CHAPTER I

INTRODUCTION

1.1 The old hazy notion of convergence of infinite series was placed on sound foundation with the appearance of Cauchy's monumental work "Cours d Analyse algebrique" in 1821 and Abel's [1] researches on the Binomial series in 1826. However, it was observed that there were certain non convergent series which, particularly in Dynamical Astronomy, furnished nearly correct results. A theory of divergent series was formulated explicitly for the first time in 1890, when Cesaro [3] published a paper on the multiplication of series. Since then the theory of series, whose sequence of partial sums oscillates, has been the centre of attraction and fascination for most of the pioneering mathematical analysts. After persistent efforts, in which a number of celebrated mathematicians took part, it was only in the closing decade of the last century and in the early years of the present century that satisfactory methods were devised so as to associate with them by processes closely connected with Cauchy's concept of convergence, certain values which may be called their "sums" in a
reasonable way. These processes of associating generalized sums known as methods of summability *(Szasz [45] and Hardy [8] provide a natural generalisation of the classical notion of convergence **(Hobson [10]) and are thus responsible for bringing within the field of applicability a wider class of erstwhile rejected series that used to be tabooed as divergent.

The idea of convergence having been thus generalised, it was quite natural to study the possibilities of extending the notion of absolute convergence. As a matter of fact, just as the notion of convergence had led to the development of its extension under the general title of summability, so also, by analogy, the concept of absolute convergence led to the formation of the various processes of absolute summability (Kogbetliantz [17]).

As the idea of ordinary and absolute convergence were instrumental to the development of ordinary and absolute summability respectively, so also, the notion of uniform convergence would have certainly insisted the analysts to think of uniform summability.

* For an account of "Summability Methods", reference may be made to Szasz [45] and Hardy [8].
** For the concept of convergence and divergence of an infinite series, see Hobson [10].
Hardy and Littlewood [7], for the first time in 1913, introduced the notion of "Strong Summability" of Fourier series. Fekete [5] in 1916 defined that a series \( \sum a_n \) is strongly summable to the sum \( s \), if

\[
\sum_{v=1}^{n} |s_v - s| = o(n),
\]

as \( n \to \infty \), where \( s_v \) is the partial sum of the series \( \sum a_n \). This type is now known as strong Cesaro summability of order 1, or \([C,1]\) summability. It is important to note that strong summability is weaker than absolute summability and stronger than ordinary summability.

In the present thesis, we have made certain investigation concerning the following.

(i) Ordinary summabilities of Fourier series, its derived series and its conjugate series.

(ii) Matrix and \( C_1 \) summability of Fourier series and its conjugate series

(iii) Almost Nörlund summability of Fourier series and its conjugate series
(iv) Uniform Matrix and generalised Nörlund summability of Fourier series.

(v) \((\bar{N},p_n)\) and strong \((\bar{N},p_n)\) summability of derived Fourier and Fourier series.

We begin by giving a resume of the results obtained hitherto, which is the backbone of the existing results obtained in the sequel.

1.2 Some of the most familiar methods of summability, and with which we shall be concerned in the sequel, are those that are known as methods of Nörlund summability, Almost Nörlund summability, \((\bar{N},p_n)\) and strong \((\bar{N},p_n)\) summability, matrix summability and uniform matrix summability, Karamata summability and the degree of approximation. It may, however, be mentioned that all these methods can be derived from two basic general processes, which are termed as.

(i) \(\wedge\) – process,
(ii) \(\phi\) – process.

\(\wedge\) – methods are based on the formation of an auxiliary sequence \(\{t_m\}\), defined by the sequence - to- sequence transformation.

\[
(1.2.1) \quad t_m = \sum_{n=0}^{m} C_{m,n} s_n, \quad (m = 0,1,2, \ldots \ldots \ldots)\]
Where $s_n$ is the $n^{th}$ partial sum of a given infinite series $\sum a_n$. The matrix $A = (C_{m,n})$, in which $C_{m,n}$ is the element in the $m^{th}$ row and $n^{th}$ column, is a Toeplitz Matrix. If $t_m$ tends to a finite limit $s$, as $m \to \infty$, the series $\sum a_n$ or the sequence $\{s_n\}$ is said to be summable by the $A$- process to the sum $s$ and this is written as:

$$A \cdot \sum a_n = s.$$

The $A$ methods are based upon the formation of a functional transform.

$t(x)$, defined by the sequence to function transformation.

$$t(x) = \sum \phi_n(x) s_n,$$

or, more generally, by the integral transformation.

$$t(x) = \int_0^\infty \phi(x,y) s(y) \, dy,$$

where $x$ is a continuous parameter and the function $\phi_n(x)$ [or $\phi(x,y)$] is defined over a suitable interval of $x$. [or $x$ and $y$].

