Chapter 4

PROPAGATION OF A CYLINDRICAL SHOCK WAVE IN AN AXISYMMETRIC ROTATING DUSTY GAS WITH HEAT CONDUCTION AND RADIATION HEAT FLUX
4.1 INTRODUCTION

The experimental studies and astrophysical observations show that the outer atmosphere of the planets rotates due to rotation of the planets. Macroscopic motion with supersonic speed occurs in an interplanetary atmosphere and shock waves are generated. Thus the rotation of planets or stars significantly affects the process taking place in their outer layers, therefore question connected with the explosions in rotating gas atmospheres are of definite astrophysical interest. Chaturani [1] studied the propagation of cylindrical shock waves through a gas having solid body rotation, and obtained the solutions by a similarity method adopted by Sakurai [2]. Nath et al [3] obtained the similarity solutions for the flow behind spherical shock waves propagating in a non-uniform rotating interplanetary atmosphere with increasing energy. Vishwakarma and Vishwakarma [4] and Vishwakarma et al [5] obtained the similarity solutions for magnetogasdynamic cylindrical shock waves propagating in a rotating medium which is perfect gas with variable density or a non-ideal gas with constant density. In all of the works, mentioned above, the ambient medium is supposed to have only one component of velocity, that is azimuthal component.

Marshak [6] studied the effect of radiation on the shock propagation by introducing the radiation diffusion approximation. Using the same mode of radiation, Elliott [7] discussed the conditions leading to self-similarity with a specified functional form of the mean free path of radiation and obtained a solution for self-similar spherical explosions. Gretler and Wehle [8] studied the propagation of blast waves with exponential heat release by taking inter-
nal heat conduction and thermal radiation in a detonating medium. Also, Abdel-Raouf and Gretler [9] obtained the non-self-similar solution for the blast waves with internal heat transfer effects. Ghoniem et al [10] obtained a self-similar solution for spherical explosions taking into account the effects of both conduction and radiation in the two limits of Rosseland radiative diffusion and Plank radiative emission.

At extreme conditions that prevail in most of the problems associated with shock waves, the assumption that the gas is ideal is no longer valid. Anisimov and Spiner [11] studied a problem of point explosion in a low density non-ideal gas by taking the equation of state in a simplified form, which describes the behaviour of the medium satisfactorily. Ranga Rao and Purohit [12] studied the self-similar flow of a non-ideal gas driven out by an expanding piston and obtained solutions by taking the equation of state suggested by Anisimov and Spiner [11]. Vishwakarma and Nath [13] obtained the similarity solutions for the flow behind an exponential shock in a non-ideal gas in both the cases, when the flow behind the shock was isothermal or adiabatic.

The study of shock waves in a mixture of a gas and small solid particles is of great importance due to its application to nozzle flow, lunar ash flow, bomb blast, coal-mine blast and many other engineering problems (see Pai et al [14], Miura and Glass [15], Higashino and Suzuki [16], Vishwakarma and Nath [17]). Pai et al [14] generalized the well-known solution of a strong explosion due to an instantaneous release of energy in gas (Sedov [18], Korobeinikov [19] to the case of two phase flow of a mixture of perfect gas and small solid particles, and brought out the essential effects due to presence of dusty particles on such a strong shock wave. Vishwakarma and Nath [17, 20],
Gretler and Regenfelder [21, 22], Steiner and Hirschler [23], Vishwakarma and Pandey [24] obtained similarity solutions for strong shock waves in radiating or non-radiating dusty gas (a mixture of small solid particles and perfect or non-ideal gas). In all of the works, mentioned above, the effects of the rotation of ambient medium are not taken into account by any of the authors in the case of dusty gas (a mixture of small solid particles and perfect or non-ideal gas).

In the present work, we obtained the self-similar solutions for the flow behind a cylindrical shock wave generated by a moving piston in an axisymmetric rotating dusty gas with heat conduction and radiation heat flux, which has a variable azimuthal fluid velocity together with a variable axial fluid velocity (Levin and Skopina [25], Nath [26]). The dusty gas is assumed to be a mixture of non-ideal gas and small solid particles. The fluid velocities in the ambient medium are assumed to obey power laws and the density in the ambient medium is assumed to be constant. The angular velocity of rotation of the ambient medium is assumed to be obeying a power law and to be decreasing as the distance from the axis increases. It is expected that such an angular velocity may occur in the atmosphere of rotating planets and stars. The piston is assumed to be moving with time according to power law (Steiner and Hirschler [23], Nath [26], Vishwakarma and Nath [27]). In order to get some essential features of shock propagation, the solid particles are considered as a pseudo-fluid and it is assumed that the equilibrium flow condition is maintained in the flow-field and that the viscous stress of the mixture is negligible (Pai et al [14], Higashino and Suzuki [16]). The heat transfer fluxes are expressed in terms of Fourier’s law for heat-conduction.
and a diffusion radiation model for an optically grey gas, which is typical of large-scale explosions. The thermal conductivity and absorption coefficient of the gas are assumed to be proportional to approximate powers of temperature and density (Ghoniem et al [10]). Also, it is assumed that the gas is grey and opaque, and the shock is isothermal. The assumption that the shock is isothermal is a result of the mathematical approximation in which the heat flux is taken to be proportional to the temperature gradient; this excludes the possibility of temperature jump (Zel’dovich and Raizer [28], Rosenau and Frankenthal [29, 30], Singh and Srivastava [31]). The counter-pressure (the pressure ahead of the shock) is taken into account. The radiation pressure and radiation energy are neglected (Elliott [7], Abdel-Raouf and Gretler [9], Ghoniem et al [10], Wang [32]). The assumption of an optically thick grey gas is physically consistent with the neglect of radiation pressure and radiation energy (Nicastro [33]). The gas ahead of the shock is assumed to be at rest.

