CHAPTER-VI
Chapter 6

Conformal Change Of Some Special Finsler Spaces With Constant Unified Main Scalar

The conformal change and conformal transformation of $n$-dimensional Finsler spaces have been studied in ([15], [42], [48]). The conformal theory of two and three-dimensional Finsler spaces have been discussed in ([39], [52]) respectively. The conformal theory of three and four-dimensional Finsler space with constant unified main scalar has been studied in ([34], [50], [53], [68]). Recently, the theory of conformal change in four and five dimensional Finsler spaces has been developed in ([54], [60]). The theory of five dimensional Finsler spaces with constant unified main scalars has been introduced in [59]. In a five-dimensional Finsler space there are
seventeen main scalars $H, I, J, K, H', I', J', K', H'', I'', J'', K'', M, M', M'', N, N'$ in which the sum of $H, I, K$ and $M$ is $LC$ [60] which is called unified main scalar. In a five-dimensional Finsler space there exist six $\nu$-connection vectors $u_i, v_i, w_i, u'_i, v'_i, w'_i$ and six $h$-connection vectors $h_i, J_i, k_i, h'_i, J'_i, k'_i$ [60]. The orthonormal frame field $(l^i, m^i, n^i, p^i, q^i)$, called the Miron frame plays an important role in five-dimensional Finsler space.

6.1 Preliminaries

Let $F^5$ be a five-dimensional Finsler space with fundamental function $L(x, y)$. The metric tensor $g_{ij}$ and Cartan $C-$tensor $C_{ijk}$ of $F^5$ are defined by

$$g_{ij} = \frac{1}{2} \partial_i \partial_j L^2, \quad C_{ijk} = \frac{1}{2} \partial_k g_{ij} = \frac{1}{4} \partial_i \partial_j \partial_k L^2.$$

Throughout the chapter, the symbols $\partial_i = \frac{\partial}{\partial x^i}$ and $\partial_i = \frac{\partial}{\partial z^i}$ have been used. The frame $\{e^i_{(a)}\}, \alpha = 1, 2, 3, 4, 5$ is called the Miron frame of $F^5$, where $e^i_{(1)} = l^i = \frac{y^i}{L}$ is called the normalized supporting element, $e^i_{(2)} = \hat{m}^i = \frac{C^i}{C}$ is called the normalized torsion vector, $e^i_{(3)} = n^i, e^i_{(4)} = p^i, e^i_{(5)} = q^i$ are constructed from $g_{ij} e^i_{(a)} e^j_{(b)} = \delta_{ab}$. Here, $C$ is the length of torsion vector $C_i = C_{ijk} g^{jk}$. The Greek letters $\alpha, \beta, \gamma, \delta$ vary from 1 to 5 throughout the chapter. Summation convention is applied for both the
Greek and Latin indices. In the Miron frame an arbitrary tensor can be expressed by scalar components along the unit vectors $l^i, m^i, n^i, p^i, q^i$.

6.2 Conformal change of some special Finsler Spaces

6.2.1 Conformal Change of C-reducible Finsler Space

Definition 6.2.1. A Finsler space of dimension $n(> 2)$, is called C-reducible if $C_{ijk}$ is written in the form \[ C_{ijk} := \frac{1}{n+1} (h_{ij} C_k + h_{jk} C_i + h_{ki} C_j), \]
where $C_i = C_{ijk} g^{jk}$ is the torsion vector and $h_{ij}$ is the angular metric tensor given by $h_{ij} = g_{ij} - l_i l_j$.

In a five-dimensional C-reducible Finsler space $C_{ijk}$ takes the value \[ C_{ijk} := \frac{1}{6} (h_{ij} C_k + h_{jk} C_i + h_{ki} C_j), \]
where $C_i = C_{ijk} g^{jk} = C m_i = C \delta_{2\alpha} e_{\alpha i} = C a_{\beta\gamma} e_{\alpha i}$ and $h_{ij} = (\delta_{\beta \gamma} - \delta_{1\beta} \delta_{1\gamma}) e_{\beta j} e_{\gamma k}$.