An infinite series $\sum a_n$ with its $n^{th}$ partial sum $s_n$ is said to converge to a finite limit $s$, if

$$\lim_{n \to \infty} s_n = s.$$
By analogy, the series \( \sum a_n \) is said to be summable by \( \wedge \)-method [or \( \phi \)-method] to the sum \( s \), if

\[
\lim_{m \to \infty} t_m = s \quad [\text{or} \quad \lim_{x \to a} t(x) = s].
\]

We know that a series \( \sum a_n \) with the sequence \( \{s_n\} \) of its partial sums is absolutely convergent, if the sequence \( \{s_n\} \) is of bounded variation, i.e.

\[
\sum_{n=1}^{\infty} \left| s_n - s_{n-1} \right| < \infty.
\]

Defining similarly, the absolute summability of an infinite series, we say that the series \( \sum a_n \) is said to be absolutely summable by \( \wedge \)-method, or simply summable \( |\wedge| \), if the corresponding auxiliary sequence \( \{t_m\} \) is of bounded variation, i.e.

\[
\sum_{m=1}^{\infty} \left| t_m - t_{m-1} \right| < \infty.
\]

Absolute summability by \( \phi \)-method, or summability \( |\phi| \), is defined in the same way with the obvious difference, that, in this case, the corresponding function \( t(x) \) should be a function of bounded variation in an interval of continuous parameter \( x \).
A summability method $Q$ is said to include another summability $P$, if the summability of a series (or a sequence) by the method $P$ to a certain sum implies its summability to the same sum by the method $Q$ as well.

A method of summability $B$ is said to be absolutely inclusive of another method $A$, if the absolute summability by the method $A$ implies absolute summability by the method $B$ and this fact is symbolically denoted as $|A| \subseteq |B|$. If any two methods of summability are absolutely inclusive of each other, then they are said to be absolutely equivalent.

The sequence-to-sequence transformation (1.2.1) is said to be an absolute conservative or absolute convergence preserving transformation, if the absolute convergence of the sequence $\{s_n\}$ implies that of the sequence $\{t_m\}$ in each case, and is said to be absolutely regular if the transformation is a absolute convergentive and the limits of $s_n$ and $t_m$ are the same, i.e. $\lim_{n \to \infty} s_n = s = \lim_{m \to \infty} t_m$.

Now we give briefly salient details of some of the methods of summability to which we have been referred and with which we shall be dealing in the present thesis.
1.3 NÖRLUND SUMMABILITY:

Although the method of summability considered under this title was first introduced by Woronoi *[55] in 1902, it is customary to associate it with the name of Nörlund [30], who, independently introduced this method in 1919.

Let \( \sum a_n \) be a given infinite series with sequence \( \{s_n\} \) of its partial sums. Let \( \{p_n\} \) be a sequence of constants, real or complex and let

\[
P_n = p_0 + p_1 + p_2 + \ldots \ldots + p_n, \quad (P_n \neq 0).
\]

Then sequence-to-sequence transformation.

\[
t_n = \frac{1}{P_n} \sum_{m=0}^{n} p_{n-m} s_m = \frac{1}{P_n} \sum_{m=0}^{n} p_m s_{n-m},
\]

defines the sequence \( \{t_n\} \) of Nörlund means of the sequence \( \{s_n\} \), generated by the sequence of constants \( \{p_n\} \). The series \( \sum a_n \) or the sequence \( \{s_n\} \) is said to be summable by Nörlund means of or summable \( (N,p_n) \) to the sum \( s \), if

* Woronoi's definition occurs in the proceeding of the t Eleventh congress of Russian Naturalists and Scientists (in Russian),
\[\lim_{n \to \infty} t_n = s\]

and is said to be absolutely summable by Nörlund means or summable \(|N,p_n|\), if

\[\sum_{n=1}^{\infty} |t_n - t_{n-1}| < \infty.\]

The condition of regularity [8] of the transformation (1.3.1) are

(i) \[p_n = o(P_n)\]

and

(ii) \[\sum_{m=0}^{n} p_m = O(|P_n|),\]

as \(n \to \infty\) \(\{p_n\}\) is real and non-negative, condition (ii) is automatically satisfied and, if in addition, \(\{p_n\}\) is non-increasing, condition (i) is also satisfied.

Mcfadden [25] has established that absolute Nörlund summabilities implies the ordinary Norlund summability but the converse implication is not necessary true.