4.2 FUNDAMENTAL EQUATIONS AND BOUNDARY CONDITIONS

We consider the medium to be a mixture of a non-ideal gas and small solid particles, which is rotating about an axis of symmetry. The equation of state of the non-ideal gas in the mixture is taken to be (Anisimov and Spiner
where \( p_g \) and \( \bar{\rho}_g \) are the partial pressure and partial density of the gas in the mixture, \( T \) is the temperature of the gas (and of the solid particles as the equilibrium flow condition is maintained), \( R^* \) is the specific gas constant and \( b \) is the internal volume of the molecules of the gas. In this equation, the deviations of an actual gas from the ideal state are taken into account, which result from the interaction between its component molecules. It is assumed that the gas is still so rarefied that triple, quadruple, etc., collisions between molecules are negligible, and their interaction is assumed to occur only through binary collisions.

The equation of state of the solid particles in the mixture is, simply,

\[
\rho_{sp} = \text{constant},
\]

where \( \rho_{sp} \) is the species density of the solid particles. Proceeding on the same lines as (Pai [34]), we obtain the equation of state of the mixture as (Vishwakarma and Nath [17])

\[
p = \frac{(1 - k_p)}{(1 - Z)}[1 + b\rho(1 - k_p)]\rho R^* T,
\]

where \( p \) and \( \rho \) are the pressure and density of the mixture, \( Z = \frac{V_{sp}}{V} \) is the volume fraction and \( k_p = \frac{m_{sp}}{m} \) is the mass fraction (concentration) of solid particles in the mixture, \( m_{sp} \) and \( V_{sp} \) being, respectively, the mass and the volumetric extensions of the solid particles in a volume \( V \) and mass \( m \) of the mixture.
The relation between $k_p$ and $Z$ is given by (Pai [34])

$$k_p = \frac{Z \rho_{sp}}{\rho}. \quad (4.2.4)$$

In equilibrium flow, $k_p$ is constant in the whole flow-field. Therefore from (4.2.4)

$$\frac{Z}{\rho} = \text{constant}. \quad (4.2.5)$$

Also, we have the relation (Pai [34])

$$Z = \frac{k_p}{G(1 - k_p) + k_p}, \quad (4.2.6)$$

where $G = \frac{\rho_{sp}}{\rho_g}$ is the ratio of density of solid particles to the species density of the gas.

The internal energy per unit mass of the mixture may be written as

$$U_m = [k_p C_{sp} + (1 - k_p)C_v]T = C_{vm}T, \quad (4.2.7)$$

where $C_{sp}$ is the specific heat of the solid particles, $C_v$ is the specific heat of the gas at constant volume and $C_{vm}$ is the specific heat of the mixture at constant volume.

The specific heat of the mixture at constant pressure is

$$C_{pm} = k_p C_{sp} + (1 - k_p)C_p, \quad (4.2.8)$$

where $C_p$ is the specific heat of the gas at constant pressure.

The ratio of the specific heats of the mixture is given by (Pai et al [14], Pai [34], Marble [35])

$$\Gamma = \frac{C_{pm}}{C_{vm}} = \gamma \frac{1 + \delta \beta' / \gamma}{1 + \delta \beta'}, \quad (4.2.9)$$

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where
\[ \gamma = \frac{C_p}{C_v}, \quad \delta = \frac{k_p}{1-k_p} \quad \text{and} \quad \beta' = \frac{C_{sp}}{C_v}. \]

Now,
\[ C_{pm} - C_{vm} = (1-k_p)(C_p - C_v) = (1-k_p)R^*, \quad (4.2.10) \]
neglecting the terms containing \( b^2 \rho^2 \) (Anisimov and Spiner [11], Singh [36]).

The internal energy per unit mass of the mixture is, therefore, given by
\[ U_m = \frac{p(1-Z)}{\rho(\Gamma - 1)[1 + b\rho(1-k_p)]}. \quad (4.2.11) \]

The fundamental equations for one-dimensional, unsteady, adiabatic axisymmetric rotational flow of the mixture of a non-ideal gas and small solid particles with heat conduction and radiation heat flux taken into account may, in Eulerian coordinates, be expressed as (Chaturani [1], Gretler and Wehle [8], Ghoniem et al [10], Levin and Skopina [25], Nath [26], Vishwakarma and Nath [27], Nath [37])
\begin{align*}
\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial r} + \rho \frac{\partial u}{\partial r} + \frac{\rho u}{r} &= 0, \\
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + \frac{1}{\rho} \frac{\partial p}{\partial r} - \frac{v^2}{r} &= 0, \\
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial r} + \frac{uv}{r} &= 0, \\
\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial r} &= 0, \\
\frac{\partial U_m}{\partial t} + u \frac{\partial U_m}{\partial r} - \frac{p}{\rho^2} \left( \frac{\partial p}{\partial t} + u \frac{\partial \rho}{\partial r} \right) + \frac{1}{\rho r} \frac{\partial}{\partial r} (rF) &= 0,
\end{align*}
(4.2.12)-(4.2.16)

where \( r \) and \( t \) are independent space and time coordinates, \( u, v \) and \( w \) are the radial, azimuthal and axial components of the fluid velocity \( \vec{q} \) in the
cylindrical coordinates \((r, \theta, z^*)\) and \(F\) is the heat flux. Also,

\[
v = Ar, \quad (4.2.17)
\]

where \(A\) is the angular velocity of the medium at radial distance \(r\) from the axis of symmetry. In this case the vorticity vector \(\vec{\zeta} = \frac{1}{2} \text{curl}\vec{q}\) has the components

\[
\zeta_r = 0, \quad \zeta_\theta = -\frac{1}{2} \frac{\partial w}{\partial r}, \quad \zeta_z^* = \frac{1}{2r} \frac{\partial}{\partial r}(rv). \quad (4.2.18)
\]

The total heat-flux \(F\), which appear in the energy equation may be decomposed as

\[
F = F_C + F_R, \quad (4.2.19)
\]

where \(F_C = \) conduction heat flux, and \(F_R = \) radiation heat flux.