In terms of scalar components $C_{ijk}$ can be written as
\[ C_{\alpha\beta\gamma} = \frac{1}{6} L C \left\{ \delta_{2\alpha} (\delta_{\beta\gamma} - \delta_{1\beta} \delta_{1\gamma}) + \delta_{2\beta} (\delta_{\gamma\alpha} - \delta_{1\gamma} \delta_{1\alpha}) + \delta_{2\gamma} (\delta_{\alpha\beta} - \delta_{1\alpha} \delta_{1\beta}) \right\}, \]  
(6.2.1)

where \( \alpha, \beta, \gamma = 1, ..., 5. \)

From (4.1.4) and (6.2.1), we get

\[ H = \frac{1}{2} L C, \quad I = K = M = \frac{1}{6} L C \quad H' = H'' = I' = I'' = K' = K'' = M' = M'' = J = J' = J'' = N = N' = 0. \]  
(6.2.2)

Hence we have the following:

**Theorem 6.2.1.** In a five-dimensional C-reducible Finsler space, the only non-zero main scalars are \( H, I, K \) and \( M \), given by (6.2.2) and the remaining main scalars vanish identically.

Using theorem (6.2.1) in the equation (4.1.17), we get

\[ u_i = v_i = w_i = w'_i = 0, \quad i = 1, ..., 5. \]  
(6.2.3)

Hence, we have the following:

**Theorem 6.2.2.** In a five-dimensional C-reducible Finsler space with non-zero constant unified main scalar, the four \( v \)-connection vectors \( u_i, v_i, w_i \), \( w'_i \) vanish identically.
Using the theorems (6.2.1) and (6.2.2), we see that (i), (iv) and (v) of the theorem (5.2.4), (i), (iii) and (v) of the theorem (5.2.5), and (i), (iii) and (iv) of the theorem (5.2.6) are satisfied identically and the remaining parts of the theorems (5.2.4), (5.2.5) and (5.2.6) are reduce to

\[ 18 - (LC)^2 = 0, \text{ provided } \sigma_2 \neq 0, \sigma_3 \neq 0, \sigma_4 \neq 0, \sigma_5 \neq 0. \]

(6.2.4)

Hence, we have the following:

**Theorem 6.2.3.** In a five-dimensional $C$-reducible Finsler space with constant unified main scalar, the $h$-connection vectors $h_i, J_i$ and $k_i$ are invariant under $\sigma-$ conformal change if and only if $LC = 3\sqrt{2}$, provided $\sigma_2 \neq 0, \sigma_3 \neq 0, \sigma_4 \neq 0, \sigma_5 \neq 0$.

**Corollary**

The main scalars in this case, are given by $H = \frac{3}{\sqrt{2}}$, $I = K = M = \frac{1}{\sqrt{2}}$ and $J = J' = J'' = H' = H'' = I' = I'' = K' = K'' = M' = M'' = N = N' = 0$.

Again, using the theorems (6.2.1) and (6.2.2), we see that (i), (ii) and (iii) of the theorem (5.2.9) are satisfied identically.
and its remaining parts are reduce to

\[ 36 - (LC)^2 = 0, \text{ provided } \sigma_4 \neq 0, \sigma_5 \neq 0. \]  

(6.2.5)

Hence we have the following:

**Theorem 6.2.4.** In a five-dimensional $C$-reducible Finsler space with constant unified main scalar, the $h$-connection vectors $k'_i$ is invariant under $\sigma$- conformal change if and only if $LC = 6$, provided $\sigma_4 \neq 0, \sigma_5 \neq 0$.

**Theorem 6.2.5.** The main scalars in this case are given by $H = 3, I = K = M = 1$ and $J = J' = J'' = H' = H'' = I' = I'' = K' = K'' = M' = M'' = N = N' = 0$.

Since a $C$-reducible Finsler space is either a Randers space or a Kropina space [52] whose metric functions are respectively given by

\[ L(x, y) = \sqrt{a_{ij}(x) y^i y^j + b_i(x) y^i} \text{ (Randers metric)} \]

\[ \text{and } L(x, y) = \frac{a_{ij}(x) y^i y^j}{b_i(x) y^i} \text{ (Kropina metric)}, \]  

(6.2.6)

therefore, the theorems (6.2.4) and (6.2.6) may be restated as

**Theorem 6.2.6.** In a five-dimensional Randers space or Kropina space
space with constant unified main scalar, the $h$-connection vectors $h_i$, $J_i$ and $k_i$ are invariant under $\sigma$— conformal change if and only if $LC = 3\sqrt{2}$, provided $\sigma_2 \neq 0, \sigma_3 \neq 0, \sigma_4 \neq 0, \sigma_5 \neq 0$.

**Theorem 6.2.7.** In a five-dimensional Randers space or Kropina space with constant unified main scalar, the $h$-connection vectors $k'_i$ is invariant under $\sigma$— conformal change if and only if $LC = 6$, provided $\sigma_4 \neq 0, \sigma_5 \neq 0$.