1.4 CESARO SUMMABILITY (Zygmund [58], p. 45)

Given a sequence \(\{s_n\}\) of partial sums of a series \(\sum a_n\), let us write.
(1.4.1) \[ \sigma_n^\alpha = \frac{s_n^\alpha}{\Lambda_n^\alpha}, \quad \alpha > -1, \]

where \( s_n^\alpha \) and \( \Lambda_n^\alpha \) are defined by the formulae.

(1.4.2) \[
\sum_{n=0}^{\infty} s_n^\alpha x^n = (1-x)^{-\alpha} \sum_{n=0}^{\infty} s_n x^n
= (1-x)^{-\alpha-1} \sum_{n=0}^{\infty} a_n x^n, \]

and

(1.4.3) \[
\sum_{n=0}^{\infty} A_n^\alpha x^n = (1-x)^{-\alpha-1}
\]

The expressions \( s_n^\alpha \) and \( \sigma_n^\alpha \) are called respectively the Cesaro sums and Cesaro means of order \( \alpha \) of the sequence \( \{s_n\} \) (or the series \( \sum a_n \)). The \( A_n^\alpha \) are termed as the Cesaro numbers of order \( \alpha \). If

(1.4.4) \[
\lim_{n \to \infty} \sigma_n^\alpha = s,
\]

where \( s \) is a finite number, the series \( \sum a_n \) is said to be summable (C,\( \alpha \)) to the sum \( s \). If \( \{\sigma_n^\alpha\} \) is simply bounded, then the series \( \sum a_n \)
is said to be bounded \((C,\alpha)\).

If the sequence \(\{\sigma_n^\alpha\}\) is of bounded variation, that is to say that if

\[
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\sum_{n=1}^{\infty} \left|\sigma_n^\alpha - \sigma_{n-1}^\alpha\right| < \infty,
\]

We say that the sequence \(\{s_n\}\), or the series \(\sum a_n\) is absolutely summable \((C,\alpha)\) ([5], [17]). or summable \(|C,\alpha|\), \(\alpha>-1\).

From the definition it is evident that the absolute convergence and summability \(|C,0|\) are equivalent properties. To Kogbetliantz [17] is due the consistency theorem in its general form for the absolute Cesaro summability which asserts that if \(\sum a_n\) is summable \(|C,\alpha|\), \(\alpha>-1\), then it is also summable \(|C,\alpha+\delta|\), for ever \(\delta>0\).

It is worth noticing that the Cesaro summability \((C,\alpha)\) is a special class of Nörlund summability \((N,p_n)\) for the case when

\[
P_n = \binom{n+\alpha-1}{\alpha-1}, \alpha>-1.
\]

### 1.5 Harmonic Summability

An infinite series \(\sum a_n\) with the sequence \(\{s_n\}\) of its partial sums is said to be summable by harmonic method, or simply summable.
to the sum $s$, where $s$ is a fixed finite number, if the sequence-to-sequence transformation

$$(1.5.1) \quad l_n = \frac{1}{\log n} \sum_{v=0}^{n} \frac{s_v}{n-v+1} \to s,$$ as $n \to \infty$. It is obvious that the transformation (1.5.1) is a special class of the transformation

$$(1.3.1) \quad \text{for } p_n = \frac{1}{n+1}.$$  

1.6 UNIFORM HARMONIC SUMMABILITY:

Let

$$(1.6.1) \quad u_0(x) + u_1(x) + u_2(x) + \ldots$$

be an infinite series, and

$$(1.6.2) \quad u_n(x) = u_0(x) + u_1(x) + u_2(x) + u_n(x),$$

if there exists a function $u = u(x)$, such that

$$(1.6.3) \quad \frac{1}{\log n} \sum_{k=0}^{n} \left( \frac{1}{k+1} \right) \{u_{n-k}(x) - u\} = o(1),$$

uniformly in a set $E$, in which $u = u(x)$ is bounded, as $n \to \infty$, then we have say that the series (1.6.1) is summable by Harmonic means uniformly in $E$ to the sum $U$. 
sequence to sequence transformation

\[(1.3.1) \quad \text{for } p_n = \frac{1}{n+1}.\]

**1.7) K\(^\lambda\)-SUMMABILITY**

Let us define, for \(n=0,1,2, \ldots\), the numbers \(\left[ \frac{n}{m} \right] \) for \(0 \leq m \leq n\), by

\[(1.7.1) \quad \frac{n-1}{\pi} (x+v) = \sum_{m=0}^{n} \left[ \frac{n}{m} \right] x^m,\]

where

\[(1.7.2) \quad \frac{n-1}{\pi} (x+v) = x(x+1)(x+2) \ldots (x+n-1).\]

The numbers \(\left[ \frac{n}{m} \right] \) are known as the absolute values of the Stirling numbers of the first kind.