According to Fourier’s law of heat conduction,

\[
F_C = -k \frac{\partial T}{\partial r}, \quad (4.2.20)
\]

where \(k\) is the coefficient of thermal conductivity and \(T\) is the absolute temperature.

Assuming local thermodynamic equilibrium and using the radiative diffusion model for an optically thick grey gas(Pomraning [38]), the radiative heat flux \(F_R\) may be obtained from the differential approximation of the radiation-transport equation in the diffusion limit as

\[
F_R = -\frac{4}{3} \left( \frac{\sigma}{\alpha_R} \right) \frac{\partial T^4}{\partial r}, \quad (4.2.21)
\]

where \(\sigma\) is the Stefan-Boltzmann constant and \(\alpha_R\) is the Rosseland mean absorption coefficient.
The thermal conductivity \( k \) and the absorption coefficient \( \alpha_R \) are assumed to vary with temperature and density. These can be written in the form of power laws, namely (Ghoniem et al [10], Vishwakarma and Nath [27])

\[
k = k_0 \left( \frac{T}{T_0} \right)^{\beta_C} \left( \frac{\rho}{\rho_0} \right)^{\delta_C}, \quad \alpha_R = \alpha_{R_0} \left( \frac{T}{T_0} \right)^{\beta_R} \left( \frac{\rho}{\rho_0} \right)^{\delta_R},
\]

where subscript ‘0’ denotes a reference state. The exponents in the above equations should satisfy the similarity requirements, if a self-similar solution is sought.

We assume that a cylindrical shock wave is propagating outwards from the axis of symmetry in the undisturbed medium (mixture of a non-ideal gas and small solid particles) with constant density, which has zero radial velocity and variable azimuthal and axial velocities. The flow variables immediately ahead of the shock front are

\[
u = 0, \quad (4.2.23)
\]
\[\rho = \rho_1 = \text{constant}, \quad (4.2.24)\]
\[v = v_1 = v_0 R^\lambda, \quad (4.2.25)\]
\[w = w_1 = w_0 R^\mu, \quad (4.2.26)\]
\[q = q_1 = 0 \quad \text{(Laumbach and Probstein [39])}, \quad (4.2.27)\]

where \( R \) is the shock radius, \( v_0, w_0, \lambda \) and \( \mu \) are constants and the subscript ‘1’ denotes the conditions immediately ahead of the shock.

The momentum equation (4.2.13) in the undisturbed state of the gas, gives

\[
p_1 = \frac{\rho_1 v_0^2 R^{2\lambda}}{2\lambda}, \quad \lambda > 0.
\]

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Ahead of the shock, the components of the vorticity vector, therefore vary as

\[ \zeta_{r_1} = 0, \]
\[ \zeta_{\theta_1} = -\frac{\mu w_0}{2} R^\mu \]
\[ \zeta_{\varphi_1} = \frac{v_0 (1 + \lambda)}{2} R^{\lambda - 1}. \]

From equations (4.2.27) and (4.2.25), we find that the initial angular velocity varies as

\[ A_1 = v_0 R^{\lambda - 1}, \]

it decreases as the distance from the axis increases, if \( \lambda < 1. \)

The disturbance is headed by an isothermal shock (the formation of isothermal shock is a result of the mathematical approximation in which the flux is taken to be proportional to the temperature gradient; this excludes the possibility of a temperature jump, see for example Zel’dovich and Raizer [28], Rosenau and Frankenthal [29, 30]) and hence, the conditions across it are

\[ \rho_2 (\hat{R} - u_2) = \rho_1 \hat{R}, \]
\[ p_2 + \rho_2 (\hat{R} - u_2)^2 = p_1 + \rho_1 \hat{R}^2, \]
\[ U_{m_2} + \frac{p_2}{\rho_2} + \frac{1}{2} (\hat{R} - u_2)^2 - \frac{F_2}{\rho_1 \hat{R}} = U_{m_1} + \frac{p_1}{\rho_1} + \frac{1}{2} \hat{R}^2, \]
\[ v_2 = v_1, \]
\[ w_2 = w_1, \]
\[ \frac{Z_2}{\rho_2} = \frac{Z_1}{\rho_1}, \]
\[ T_2 = T_1, \]
where subscript ‘2’ denotes the conditions immediately behind the shock and 
\[ \dot{R} = \frac{dR}{dt} \]
denotes the velocity of the shock front.

From equations (4.2.33) to (4.2.39), we obtain

\[ u_2 = (1 - \beta)\dot{R}, \quad (4.2.40) \]
\[ \rho_2 = \frac{\rho_1}{\beta}, \quad (4.2.41) \]
\[ Z_2 = \frac{Z_1}{\beta}, \quad (4.2.42) \]
\[ p_2 = \left[ (1 - \beta) + \frac{1}{\gamma M^2} \right] \rho_1 \dot{R}^2, \quad (4.2.43) \]
\[ F_2 = (1 - \beta) \left[ \frac{Z_1 + \bar{b}(1 - k_p)}{\gamma M^2(\beta - Z_1)(1 + b(1 - k_p))} - \frac{1}{2}(1 + \beta) \right] \rho_1 \dot{R}^3, \quad (4.2.44) \]
\[ v_2 = v_0 R^\lambda, \quad (4.2.45) \]
\[ w_2 = w_0 R^\mu, \quad (4.2.46) \]

where \( M = \left( \frac{\rho_1 \dot{R}^2}{\gamma p_1} \right)^{\frac{1}{2}} \) is the shock - Mach number referred to the frozen speed of sound \( \left( \frac{\gamma p_1}{\rho_1} \right)^{\frac{1}{2}} \), and \( \bar{b} = b\rho_1 \) is the parameter of non-idealness of the gas. The quantity \( \beta (0 < \beta < 1) \) is obtained by the relation

\[ \beta^3 - \left[ Z_1 + 1 + \frac{1}{\gamma M^2} \right] \beta^2 + \left[ \frac{Z_1 \bar{b}(1 - k_p)(1 + \gamma M^2) + Z_1 \gamma M^2 + 1}{\gamma M^2(1 + b(1 - k_p))} \right] \beta \]
\[ + \frac{(1 - Z_1)\bar{b}(1 - k_p)}{\gamma M^2(1 + b(1 - k_p))} = 0. \quad (4.2.47) \]