Using the theorem (6.2.2) in the equation (4.1.19), we get

$$S' = -\frac{1}{4} (LC)^2$$

which is constant.

Hence, we have the following:

**Theorem 6.2.8.** In a five-dimensional $C$-reducible Finsler space with constant unified main scalar, the $v$-scalar curvature $S$ is constant.

### 6.2.2 Conformal Change of semi-C-reducible Finsler space

Next, we consider a five-dimensional semi-C-reducible Finsler space with constant coefficients $p$ and $q$ and with constant unified main scalar.

**Definition 6.2.2.** A Finsler space of dimension $(n \geq 2)$, is called
semi-C-reducible if \( C_{ijk} \) is written in the form [35]

\[
C_{ijk} := \frac{p}{n+1} (h_{ij} C_k + h_{jk} C_i + h_{ki} C_j) + \frac{q}{C^2} C_i C_j C_k,
\]

where \( p + q = 1 \), \( C_i = C_{ijk} g^{jk} \) is the torsion vector and \( h_{ij} \) is the angular metric tensor given by \( h_{ij} = g_{ij} - l_i l_j \).

In a five-dimensional semi-C-reducible Finsler space \( C_{ijk} \) takes the value

\[
C_{ijk} := \frac{p}{6} (h_{ij} C_k + h_{jk} C_i + h_{ki} C_j) + \frac{q}{C^2} C_i C_j C_k,
\]

where \( C_i = C_{ijk} g^{jk} = C m_i = C \delta_{2a i} e_{\alpha} i = C a \beta \beta e_{\alpha} i \) and \( h_{ij} = (\delta_{\beta \gamma} - \delta_{1 \beta \delta_{1 \gamma}}) e_{\beta j} e_{\gamma k} \).

In terms of scalar components \( C_{ijk} \) can be written as

\[
C_{\alpha \beta \gamma} = \frac{1}{6} p L C \{ \delta_{2a} (\delta_{\beta \gamma} - \delta_{1 \beta \delta_{1 \gamma}}) + \delta_{2 \beta} (\delta_{\gamma \alpha} - \delta_{1 \gamma \delta_{1 \alpha}}) + \delta_{2 \gamma} (\delta_{\alpha \beta} - \delta_{1 \alpha \delta_{1 \beta}}) \} + q L C \delta_{2a} \delta_{2 \beta} \delta_{2 \gamma}, \tag{6.2.7}
\]

where \( \alpha, \beta, \gamma = 1, ..., 5 \).

From (4.1.4) and (6.2.7), we get

\[
H = \frac{1}{2} (p + 2q) L C, \quad I = K = M = \frac{1}{6} p L C \quad \text{and} \quad H' = H'' = I' = I'' = K' = K'' = M' = M'' = J = J' = J'' = N = N' = 0. \tag{6.2.8}
\]
Hence, we have the following:

**Theorem 6.2.9.** In a five-dimensional semi-C-reducible Finsler space with constant coefficients, the only non-zero main scalars are $H, I, K$ and $M$, given by (6.2.8) and the remaining main scalars vanish identically.

Using theorem (6.2.9) in the equation (4.1.19), we get (6.2.3). Hence we, have the following:

**Theorem 6.2.10.** In a five-dimensional semi-C-reducible Finsler space with constant coefficients and non-zero constant unified main scalar, the four $v$-connection vectors $u_i, v_i, w_i$ and $w'_i$ vanish identically.

Using the theorems (6.2.9) and (6.2.10), we see that $(i), (iv)$ and $(v)$ of the theorem (5.2.4), $(i), (iii)$ and $(v)$ of the theorem (5.2.5), and $(i), (iii)$ and $(iv)$ of the theorem (5.2.6) are satisfied identically and the remaining parts of the theorems (5.2.4), (5.2.5) and (5.2.6) are reduce to

$$18 - p(2p + 3q)(L^2) = 0, \text{ provided } \sigma_2 \neq 0, \sigma_3 \neq 0, \sigma_4 \neq 0, \sigma_5 \neq 0.$$

(6.2.9)

Since $p + q = 1$, we have the following:
Theorem 6.2.11. In a five-dimensional semi-C-reducible Finsler space with constant coefficients and constant unified main scalar, the h-connection vectors $h_i$, $J_i$ and $k_i$ are invariant under $\sigma$–conformal change if and only if $LC = \frac{3\sqrt{2}}{\sqrt{p(3-p)}}$ provided $\sigma_2 \neq 0$, $\sigma_3 \neq 0$, $\sigma_4 \neq 0$, $\sigma_5 \neq 0$.