Let \(\{s_n\}\) denote the sequence of partial sums of infinite series \(\sum a_n\) and let

\[(1.7.3) \quad s_n^\lambda = \frac{\Gamma\lambda}{\Gamma\lambda + n} \sum_{m=0}^{n} \left[ \frac{n}{m} \right] \lambda^m s_m,\]

denote the \(n\)th \(K^\lambda\)-mean of order \(\lambda > 0\). If

\[(1.7.4) \quad s_n^\lambda \to s,\]

as \(n \to \infty\), where \(s\) is a fixed finite quantity, then the sequence \(\{s_n\}\) or
the series $\sum a_n$ is said to be summable by Karamata method $K^\lambda$ or order $\lambda>0$ to the sum $s$. The method $K^\lambda$ are regular for $\lambda>0$.

The method $K^\lambda$ were first introduced by Karanata [13], Lototsky [23] reintroduced the special case $\lambda=1$, only after the paper of Agnew [2], an intensive study of those and similar methods took place.

1.8 ALMOST NÖRLUND SUMMABILITY.

Let $\sum a_n$ be an infinite series with $\{S_n\}$ as the sequence of its $n$-th partial sums. Lorentz [22] (1948) has given the following definition:

A bounded sequence $\{S_n\}$ is said to be almost convergent to a limit $S$. If

$$ (1.8.1) \quad \lim_{n \to \infty} \frac{1}{n+1} \sum_{v=m}^{n+m} S_v = S $$

uniformly with respect to $m$.

Let $\{P_n\}$ be a sequence of non-zero real constants and

$$ (1.8.2) \quad P_n = p_0 + p_1 + p_2 + \cdots + p_n, P_{n-1} = P_{n-1} = 0. $$

We define that the series $\sum a_n$ or sequence $\{S_n\}$ is said to be almost
\[(N,P_n)\] summable to \(S\) if

\[(1.8.3)\]
\[t_{n,m} = \frac{1}{p_n} \sum_{v=0}^{n} p_{n-v} s_{v,m} , \text{ tends to } S\]
as \(n \to \infty\), uniformly with respect to \(m\).

\[(1.8.4)\]
\[S_{v,m} = \frac{1}{v+1} \sum_{v=k}^{v+m} S_k\]

### 1.9 MATRIX SUMMABILITY

Let \(\sum u_n\) be a given infinite series with sequence of partial sums \(\{s_n\}\). Let \(\text{LT} = (a_{n,k})\) be an infinite triangular matrix with real constants. Then sequence-to-sequence transformation

\[(1.9.1)\]
\[t_n = \sum_{k=0}^{n} a_{n,k} s_k , \quad n = 0, 1, 2, \ldots \ldots\]
defines the \(T\)-transformation of the sequence \(\{s_n\}\).

The series \(\sum u_n\) is said to be \(T\)-summable to \(s\) if \(\lim_{t \to \infty} t_n = s\).

The necessary and sufficient condition for \(T\)-method to be regular (i.e. \(\lim_{n \to \infty} s_n = s \Rightarrow \lim_{n \to \infty} t_n = s\)) are that

1. There is a constant \(A\),
\[\sum_{k=0}^{\infty} |a_{n,k}| < A , \text{ for every } n ;\]
2. For every k \( \lim_{n \to \infty} a_{n,k} = 0 \); 

3. \( \lim_{n \to \infty} \sum_{k=0}^{n} a_{n,k} = 1 \)

If matrix element \( a_{n,k} = 0 \), for every \( k > n \)

then the matrix is called triangular matrix. The matrix \( T \)-reduces to Nörlund matrix generated by the sequence of coefficients \( \{p_n\} \) if

\[
a_{n,k} = \begin{cases} 
\frac{p_{n-k}}{P_n} & \text{if } k \leq n; \\
0 & \text{if } k > n;
\end{cases}
\]

where \( P_n = \sum_{r=0}^{n} p_r \neq 0 \).

If the method of summability \( \Pi II \) is applied to Cesaro means of order one, another method of summability \( \Pi III. C \) is obtained.