The expression for the initial volume fraction of the solid particles \( Z_1 \) is given by, from equation (4.2.6),

\[ Z_1 = \frac{V_{sp}}{V_1} = \frac{k_p}{(1 - k_p)G_1 + k_p}, \quad (4.2.48) \]
where \( G_1 = \frac{\rho_{sp}}{\rho_{g1}} \) is the ratio of the species density of solid particles to the initial species density of the gas in the mixture.

Following Levin and Skopina [25] and Nath [26, 37], we obtain the jump conditions for the components of vorticity vector across the shock as

\[
\begin{align*}
\zeta_\theta_2 &= \frac{\zeta_\theta_1}{\beta}, \\
\zeta_Z^{*2} &= \frac{\zeta_Z^{*1}}{\beta}.
\end{align*}
\] (4.2.49)

\[
\begin{align*}
\zeta_\theta_2 &= \frac{\zeta_\theta_1}{\beta}, \\
\zeta_Z^{*2} &= \frac{\zeta_Z^{*1}}{\beta}.
\end{align*}
\] (4.2.50)

### 4.3 SELF-SIMILARITY TRANSFORMATIONS

The inner boundary of the flow-field behind the shock is assumed to be an expanding piston. In the framework of self-similarity (Sedov [18]) the velocity \( u_p = \frac{dr_p}{dt} \) of the piston is assumed to follow a power law, which results in (Vishwakarma and Nath [17, 27], Steiner and Hirschler [23], Nath [26])

\[
\begin{align*}
u_p = \frac{dr_p}{dt} &= U_0 \left( \frac{t}{t_0} \right)^n,
\end{align*}
\] (4.3.1)

where \( r_p \) is the radius of the piston, \( t_0 \) denotes a reference time, \( U_0 \) is the piston velocity at \( t = t_0 \) and \( n \) is a constant. The consideration of ambient pressure and ambient angular velocity imposes a restriction on \( n \) as \( 0 < n < \infty \) (see equations (4.3.5) and (4.3.6)). Thus, the piston is continuously accelerated, and a shock is formed which arrives at the strong shock limit at large times. Concerning the shock boundary conditions, self-similarity
requires that the velocity of the shock $\dot{R}$ is proportional to the velocity of the piston:

$$\dot{R} = \frac{dR}{dt} = CU_0 \left( \frac{t}{t_0} \right)^n,$$

(4.3.2)

where $C$ is a constant. The time and space coordinate can be transformed into a dimensionless similarity variable as follows:

$$\eta = \frac{r}{R} = \left[ \frac{(n + 1)t_0^n}{CU_0} \right] \left( \frac{r}{t_0^{n+1}} \right).$$

(4.3.3)

Evidently, $\eta = \eta_p = \frac{r_p}{R}$ at the piston and $\eta = 1$ at the shock front. To obtain the similarity solutions, we write the unknown variables in the following form (Vishwakarma and Nath [17], Steiner and Hirschler [23], Nath[26, 40])

$$u = \frac{r}{t} U(\eta), \quad v = \frac{r}{t} \phi(\eta), \quad w = \frac{r}{t} W(\eta), \quad \rho = \rho_1 D(\eta), \quad p = \frac{r^2}{t^2} \rho_1 P(\eta),$$

$$Z = Z_1 D(\eta), \quad F = \frac{r^3}{t^3} \rho_1 Q(\eta),$$

(4.3.4)

where $U, \phi, W, D, P,$ and $Q$ are functions of $\eta$ only.

For the existence of similarity solutions, $M$ should be a constant; therefore

$$\lambda = \frac{n}{n + 1}.$$

(4.3.5)

Thus

$$M^2 = \frac{2n(n + 1) \frac{\gamma v_0^2}{n+1} \left( \frac{CU_0}{t_0^n} \right)^{n+1}}{\gamma v_0^2} = \text{constant},$$

(4.3.6)

where $0 < n < \infty$. Equation (4.3.6) shows that the solutions of the present problem cannot be reduced to the case in which the ambient medium is non-rotating (i.e. the case in which $v_0 = 0$).
Also, the total energy of the disturbance is given by

\[ E = 2\pi \int_{r_p}^{R} \rho \left[ U_m + \frac{1}{2}(u^2 + v^2 + w^2) \right] r dr. \]  \hspace{1cm} (4.3.7)

Applying the similarity transformations (4.3.4) and equation (4.2.11) in equation (4.3.7), we obtain

\[ E = 2\pi \rho_1 \left[ \frac{CU_0}{(n+1)\eta_0} \right] \frac{2}{R^{2(n+1)}} \int_{\eta_p}^{1} \frac{P(1 - Z_1 D)}{(\Gamma - 1)\{1 + bD(1 - k_p)\}} + \frac{1}{2} D(U^2 + \phi^2 + W^2) \eta^3 d\eta. \]  \hspace{1cm} (4.3.8)

Hence, the total energy of the shock wave is non-constant and varies as \( R^{-2(n+1)} \).

