delta^i_j.
\]

The induced connection \(IC = (G^\alpha_{\beta\gamma}, G^\alpha_\beta, C^\alpha_{\beta\gamma})\) of \(F^{n-1}\) induced from the Cartan’s connection \(C = (\Gamma^\alpha_{jk}, \Gamma^\alpha_{0k}, C^\alpha_{jk})\) is given by [37]
\[
\Gamma^\alpha_{\beta\gamma} = B^\alpha_i (B^i_{\beta\gamma} + \Gamma^\alpha_{jk} B^j_\beta B^k_\gamma) + M^\alpha_\beta H_\gamma, \\
C^\alpha_\beta = B^\alpha_i (B^i_{0\beta} + \Gamma^\alpha_{ij} B^j_\beta), \\
C^\alpha_{\beta\gamma} = B^\alpha_i C^i_{jk} B^j_\beta B^k_\gamma,
\]
where,
\[
M_{\beta\gamma} = N_i C^i_{jk} B^j_\beta B^k_\gamma, \quad M^\alpha_\beta = g^{\alpha\gamma} M_{\beta\gamma}, \\
H_\beta = N_i (B^i_{0\beta} + G^i_{0\beta} B^i_\beta), \quad (2.2.3)
\]
and \(B^i_{\beta\gamma} = \partial B^i_\beta / \partial u^\gamma, \quad B^i_{0\beta} = B^i_{\alpha\beta} u^\alpha\). The quantities \(M_{\beta\gamma}\) and \(H_\beta\) are called the second fundamental \(v\)-tensor and normal curvature vector respectively [37]. The second fundamental \(h\)-tensor \(H_{\beta\gamma}\) is
defined as [37]

\[ H_{\beta\gamma} = N_i(B_{\beta\gamma}^i + \Gamma_{jk}^i B_{\beta}^j B_{\gamma}^k) + M_\beta H_\gamma, \quad (2.2.4) \]

where

\[ M_\beta = N_i C_j^i B_{\beta}^j N_k. \quad (2.2.5) \]

The relative \( h \) and \( v \)-covariant derivatives of projection factor \( B_\alpha^i \) with respect to \( ICT \) are given by

\[ B_{\alpha|\beta} = H_\alpha^i N^i, \quad B_{\alpha|\beta} = M_\alpha^i N^i. \quad (2.2.6) \]

The equation (2.1.3) shows that \( H_{\beta\gamma} \) is generally not symmetric and

\[ H_{\beta\gamma} - H_{\gamma\beta} = M_\beta H_\gamma - M_\gamma H_\beta. \quad (2.2.7) \]

The above equation yield

\[ H_{0\gamma} = H_\gamma, \quad H_{\gamma 0} = H_\gamma + M_\gamma H_0. \quad (2.2.8) \]

We use following lemmas which are due to Matsumoto [37] as follows:
Lemma 2.2.1. The normal curvature $H_0 = H_\beta v^\beta$ vanishes if and only if the normal curvature vector $H_\beta$ vanishes.

Lemma 2.2.2. A hypersurface $F^{n-1}$ is a hyperplane of the 1st kind if and only if $H_\alpha = 0$.

Lemma 2.2.3. A hypersurface $F^{n-1}$ is a hyperplane of the 2nd kind with respect to the connection $C_\Gamma$ if and only if $H_\alpha = 0$ and $H_{\alpha\beta} = 0$.

Lemma 2.2.4. A hypersurface $F^{n-1}$ is a hyperplane of the 3rd kind with respect to the connection $C_\Gamma$ if and only if $H_\alpha = 0$ and $H_{\alpha\beta} = M_{\alpha\beta} = 0$.

2.3 Hypersurface $F^{n-1}(c)$ of the special Finsler space

Let us consider a special Finsler metric $L = \alpha + \frac{\beta^{n+1}}{(\alpha - \beta)^n}$ with a gradient $b_i(x) = \partial_i b$ for a scalar function $b(x)$ and a hypersurface $F^{n-1}(c)$ given by the equation $b(x) = c$ (constant)[28]. From parametric equation $x^i = x^i(u^\alpha)$ of $F^{n-1}(c)$, we get $\partial_\alpha b(x(u)) = 0 = b_i B_\alpha^i$, so that $b_i(x)$ are regarded as covariant components of a normal vector field of $F^{n-1}(c)$. Therefore, along the $F^{n-1}(c)$ we
have

\[ b_i B^i_\alpha = 0 \quad \text{and} \quad b_i y^i = 0. \] (2.3.1)

The induced metric \( L(u, v) \) of \( F^{n-1}(c) \) is given by

\[ L(u, v) = \alpha_{\alpha\beta} u^\alpha v^\beta, \quad a_{\alpha\beta} = a_{ij} B^i_\alpha B^j_\beta \] (2.3.2)

which is the Riemannian metric.