**1.10 UNIFORM MATRIX SUMMABILITY.**

Let \( T = (a_{n,k}) \) be an infinite triangular matrix satisfying the silverman-Toeplitz (1913) \([49]\) condition of regularity.

ie. \[
\sum_{k=0}^{n} a_{n,k} \to 1 \quad \text{as} \quad n \to \infty
\]

\[
a_{n,k} = 0 \quad \text{for } k > n
\]

and \[
\sum_{k=0}^{n} \left| a_{n,k} \right| \leq M \quad \text{a finite constant.}
\]
Let $\sum_{m=0}^{\infty} u_m(x)$ be an infinite series such that

$$s_k(x) = u_0(x) + u_1(x) + \cdots + u_k(x) = \sum_{v=0}^{k} u_v(x).$$

If there exists a function $s=s(x)$ such that

$$t_n(x) = \sum_{k=0}^{n} a_{n,k} \{s_k(x) - s\}$$

$$= \sum_{k=0}^{n} a_{n,n-k} \{s_{n-k}(x) - s\}$$

$$= o(1), \quad n \to \infty$$

uniformly in a set $E$ in which $s=s(x)$ is bounded, then we say that the series $\sum_{m=0}^{\infty} u_m(x)$ summable (T) uniformly in $E$ to the sum $s$.

### 1.11. Uniform Generalised Nörlund Summability

Let $\sum a_n$ be a given infinite series with the sequence of partial sums $\{S_n\}$. Let $\{R_n\}$ be a sequence of constant real or complex and let us write

$$R_n = p_0 q_{n-x} + \cdots + p_n q_0.$$ 

The sequence to sequence transformation.
defined the sequence \( \{t_n\} \) of generalised Nörlund means of the sequence \( \{S_n\} \), generated by the sequence of coefficients \( \{R_n\} \). The series \( \sum a_n \) is said to be summable \( (N, p_n, q_n) \) to the sum \( s \) if \( \lim_{n \to \infty} t_n \) exist and equals \( s \).

The conditions for regularity of the method of summability \( (N, p_n, q_n) \) defined by (a), (b), (c) are

(a) \( p_{n-k} q_k = o(R_n) \) as \( n \to \infty \) (\( k \) fixed)

(b) \( (|p|^* |q|)_n = o(R_n) \) as \( n \to \infty \) condition (a) is equivalent to the condition that for (fixed) \( k \) for which \( q_k \neq 0 \).

(c) \( p_{n-k} = o(R_n) \) as \( n \to \infty \)

but (c) does not hold for those values of \( k \) (if any) for which \( q_k = 0 \).

Let \( u_0(x) + u_1(x) + \ldots \ldots \ldots \ldots \]

be any infinite series and

\[
U_1(x) = U_0(x) + U_1(x) + \ldots \ldots \ldots \ldots U_1(x)
\]

Let \( \{R_n\} \) be a sequence of constants real or complex and let us write

\[
R_n = p_0 q_{n-k} + \ldots \ldots \ldots \ldots \ldots p_n q_0
\]

if there exists a function \( U = U(x) \) such that
Uniformly in a set $E$ in which $u=(x)$ is bounded, then we shall say that the series (1.11.3) summable $(N,p_n,q_n)$ uniformly in $E$ to the sum $u$.

1.12.A $(\overline{N}, P_n)$ SUMMABLE:

Let $(p_n)$ be a sequence of real constants with $p_0 > 0$, $p_n > 0$ and

$$p_n = \sum_{v=0}^{n} p_v \neq 0,$$

such that $p_n \rightarrow 0$ as $n \rightarrow \infty$.

An infinite series $\sum a_n$ with the sequence $\{s_n\}$ of its partial sums is said to be summable $(\overline{N}, p_n)$ to the sum $s$ if the sequence-to-sequence transformation

$$t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v,$$

called as discontinuous Riesz mean or Riesz's typical mean ([45], p.142) tends to a finite limits as $n \rightarrow \infty$ ([8] p. 137).

The transformation (1.12.2) is obtained after the use of the so-called Hadamard type composition of the two sequence $\{s_n\}$.
and \(\{p_n\}\) [45]. The method \((N, p_n)\) is regular ([8], the 1.22, p.36).

The series \(\sum a_n\) is said to be summable \((N, p_n)\) if

\[
\sum_{n=1}^{m} \left| \sum_{v=0}^{n} P_v a_v \right| = O(1), \quad \text{as } m \to \infty.
\]

For \(p_n = \frac{1}{n+1}\), the transformation (1.22.2) reduces to Riesz Logarithmic summability.

1.12.b. STRONG NÖRLUND SUMMABILITY:

Following the lines of Fekete [8] or Hardy and Littlewood [7], we define that if

\[
\sum_{m=0}^{n} p_m \left| S_{n-m} - S \right| = o(P_n)
\]

as \(n\to\infty\), then the sequence \(\{s_n\}\) of partial sums, of a given infinite series \(\sum a_n\) is said to be strongly summable \((N, p_n)\), or simply summable \([N, p_n]\) to the fixed finite sum \(S\).

It is worth noticing that strong summability implies ordinary summability.