With the help of transformations (4.3.4), equations (4.2.12) to (4.2.16) can be transformed and simplified to

\[ \{U - (n + 1)\} \frac{dU}{d\eta} + \frac{2PU}{\eta} + \frac{2DU}{\eta} = 0, \]  \hspace{1cm} (4.3.9)

\[ \{U - (n + 1)\} \frac{dU}{d\eta} + \frac{U(U - 1)}{\eta} + 2P \frac{dU}{D \eta} + \frac{1}{D \eta} \frac{dP}{d\eta} - \frac{\phi^2}{\eta} = 0, \]  \hspace{1cm} (4.3.10)

\[ \{U - (n + 1)\} \frac{d\phi}{d\eta} + (2U - 1)\phi = 0, \]  \hspace{1cm} (4.3.11)

\[ \{U - (n + 1)\} \frac{dW}{d\eta} + \frac{(U - 1)W}{\eta} = 0, \]  \hspace{1cm} (4.3.12)

\[ \frac{dP}{d\eta} + L \frac{dD}{d\eta} + S \frac{dQ}{d\eta} + 2\frac{P(U - 1)}{\{U - (n + 1)\} \eta} + \frac{4SQ}{\eta} = 0, \]  \hspace{1cm} (4.3.13)

where

\[ L = \frac{P[(Z_1 D - 2)b(1 - k_p)D - 1 - (\Gamma - 1)\{1 + b(1 - k_p)D\}^2]}{D(1 - Z_1 D)\{1 + b(1 - k_p)D\}} \]

and

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Using equations (4.2.3) and (4.3.4) in equation (4.3.19), we obtain

\[
S = (\Gamma - 1)\{1 + \bar{b}(1 - k_p)D\} \big/ \left\{U - (n + 1)(1 - Z_1D)\right\}.
\]

From equations (4.3.9) to (4.3.13), we have

\[
\frac{dU}{d\eta} = - \frac{\{U - (n + 1)\}}{D} \frac{dD}{d\eta} - \frac{2U}{\eta}, \tag{4.3.14}
\]

\[
\frac{dP}{d\eta} = \{U - (n + 1)\}^2 \frac{dD}{d\eta} + \frac{U D}{\eta} \left[2\{U - (n + 1)\} - (U - 1)\right] - \frac{2P}{\eta} + \frac{\phi^2 D}{\eta}, \tag{4.3.15}
\]

\[
\frac{d\phi}{d\eta} = - \frac{\{U - (n + 1)\}}{\eta^2}, \tag{4.3.16}
\]

\[
\frac{dW}{d\eta} = - \frac{(U - 1)W}{\eta\{U - (n + 1)\}}. \tag{4.3.17}
\]

\[
\frac{dQ}{d\eta} = - \left[\frac{\{U - (n + 1)\}^2 + L}{S}\right] \frac{dD}{d\eta} - \frac{UD}{S\eta} \left[2\{U - (n + 1)\} - (U - 1)\right] + \frac{2P}{S\eta} - \frac{\phi^2 D}{S\eta} - \frac{2(U - 1)P}{S\eta(U - (n + 1))} - \frac{4Q}{\eta}. \tag{4.3.18}
\]

By use of equations (4.2.20),(4.2.21),(4.2.22) in equation (4.2.19), we obtain

\[
F = - \left[k_0 \left(CU_{n_0} \rho_{n_0} \delta_C \rho^3 \rho^3 \frac{n}{3} \right) \frac{\partial T}{\partial T} + 16\sigma T_0 \frac{\beta R_0 \rho_0 \delta R}{3 \alpha R_0} T^3 - \beta R \rho^{-\beta R} \right] \frac{\partial T}{\partial T}. \tag{4.3.19}
\]

Using equations (4.2.3) and (4.3.4) in equation (4.3.19), we obtain

\[
Q = - \left[k_0 \left(\frac{C U_{n_0}}{\Delta_{n_0}} \rho_1 \beta - \delta - 1 \eta \right) \frac{1}{\eta^2} \frac{\partial T}{\partial T} + 16 \sigma T_0 \frac{\beta R_0 \rho_0 \delta R_1 - \delta R - 1}{\delta R} \left(C U_{n_0} \rho^3 \rho^3 \frac{n}{3} \right) \frac{\partial T}{\partial T} \right] \times \left[\frac{2P(1 - Z_1D)}{n} + \frac{d}{d\eta} \left\{P(1 - Z_1D)\right\} \{D + \bar{b}(1 - k_p)D^2\} - \{1 + \bar{b}(1 - k_p)D\} \left\{P(1 - Z_1D)\right\} \frac{dD}{d\eta} \right]. \tag{4.3.20}
\]
Equation (4.3.20) shows that the similarity solution of the present problem exist only when

$$\beta_C = 1 + \frac{1}{2n} \quad \text{and} \quad \beta_R = 2 - \frac{1}{2n}.$$  

Therefore equation (4.3.20) becomes

$$Q = -X \left[ \frac{2P(1 - Z_1 D)}{\eta D\{1 + b(1 - k_p)D\}} + \frac{(1 - Z_1 D)}{D\{1 + b(1 - k_p)D\}} \frac{dP}{\eta} - \frac{P\{1 + \bar{b}(1 - k_p)D(2 - Z_1 D)\} dD}{D^2\{1 + \bar{b}(1 - k_p)D\}^2 \frac{d\eta}{\eta}} \right] \quad (4.3.21)$$

where

$$X = (n+1)^{-\frac{1}{n}} \left[ \Gamma_C D^{\delta_C - 1 - \frac{1}{2n}} + \Gamma_R D^{-\delta_R - 1 - \frac{1}{2n}} \right] (1 - k_p)^{-(2 + \frac{1}{n})} \eta^{1 + \frac{1}{n}} \left[ \frac{P(1 - Z_1 D)}{1 + b(1 - k_p)D} \right]^{1 + \frac{1}{n}}$$

and $\Gamma_C$ and $\Gamma_R$ are the conductive and radiative non-dimensional heat transfer parameters, respectively. The parameters $\Gamma_C$ and $\Gamma_R$ depend on the thermal conductivity $k$ and the mean free path of radiation $\frac{1}{\alpha_R}$, respectively, and also on the exponent $n$, and they are given by

$$\Gamma_C = \frac{k_0 \rho_1^{\delta_C - 1}}{t_0 T_0^{\delta_C} R^{\frac{\alpha}{\alpha}} \left( \frac{CU_u}{\sqrt{RT}} \right)} \quad \text{and} \quad \Gamma_R = \frac{16\sigma \rho_0^{\delta_R} T_0^{2}}{3\alpha R_0 t_0 R^{\frac{\alpha}{\alpha}} \rho_1^{\delta_R + 1} \left( \frac{CU_u}{\sqrt{RT}} \right) \frac{1}{n}}.$$