At a point of \( F^{n-1}(c) \), from (2.1.2), (2.1.3) and (2.1.5), we have

\[ p = 1, \quad q_0 = 0, \quad q_1 = 0, \quad q_2 = -\alpha^{-2}, \quad p_0 = 0, \quad p_1 = 0 \] (2.3.3)

\[ p_2 = 0, \quad \zeta = 1, \quad S_0 = 0, \quad S_1 = 0, \quad S_2 = 0. \]

Therefore, from (2.1.4) we get

\[ g^{ij} = a^{ij}. \] (2.3.4)

Thus along \( F^{n-1}(c) \), (2.3.3) and (2.3.1) leads to

\[ g^{ij} b_i b_j = b^2 \]

\[ = bb \]

\[ = b N^i b_j \]
Therefore, we get

\[ b_i(x(u)) = \sqrt{b^2} N_i, \quad b^2 = a^{ij} b_i b_j. \quad (2.3.5) \]

i.e. \( b_i(x(u)) = b N_i \), where \( b \) is length of vector \( b^i \).

Again from (2.3.4) and (2.3.5) we get

\[ b^i = b N^i. \quad (2.3.6) \]

Thus we have

**Theorem 2.3.1.** In the special Finsler hypersurface \( F^{n-1}(c) \), the induced Riemannian metric is given by (2.3.2) and the scalar function \( b(x) \) is given by (2.3.5) and (2.3.6).

The angular metric tensor and metric tensor of \( F^n \) are given by

\[ h_{ij} = a_{ij} - \frac{Y_i Y_j}{a^2}, \quad g_{ij} = a_{ij}. \quad (2.3.7) \]
From (2.3.1), (2.3.7) and (2.2.2) it follows that if \( h^{(a)}_{\alpha\beta} \) denote the angular metric tensor of the Riemannian \( a_{ij}(x) \), then along \( F^{n-1}(c) \), \( h_{\alpha\beta} = h^{(a)}_{\alpha\beta} \).

From (2.1.3), we get

\[
\frac{\partial p_0}{\partial \beta} = \frac{1}{(\alpha - \beta)^{4n+4}} \left\{ n(n+1)\alpha^3 \left\{ (n-1)\beta^n(\alpha - \beta)^{3n+2} 
+ (n+2)\beta^{n-1}(\alpha - \beta)^{3n+1} \right\} + \left\{ n(n+1)\alpha^2 + (n+1)^2\alpha^2 \right\} 
\right. \\
\left. 
\cdot \left\{ 2n(\alpha - \beta)^{2n+2}\beta^{2n-1} + 2(n+1)(\alpha - \beta)^{2n+1}\beta^{2n} - 2\alpha(n+1) \right\} 
\right. \\
\left. 
\cdot \left\{ (2n+1)\beta^{2n}(\alpha - \beta)^{2n+2} + (2n+2)(\alpha - \beta)^{2n+1}\beta^{2n+1}(\alpha - \beta)^{2n+2} 
\cdot (2n+2)\beta^{2n+1} + (2n+2)(\alpha - \beta)^{2n+1}\beta^{2n+2} \right\} \right\}.
\]

Thus along \( F^{n-1}(c) \), \( \frac{\partial p_0}{\partial \beta} = 0 \) and therefore (2.1.6) gives \( \gamma_1 = 0 \), \( m_i = b_i \).

Then the \( hv \)-torsion tensor becomes

\[
C_{ijk} = 0. \tag{2.3.8}
\]

in a special Finsler hypersurface \( F^{n-1}(c) \).

Therefore, (2.2.3), (2.2.5) and (2.3.8) give

\[
M_{\alpha\beta} = 0, \quad M_{\alpha} = 0. \tag{2.3.9}
\]
From (2.2.7) it follows that $H_{\alpha\beta}$ is symmetric. Thus we have

**Theorem 2.3.2.** The second fundamental $\nu$-tensor of special Finsler hypersurface $F^{n-1}(c)$ vanishes and the second fundamental $h$-tensor $H_{\alpha\beta}$ is symmetric.

Next from (2.3.1), we get $b_{i|\beta}B^i_\alpha + b_iB^i_{\alpha|\beta} = 0$. Therefore, from (2.2.6) and using $b_{i|\beta} = b_{ij}B^j_\beta + b_i|j\beta N^j\beta$, we get

$$b_{i|j}B^i_\alpha B^j_\beta + b_{ij}B^i_\alpha N^j\beta + b_iH_{\alpha\beta}N^i = 0. \quad (2.3.10)$$

Since $b_{i|j} = -b_iC^h_{ij}$, we get $b_{ij}B^i_\alpha N^j = 0$.