1.13 (a). TRIANGULAR MATRIX SUMMABILITY:

Let \(\{s_n\}\) be the sequence of \(n^{th}\) partial sum \(s\) of an infinite series
\[
\sum a_n \text{ and let } ((\lambda_{n,k})) (n=0,1,\ldots, k=0,1,\ldots n, \lambda_{n,0}=1) \text{ be a triangular matrix of real or complex numbers if }
\]

(1.13.1.a) \[ t_n = \sum_{v=0}^{n} \lambda_{n,v} a_v = \sum_{v=0}^{n} \lambda_{n,v} s_v, \]
where \[ \Delta \lambda_{n,m} = \lambda_{n,m} - \lambda_{n,m+1}, \]
and \[ \Delta^2 \lambda_{n,m} = \Delta \lambda_{n,m} - \Delta \lambda_{n,m+1}, \]
tends to a finite limits as \( n \to \infty \), then the series \( \sum a_n \) is said to be summable by triangular. Matrix method, or simply summable \( (\lambda) \) to the sum \( s \) \[ [8] \text{ page 137.} \]

The necessary and sufficient condition for the regularity of the summability \( (\lambda) \) are that

(i) there exists a constant \( M \) such that
\[
\sum_{v=0}^{n} \left| \Delta \lambda_{n,v} \right| < M, \text{ for every } n,
\]
(ii) for every \( v \), \( \lim_{n \to \infty} \Delta \lambda_{n,v} = 0 \), and
(iii) \( \lim_{n \to \infty} \sum_{v=0}^{\infty} \Delta \lambda_{n,v} = 1 \) \[ ([8], \text{p.30, the 0.1}) \]
as \( n \to \infty \), then the sequence \( \{s_n\} \) of partial sums of a given infinite series \( \sum a_n \) is said to be strongly summable \( (N,p_n) \), or simply
summable \([N,p_n]\) to the fixed finite sum \(S\).

It is worth noticing that strong summability implies ordinary summability.

**1.13b. APPROXIMATION TO FUNCTIONS BY TRIGONOMETRIC POLYNOMIALS:**

Zygmund ([58], p143.) has defined the approximation to functions by trigonometric polynomials as follows.

Given a periodic and continuous function \(f(x)\), the derivation \(\delta(F,T)\) of a trigonometric polynomial \(T(x)\). From \(f\) is defined by the formula.

\[
(1.13.1b) \quad \delta(f,T) = \max \left| f(x) - T(x) \right|
\]

The lower bound of the numbers \(\delta(F,T)\) for all polynomials.

\[
(1.13.2b) \quad T(x) = \frac{1}{2} a_0 + \sum_{\gamma=1}^{n} (a_\gamma \cos \gamma x + b_\gamma \sin \gamma x)
\]

of given order \(n\) will be denoted by \(E_n(f)\) and called the best approximation of order \(n\).

By the theorem of Weierstrass ([58]) \(E_n(F)\) tends (monotonically) to zero as \(n \to \infty\), Weierstrass's theorem reads as follows:

If \(f\) is periodic and continuous, then for every \(\varepsilon > 0\) there is
at trigonometric polynomials $T(x)$ such that.

\[(1.13.3b) \quad |f(x) - T(x)| < \varepsilon, \text{ for all } x.\]

1.14. FOURIER SERIES, ITS CONJUGATE SERIES, ITS DERIVED SERIES:

Let $f(t)$ be a $2\pi$ periodic and Lebesgue, integrable function of $t$ in the interval $(-\pi, \pi)$ and then, periodically extended beyond this interval to the left and to the right so as to satisfy. The functional equation.

\[(1.14.1) \quad f(x \pm 2\pi) = f(x).\]

Then the Fourier series corresponding to the function $f(t)$ is defined, by the corresponding.

\[(1.14.2) \quad f(t) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)\]

\[= \sum_{n=0}^{\infty} A_n(t),\]

where the coefficients $a_0, a_n, b_n$ are known as Fourier coefficients given by

\[(1.14.3) \quad a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \, dt\]
\[ (1.14.4) \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos nt \, dt \] 
and 
\[ (1.14.5) \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt \, dt. \]

The notation in (1.14.2) is due to Hurwitz [11] and the formulae (1.14.2) and (1.14.5) are known as Euler-Fourier formulae.

The series
\[ (1.14.6) \quad \sum_{n=1}^{\infty} (b_n \cos nt - a_n \sin nt) = \sum_{n=1}^{\infty} B_n(t) \]
is called the conjugate series of the Fourier series (1.13.2).

The series
\[ (1.14.7) \quad \sum_{n=1}^{\infty} n(b_n \cos nt - a_n \sin nt) = \sum_{n=1}^{\infty} nB_n(t). \]

Which is obtained by differentiating the series (1.14.2) term by term is called the first differentiated series or the derived series of the Fourier series (1.14.2).