From equations (4.3.15) and (4.3.21), we obtain

$$\frac{dD}{d\eta} = - \frac{D\{1 + \bar{b}(1 - k_p)D\}}{D\{1 + \bar{b}(1 - k_p)D\}\{1 - Z_1 D\}\{U - (n + 1)\}^2 - P\{1 + \bar{b}(1 - k_p)D(2 - Z_1 D)\}} \times \left[ \frac{QD\{1 + \bar{b}(1 - k_p)D\}}{X} + \frac{2P(1 - Z_1 D)}{\eta} \right] \frac{dD}{d\eta} \left(1 - Z_1 D\right) \frac{U D}{\eta} \left(\frac{U}{2n} - 1\right) \frac{2P}{\eta} + \frac{\phi^2 D}{\eta} \right] \right]. \quad (4.3.22)$$

Also, applying the similarity transformations on equation (4.2.18), we obtain the non-dimensional components of the vorticity vector \( l_r = \frac{\zeta_r}{R/R}, l_\theta = \frac{\zeta_\theta}{R/R}, l_z = \frac{\zeta_z}{R/R} \) in the flow-field behind the shock as

$$l_r = 0, \quad (4.3.23)$$

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\[ l_\theta = \frac{nW}{2(n+1\{U - (n+1)\}}, \quad (4.3.24) \]

\[ l_{Z^*} = -\frac{(2n+1)\phi}{2(n+1\{U - (n+1)\}}, \quad (4.3.25) \]

Using the self-similarity transformations (4.3.4) and equation (4.3.2), shock conditions (4.2.40) to (4.2.46) can be rewritten as

\[ U(1) = (n+1)(1 - \beta), \quad (4.3.26) \]

\[ g(1) = \frac{1}{\beta}, \quad (4.3.27) \]

\[ P(1) = (n+1)^2 \left[ (1 - \beta) + \frac{1}{\gamma M^2} \right], \quad (4.3.28) \]

\[ Q(1) = (n+1)^3(1 - \beta) \left[ \frac{Z_1 + \tilde{b}(1 - k_p)}{\gamma M^2(\beta - Z_1)\{1 + \tilde{b}(1 - k_p)\}} - \frac{1}{2}(1 + \beta) \right], \quad (4.3.29) \]

\[ \phi(1) = (n+1)^{\frac{1}{\pi+1}} \sqrt{\frac{2n(n+1)^{\frac{n+1}{\pi+1}}}{\gamma M^2}}, \quad (4.3.30) \]

\[ W(1) = \frac{w_0}{v_0}(n+1)^{\frac{1}{\pi+1}} \sqrt{\frac{2n(n+1)^{\frac{n+1}{\pi+1}}}{\gamma M^2}}, \quad (4.3.31) \]

where \( \lambda = \mu \).

The piston path coincides at \( \eta_p = \frac{r_p}{R} \) with a particle path. Using equations (4.3.1) and (4.3.4), the relation

\[ U(\eta_p) = n + 1 \quad (4.3.32) \]

can be derived. In addition to the shock conditions (4.3.26) to (4.3.31), the kinematic condition (4.3.32) at the piston surface must be satisfied.
For an isentropic change of state of the mixture of non-ideal gas and small solid particles, under the thermodynamic equilibrium condition, we may calculate the equilibrium sound speed of the mixture, as follows

\[
 a_{isen} = \left( \frac{\partial p}{\partial \rho} \right)_{s}^{\frac{1}{2}} = \left[ \frac{\left\{ \Gamma + (2\Gamma - Z)b\rho(1 - k_{p}) \right\}p}{(1 - Z)\left\{ 1 + b\rho(1 - k_{p}) \right\}\rho} \right]^{\frac{1}{2}}, \tag{4.3.33}
\]

neglecting \( b^2\rho^2 \), where subscript ‘s’ refers to the process of constant entropy.

In addition, the isothermal speed of sound may also play a role, when thermal radiation is taken into account. The isothermal sound speed in the mixture is

\[
 a_{iso} = \left( \frac{\partial p}{\partial \rho} \right)_{T}^{\frac{1}{2}} = \left[ \frac{\left\{ 1 + (2 - Z)b\rho(1 - k_{p}) \right\}p}{(1 - Z)\left\{ 1 + b\rho(1 - k_{p}) \right\}\rho} \right]^{\frac{1}{2}}, \tag{4.3.34}
\]

where the subscript \( T \) refers to the process of constant temperature.