Thus (2.3.10) gives

$$bH_{\alpha\beta} + b_{i|j}B^i_\alpha B^j_\beta = 0. \quad (2.3.11)$$

It is noted that $b_{i|j}$ is symmetric. Furthermore, contracting (2.3.11) with $\nu^\beta$ and then with $\nu^\alpha$ and using (2.2.1), (2.2.8) and (2.3.9), we get

$$bH_\alpha + b_{i|j}B^i_\alpha y^j = 0, \quad (2.3.12)$$

$$bH_0 + b_{i|j}y^i y^j = 0. \quad (2.3.13)$$

In view of Lemmas (2.2.1) and (2.3.2), the hypersurface $F^{n-1}(c)$ is hyperplane of the first kind if and only if $H_0 = 0$. Thus from...
(2.3.12) it follows that $F^{n-1}(c)$ is a hyperplane of the first kind if and only if $b_{ij}y^iy^j = 0$. Here $b_{ij}$ being the covariant derivative with respect to $CT$ of $F^n$ depends on $y^i$. Since $b_i$ is a gradient vector, from (2.1.7) we have $E_{ij} = b_{ij}$, $F_{ij} = 0$ and $F^i_j = 0$. Thus (2.1.8) reduces to

\[
D^i_{jk} = B^i_{jk} + B^i_j b_{0k} + B^i_k b_{0j} - b_{0m} g^{im} B_{jk} - C^i_{jm} A^m_k - C^i_{km} A^m_j + C_{jkm} A^m_s g^{is} + \lambda s (C^i_{jm} C^m_{sk} + C^i_{km} C^m_{sj} - C^m_{jk} C^i_{ms}),
\]

(2.3.14)

In view of (2.3.3) and (2.3.4), the relations in (2.1.9) become to

\[
B_i = 0, \quad B^i = 0, \quad B^i_j = 0, \quad B_{ij} = 0, \quad A^m_k = 0, \quad \lambda^m = 0.
\]

(2.3.15)

By virtue of (2.3.15) we have $B^i_0 = 0, B_{i0} = 0$ which leads $A^m_0 = 0$. Therefore we have

\[
D^i_{j0} = 0, \quad D^i_{00} = 0.
\]
Thus from the relation (2.3.1), we get

\[ b_l D^i_{j0} = 0, \quad (2.3.16) \]
\[ b_l D^i_{00} = 0. \quad (2.3.17) \]

From (2.3.8) it follows that

\[ b_l C^i_{jm} B^j_{\alpha} = b^2 M_{\alpha} = 0. \]

Therefore, the relation \( b_{ij} = b_{ij} - b_r D^r_{ij} \) and equations (2.3.16), (2.3.17) give

\[ b_{ij} y^i y^j = b_{00}. \]

Consequently, (2.3.12) and (2.3.13) may be written as

\[ b H^i_{\alpha} + b_{i0} B^i_{\alpha} = 0, \quad b H_0 + b_{00} = 0. \quad (2.3.18) \]

Thus the condition \( H_0 = 0 \) is equivalent to \( b_{00} = 0 \), where \( b_{ij} \) does not depend on \( y^i \). Since \( y^i \) is to satisfy (2.3.1), the condition is written as

\[ b_{ij} y^i y^j = (b_l y^i)(c_j y^j) \]
for some $c_j(x)$, so that we have

$$2b_{ij} = b_i c_j + b_j c_i. \quad (2.3.19)$$

From (2.2.1) and (2.3.19) it follows that $b_{00} = 0$, $b_{ij} B^i_\alpha B^j_\beta = 0$, $b_{ij} B^i_\alpha y^i = 0$. Hence (2.3.18) gives $H_\alpha = 0$. Again from (2.3.19) and (2.3.15) we get $b_{i0} b^i = \frac{v^2}{2}$, $\lambda^m = 0$, $A^i_j B^j_\beta = 0$ and $B_{ij} B^i_\alpha B^j_\beta = 0$. Thus (2.1.4), (2.2.4), (2.2.5), (2.2.6), (2.2.7) and (2.2.14) gives $b_r D^i_{ij} B^i_\alpha B^j_\beta = 0$.

Thus we have

**Theorem 2.3.3.** The special Finsler hypersurface $F^{n-1}(c)$ is hyperplane of first kind if and only if (2.3.19) holds.

*Using (2.3.8), (2.3.13) and (2.3.15), we have*

Substituting (2.3.19) in (2.3.11) and using (2.3.1), we get

$$H_{\alpha\beta} = 0. \quad (2.3.20)$$

**Theorem 2.3.4.** If the special Finsler hypersurface $F^{n-1}(c)$ is a hyperplane of the first kind then it becomes a hyperplane of the second kind too.

Hence from (2.2.8), (2.3.20), Theorem (2.3.2), and Lemma (2.2.4) we have
Theorem 2.3.5. The special Finsler hypersurface $F^{n-1}(c)$ is hyperplane of the third kind if and only if it is a hyperplane of first kind.