It is well known (Young [56]) that the derived series of a Fourier series (1.14.2) may not itself be a Fourier series. It is only when the function is an integral that its derived series is a Fourier series.
1.15 $K^\lambda$-SUMMABILITY OF A FOURIER SERIES:

Let $f(t)$ be a $2\pi$-periodic and Lebesgue integrable function of $t$ in the interval $(-\pi, \pi)$. Then the Fourier series of $f(t)$ is given by

\[ f(t) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \left( a_n \cos nt + b_n \sin nt \right) \]

\[ = \sum_{n=1}^{\infty} A_n(t). \]

Let us write

\[ \phi(t) = f(x+t) + f(x-t) - 2f(x) \]

\[ \tau = \left\lfloor \frac{1}{t} \right\rfloor = \text{the integral part of } \frac{1}{t}. \]

The method $K^\lambda$ were first introduced by Karamata. [13]. Lototsky [23]. reintroduced the special case $\lambda=1$. Only after the paper of Agnew [2] an intensive study of those and similar methods took place.

Vuckovic [54] established the following theorem on the summability of Fourier series by Karamata methods (1.7)

THEOREM A. If

\[ \phi(t) = o \left( \frac{1}{\log 1/t} \right), \ t \to +0, \]

then the series (1.15.1) is summable $K^\lambda$ to the sum $f(x)$ at the point $t=x$, for every $\lambda>0$. 

(25)
Kathal [14] generalised theorem A by proving the following:

**THEOREM B.** If

\[(1.15.3) \quad \Phi(t) = \int_0^t \left| \phi(u) \right| \, du = o\left( \frac{1}{\log 1/t} \right),\]

as \( t \to +0 \), then the series (1.15.1) is summable \( K^\lambda (\lambda > 0) \) to the sum \( f(x) \) at the point \( t=x \).

We [50], in Chapter II, have generalised Theorem B under very general conditions, by establishing the following:

**THEOREM** If

\[(1.15.4) \quad \Phi(t) = \int_0^t \left| \phi(u) \right| \, du = o\left( \frac{\varepsilon(1/t)t}{\log 1/t} \right),\]

Where \( \varepsilon(t) \) is positive function of \( t \) as \( t \to 0 \) such that \( \frac{\varepsilon(n)}{\log n} \to o(1) \) as \( n \to \infty \) and

\[(1.15.5) \quad \varepsilon(n) \to O[\log(n)]\]

then the Fourier series is summable \( K^\lambda \) to the sum \( f(x) \) at the point \( t=x \).

It is important to note that our theorem includes theorems A and B as special cases.
1.16. ON $K^\lambda$-SUMMABILITY OF CONJUGATE SERIES OF FOURIER SERIES:-

Let the Fourier series of $f(t)$ be given by

\begin{align*}
(1.16.1) \quad f(t) &\sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \\
&= \sum_{n=1}^{\infty} A_n(t).
\end{align*}

and then

\begin{align*}
(1.16.2) \quad \bar{f}(t) &\sim \sum_{n=1}^{\infty} (a_n \sin nt - b_n \cos nt) \\
&= \sum_{n=1}^{\infty} B_n(t).
\end{align*}

(1.16.2) Is Known as conjugate series of Fourier series (1.16.1) We write

\begin{align*}
(1.16.3) \quad \phi(t) &= f(x+t) + f(x-t) - 2f(x) \\
\psi(t) &= f(x+t) - f(x-t).
\end{align*}

\[ K(t) = \sum_{m=0}^{n} \left[ \frac{\lambda^m}{m!} \right] \cos(m+1/2)t \]

\[ \Gamma(\lambda+n)\sin(t/2) \]

THEOREM: If
\( \phi(t) = \int_0^t |\phi(u)| \, du = o \left( \frac{t}{\log 1/t} \right) \) as \( t \to +0 \)

then the Fourier series (1.15.1) is summable \( K^\lambda (\lambda > 0) \) to the sum \( f(x) \) at the point \( t = x \).