Corollary:

The main scalars in this case are given by $H = \frac{3(2-p)}{\sqrt{2p(3-p)}}$, $I = K = M = \sqrt{\left\{ \frac{p}{2(3-p)} \right\}}$ and $J = J' = J'' = H' = H'' = I' = I'' = K' = K'' = M' = M'' = N = N' = 0$.

Again, using the theorems (6.2.9) and (6.2.10), we see that (i), (ii) and (iii) of the theorem (5.2.9) are satisfied identically and its remaining parts reduce to

$$36 - (pLC)^2 = 0, \text{provided} \sigma_4 \neq 0, \sigma_5 \neq 0.$$ (6.2.10)

Hence we have the following:

Theorem 6.2.12. In a five-dimensional semi-C-reducible Finsler space with constant coefficients and constant unified main scalar, the h-connection vector $k_i$ is invariant under $\sigma$–conformal change if and only if $LC = \frac{6}{p}$, provided $\sigma_4 \neq 0, \sigma_5 \neq 0$.  

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Corollary

The main scalars in this case are given by $H = \frac{3(p+2q)}{p} = \frac{3(2-p)}{p}$, $I = K = M = 1$ and $J = J' = J'' = H' = H'' = I' = I'' = K' = K'' = M' = M'' = N = N' = 0$.

Using the theorem (6.2.9) in the equation (4.1.19), we get
$$S = -\frac{1}{4}p(p+2q)(LC)^2 = -\frac{1}{4}p(2-p)(LC)^2,$$ which is constant.

Hence, we have the following:

**Theorem 6.2.13.** In a five-dimensional semi-$C$-reducible Finsler space with constant coefficients and constant unified main scalar, the $v$-scalar curvature $S$ is constant.

### 6.2.3 Conformal Change of $C^2$-like Finsler space

Next, we consider a five-dimensional $C^2$-like Finsler space with constant unified main scalar.

**Definition 6.2.3.** A Finsler space of dimension $n$, is called $C^2$-like if $C_{ijk}$ is written in the form [32], [35]

$$C_{ijk} := \frac{1}{C^2} C_i C_j C_k,$$

where $C_i = C_{ijk} g^{jk}$ is the torsion vector.
In terms of scalar components $C_{ijk}$ can be written as

$$C_{\alpha\beta\gamma} = LC\delta_{2\alpha}\delta_{2\beta}\delta_{2\gamma},$$  \hspace{1cm} (6.2.11)

where, $\alpha, \beta, \gamma = 1, \ldots, 5$.

From (4.1.4) and (6.2.11), we get

$$H = LC, \ I = K = M = H' = H'' = I' = I'' = K' = K'' = M' = M'' = J = J' = J'' = N = N' = 0.$$  \hspace{1cm} (6.2.12)

Hence, we have the following:

**Theorem 6.2.14.** In a five-dimensional $C^2$-like Finsler space the only non-zero main scalar is $H$ and the remaining main scalars vanish identically, given by (6.2.12).

Using theorem (6.2.14) in the equation (4.1.17), we get

$$u_i = v_i = w_i = 0, \ i = 1, \ldots, 5.$$  \hspace{1cm} (6.2.13)

Hence, we have the following:

**Theorem 6.2.15.** In a five-dimensional $C^2$-like Finsler space with non-zero constant unified main scalar, the three $v$-connection vectors $u_i, v_i, w_i$ vanish identically.
Using the theorems (6.2.14) and (6.2.15), we see that (i), (iv) and (v) of the theorem (5.2.4), (i), (iii) and (v) of the theorem (5.2.5), and (i), (iii) and (iv) of the theorem (5.2.6) are satisfied identically and the remaining parts of the theorems (5.2.4), (5.2.5) and (5.2.6) give

\[ \sigma_2 = 0, \sigma_3 = 0, \sigma_4 = 0, \sigma_5 = 0. \]  

(6.2.14)

Hence, we have the following:

**Theorem 6.2.16.** In a five-dimensional C2-like Finsler space with constant unified main scalar, the h-connection vectors $h_i, J_i$ and $k_i$ are invariant under $\sigma$-conformal change if and only if $\sigma_2 = 0, \sigma_3 = 0, \sigma_4 = 0, \sigma_5 = 0$.

Using the theorem (6.2.12) in the equation (4.1.19), we get

\[ S = 0. \]

Hence, we have the following:

**Theorem 6.2.17.** In a five-dimensional C2-like Finsler space with constant unified main scalar, the $v$-scalar curvature $S$ vanishes identically.