By using similarity transformations (4.3.4) in equation (4.3.34), we get the expression for reduced isothermal speed of sound as

\[
 \frac{a_{iso}}{R} = \left[ \frac{\left\{ 1 + (2 - Z)\bar{b}D(1 - k_{p}) \right\}P}{(1 - Z)\bar{D}\left\{ 1 + \bar{b}D(1 - k_{p}) \right\}} \right]^{\frac{1}{2}} \frac{\eta}{(n + 1)}. \tag{4.3.35}
\]

The adiabatic compressibility of the mixture of non-ideal gas and small solid particles may be calculated as (c. f. Moelwyn-Hughes [41])

\[
 C_{adi} = -\rho \left( \frac{\partial \left( \frac{1}{\rho} \right)}{\partial p} \right)_{s} = \frac{1}{\rho a_{isen}^2} = \frac{(1 - Z)[1 + b\rho(1 - k_{p})]}{[\Gamma + (2\Gamma - Z)b\rho(1 - k_{p})]p}. \tag{4.3.36}
\]

Using similarity transformations (4.3.4) in equation (4.3.36), we get the expression for the adiabatic compressibility as

\[
 (C_{adi})p_1 = \frac{(1 - Z_{1}D)[1 + \bar{b}D(1 - k_{p})](n + 1)^2}{[\Gamma + (2\Gamma - Z_{1}D)bD(1 - k_{p})]n^2\gamma M^2\rho}. \tag{4.3.37}
\]
For exhibiting the numerical solutions, it is convenient to write the field variables in non-dimensional form as

\[ \frac{u}{u_2} = \frac{U(\eta)}{U(1)\eta}, \quad \frac{v}{v_2} = \frac{\phi(\eta)}{\phi(1)\eta}, \quad \frac{w}{w_2} = \frac{W(\eta)}{W(1)\eta}, \quad \frac{Z}{Z_2} = \frac{\rho}{\rho_2} = \beta D(\eta), \]

\[ \frac{p}{p_2} = \frac{P(\eta)}{P(1)\eta^2}, \quad \frac{F}{F_2} = \frac{Q(\eta)}{Q(1)\eta^3}. \]  

(4.3.38)

4.4 RESULTS AND DISCUSSION

The distribution of the flow variables between the shock front \((\eta = 1)\) and the piston \((\eta = \eta_p)\) is obtained by numerical integration of equations (4.3.14) to (4.3.18) and (4.3.22) with boundary conditions (4.3.26) to (4.3.31) by the Runge-Kutta method of fourth order. For the purpose of numerical integration, the values of the constant parameters are taken to be (Pai et al [14], Miura and Glass [15], Vishwakarma and Nath [17, 27], Steiner and Hirschler [23]) \(\gamma = 1.4; \quad k_p = 0, 0.1, 0.4; \quad G_1 = 50, 100; \quad \beta' = 1; \quad \bar{b} = 0, 0.01, 0.02; \quad M^2 = 25; \quad \delta_C = 1; \quad \delta_R = 2; \quad \Gamma_C = 1; \quad \Gamma_R = 10; \quad n = 2\) and \(\frac{w_0}{v_0} = 0.5\). The value \(\gamma = 1.4, \beta' = 1\) may correspond to the mixture of air and glass particles (Miura and Glass [42]). The value \(k_p = 0\) corresponds to the dust-free case and \(k_p = 0, \bar{b} = 0\) to the perfect gas case. Also, \(n = 2\) corresponds to an accelerated piston. The value \(M = 5\) of the shock Mach number is appropriate, because we have treated the flow of a mixture of a non-ideal gas and a pseudo-fluid (small solid particles) at a velocity and temperature equilibrium. The assumption of velocity and temperature equilibrium may be a good approximation for strong shock waves, because the
thickness of the relaxation zone behind the shock front becomes very small for high Mach numbers (Gretler and Regenfelder [43]). The set of values \( \delta_C = 1, \delta_R = 2 \) is representative of the case of a high-temperature, low density medium (Ghoniem et al. [10]). Also, the set of values \( \Gamma_C = 1, \Gamma_R = 10 \) is the representative of the case in which there is heat transfer by both the conduction and the radiative diffusion.

Values of the density ratio across the shock front \( \beta \) and the piston position \( \eta_p \) are tabulated in Table 1 for different values of \( k_p, G_1 \) and \( \bar{b} \) with \( \gamma = 1.4, \beta' = 1, M^2 = 25, \frac{w_0}{v_0} = 0.5, \Gamma_C = 1, \Gamma_R = 10, \delta_C = 1, \delta_R = 2, \) and \( n = 2. \)
Table 1. Density ratio $\beta(=\frac{\rho_1}{\rho_2})$ and position of the piston $\eta_p$ for different values of $k_p$, $G_1$ and $\bar{b}$ with $n = 2$, $\gamma = 1.4$, $\beta' = 1$, $M^2 = 25$, $\frac{u_0}{v_0} = 0.5$, $\Gamma_C = 1$, $\Gamma_R = 10$, $\delta_C = 1$ and $\delta_R = 2$

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<th>$1 - \beta = \frac{u_2}{R}$</th>
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Fig. 1. Variation of reduced radial component of fluid velocity $\frac{u_1}{u_2}$ in the flow-field behind the shock front.
Fig. 2. Variation of reduced azimuthal component of fluid velocity $\frac{V}{V_2}$ in the flow-field behind the shock front.
Fig. 3. Variation of reduced axial component of fluid velocity $\frac{W}{W_2}$ in the flow-field behind the shock front
Fig. 4. Variation of reduced density $\frac{\rho}{\rho_0}$ in the flow-field behind the shock front.
Fig. 5. Variation of reduced pressure $\frac{p}{p_2}$ in the flow-field behind the shock front
Fig. 6. Variation of reduced total heat flux $\frac{5}{2}$ in the flow-field behind the shock front.
Fig. 7. Variation of azimuthal component of vorticity vector $l_\theta$ in the flow-field behind the shock front
Fig. 8. Variation of axial component of vorticity vector $l_z$ in the flow-field behind the shock front.
Fig. 9. Variation of reduced isothermal speed of sound $\frac{a}{R}$ in the flow-field behind the shock front.
Fig. 10. Variation of adiabatic compressibility \((C_{\text{ad}})\rho\) in the flow-field behind the shock front
Figs. 1-6 show the variation of the reduced flow variables \( \frac{u}{u_2}, \frac{v}{v_2}, \frac{w}{w_2}, \frac{\rho}{\rho_2}, \frac{p}{p_2}, \frac{F}{F_2}, l_\theta, l_{Z^*}, \frac{a_{iso}}{R}, \) and the adiabatic compressibility \((C_{adi})_p\) with \(\eta\) at various values of the parameters \(k_p, G_1\) and \(\bar{b}\). It is shown that, as we move inward from the shock front towards the piston, the reduced radial component of fluid velocity \(\frac{u}{u_2}\), the reduced density \(\frac{\rho}{\rho_2}\), the reduced pressure \(\frac{p}{p_2}\) and the reduced axial component of vorticity vector \(l_{Z^*}\) increase. These flow variables have higher values at the piston than at the shock front. In fact, since the similarity solutions of the present problem exist only for accelerating piston \((0 < n < \infty)\), the velocity of the piston is higher than the fluid velocity behind the shock. This fact can be seen from Table 1, which displays that \(\eta_p(= \frac{1}{\epsilon} = \frac{u_p}{F})\) is greater than \(1 - \beta (= \frac{u}{R})\). Therefore, the mass is more concentrated near the piston than at the shock front. In the figures, it is also seen that the reduced azimuthal component of fluid velocity \(\frac{v}{v_2}\), the reduced axial component of fluid velocity \(\frac{w}{w_2}\), the reduced total heat flux \(\frac{F}{F_2}\), the reduced azimuthal component of vorticity vector \(l_\theta\), the reduced isothermal speed of sound \(\frac{a_{iso}}{R}\) and the adiabatic compressibility \((C_{adi})_p\) decrease, in general, as we move inward from the shock front. The behaviour of the heat flux profiles is similar to those obtained by Elliott [7], Ghoniem et al [10] and Vishwakarma and Nath [27].

It is found that the effects of an increase in the value of the parameter of non-idealness \(\bar{b}\) of the gas are

(i) to increase the value of \(\beta\) (i.e. to decrease the shock strength, see Table 1);

(ii) to decrease \(\eta_p\) i.e. to increase the distance of the piston from the shock

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front (see Table 1). Physically it means that the gas behind the shock is less compressed, i.e. the shock strength is decreased, which is same as in (i) above.

(iii) to increase the flow variables \( \frac{u}{u_2}, \frac{v}{v_2}, \frac{w}{w_2}, \frac{F}{F_2}, l_{\theta} \) and isothermal speed of sound \( \frac{a_{\text{iso}}}{R} \) at any point in the flow-field behind the shock front (see Figs. 1-3, 6, 7 and 9);

(iv) to decrease the flow variables \( \frac{\rho}{\rho_2}, \frac{p}{p_2} \) and \( l_{Z^*} \) (see Figs. 4, 5 and 8);

(v) to decrease the adiabatic compressibility \( (C_{\text{adi}})p_1 \) (see Fig. 10).

The effects of an increase in the value of \( G_1 \) are

(i) to decrease the value of \( \beta \) (i.e. to increase the shock strength, see Table 1);

(ii) to increase \( \eta_{p} \) i.e., to decrease the distance of the piston from the shock front (see Table 1). This means that an increase in the ratio of the species density of solid particles to the initial species density of the gas has an effect to increase the shock strength, which is same as indicated in (i) above.

(iii) to decrease the flow variables \( \frac{u}{u_2}, \frac{v}{v_2}, \frac{w}{w_2}, \frac{F}{F_2}, l_{\theta}, \frac{a_{\text{iso}}}{R} \) and to increase the flow variable \( \frac{\rho}{\rho_2}, \frac{p}{p_2}, l_{Z^*} \) (see Figs. 1-9);

(iv) to increase the adiabatic compressibility \( (C_{\text{adi}})p_1 \). At constant ‘\( k_p \)’ an increase in \( G_1 \) results a high decrease in \( Z_1 \), i.e. the volume fraction of solid particles in the undisturbed medium becomes, comparatively, very
small. This causes comparatively more compression of the mixture in the region between the shock and the piston, which displays the above effects.

The effects of an increase in the value of $k_p$ are

(i) to increase the value of $\beta$ (i.e. to decrease the shock strength, see Table 1);

(ii) to decrease $\eta_p$ i.e. to increase the distance of the piston from the shock front (see Table 1). This means that an increase in the mass concentration of solid particles has an effect to decrease the shock strength, which is same as indicated in (i) above;

(iii) to decrease the flow variables $\frac{u}{u_2}$, $\frac{\rho}{\rho_2}$, $\frac{p}{p_2}$, $\frac{E}{E_2}$ and $l_{2^*}$, but to increase the flow variables $\frac{v}{v_2}$, $\frac{w}{w_2}$, $l_{\theta}$ and $\frac{\alpha_{ma}}{R}$ at any point behind the shock front;

(iv) to decrease the adiabatic compressibility $(C_{\text{adi}})p_1$ when $G_1 = 50$ and to decrease it, when $G_1 = 100$, $\bar{b} \neq 0$.

Physical interpretations of these effects are as follows:

In the case of $G_1 = 50$, an increase in $k_p$ reduces the compressibility of the medium due to increase in inertia of the mixture. This causes an increase in the distance between the shock front and the piston, a decrease in the shock strength and the above behaviour of the flow variables. Similar effects can be obtained for $G_1 = 100$, $\bar{b} = 0$.

In the case of $G_1 = 100$, $\bar{b} \neq 0$, Fig. 10 shows an increase in $(C_{\text{adi}})p_1$ due to increase in $k_p$. This increase in $(C_{\text{adi}})p_1$ is not because of the increase in
the compressibility ($C_{\text{adi}}$), but is the result of increase in $p_1$ due to increase in the mass of the mixture.
REFERENCES