**THEOREM:** Let \( \{p_n\} \) be a non-negative, monotonic non-increasing sequence of real constants that

\[ P_n \to \infty \text{ as } n \to \infty \]

where \( P_n = \sum_{v=0}^{n} p_v \)

If \( \alpha(t) \) denote a positive, monotonic, non-increasing function of \( t \), where

\[ \alpha(n) \log n = O(p_n), \text{ as } n \to \infty, \]

then the conjugate series (1.16.7) of Fourier series is \( K^\lambda \)-summable to

\[ -\frac{1}{2\pi} \int_0^n \psi(t) \cot \left( \frac{1}{2} t \right) \, dt, \] at every point \( x \), where. This integral exists.
1.17 ALMOST NÖRLUND SUMMABILITY OF CONJUGATE SERIES OF A FOURIER SERIES:

The conjugate series of a Fourier series (1.14.6) is given by

\[ \sum_{n=1}^{\infty} (b_n \cos nt - a_n \sin nt) = \sum_{n=1}^{\infty} B_n(t). \]

\( \psi(t) \) and \( \varphi(t) \) is given by (1.16.3).

\[ \tau_{n,m} = \frac{1}{p_n} \sum_{v=0}^{n} P_{n-v} \bar{S}_{v,m}. \]

\[ \bar{N}_{n,m} = \frac{1}{2\pi} \frac{1}{p_n} \sum_{v=0}^{n} P_{n-v} x \frac{\sin(v + m + 1)t - \sin mt}{2(v + 1)\sin^2 t / 2}. \]

\[ \tau = \left[ \frac{1}{t} \right] = \text{the integral part of} \ \frac{1}{t}. \]

Pati [34], (1961) has established the following theorem on the Nörlund summability of a Fourier series.

**THEOREM:** If \((N,p_n)\) be a regular Nörlund method defined by a real, non-negative monotonic non-increasing sequence of coefficient \(\{p_v\}\) such that \(P_n \to \infty\) as \(n \to \infty\).

\[ P_n = \sum_{v=\infty}^{n} p_v \to \infty \text{ as } n \to \infty. \]
Then if
\[ \phi(t) = \int_0^t |\phi(u)| \, du \]

(1.17.4)
\[ = o \left( \frac{1}{P_t} \right) \text{ as } t' \to +0 \]

The conjugate Fourier series (1.14.6) is summable \((N, p_n)\) to \(f(x)\) at the point \(t = x\).

The object of present chapter is to generalize the above result for almost Nörlund summability of conjugate series of Fourier series in the following form.

THEOREM: Let \(\{p_n\}\) be a real non-negative monotonic, nonincreasing sequence of coefficients such that \(P_n \to \infty\) as \(n \to \infty\).

If
\[ \Psi(t) = \int_0^t |\psi(u)| \, du = o \left( \frac{t^{\lambda(1/\ell)}}{P_t} \right) \]
as \(t \to +0\),
and
\[ \int_{\theta t^{\lambda(n+m)}}^{\theta t^{\lambda(n+m)}} \frac{|\psi(u)|}{u} \, du = o(1), \quad \text{as } n \to \infty \]

where \(0 < \delta < 1\), uniformly with respect to \(m\), then the series (1.14.6) is almost \((N, p_n)\) summable to
\[
\frac{1}{2\pi} \int_0^\infty \psi(t) \cot \frac{t}{2} \, dt,
\]

at every point where this integral exists, provided

(1.17.7) \( \lambda(n) \log n = O(n^\rho), \) as \( n \to \infty. \)

**1.18 ON MATRIX SUMMABILITY OF FOURIER SERIES AND ITS CONJUGATE SERIES:**

The Fourier series and its conjugate Fourier series is given by (1.14.2) and (1.14.6). The value of \( \phi(t) \) and \( \psi(t) \) is also given by (1.16.3).

\[(1.18.1) \quad N_n(t) = \frac{1}{2\pi} \sum_{k=0}^{n} a_{n,k} \frac{\cos(k+1/2)t}{\sin t/2}\]

\[(1.18.2) \quad \overline{N}_n(t) = \frac{1}{2\pi} \sum_{k=0}^{n} a_{n,k} \frac{\sin(k+1/2)t}{\sin t/2}\]

\[A_{n,\tau} = \sum_{k=1}^{n} a_{n,k} \text{ and } \tau = [1/t] \text{ an integral part of } 1/t\]

Iyengar [12] (1943) proved that, if

(1.18.3) \( \phi(t) = o[\log (1/t)^{-1}] \) below as \( t \to 0, \) then the series (1.14.2) is summable by harmonic means [1.5] to \( f(x). \)

Siddiqi [41] (1948) has improved the above result using stringent condition for harmonic means which was further improved recently by Pandey [33] (1983) for Nörlund means, He proved.
THEOREM A. If

\[ \phi(t) = o[t \alpha(t)] \text{ as } t \to +0, \]

where \( \alpha(t) \) is a non-negative, non-decreasing function of \( t \) such that

\[ \alpha(1/n) = o(1) \text{ as } n \to \infty \]

and

\[ \int_{1/n}^{1} \alpha(t) \frac{dt}{t} = O(p_n), \quad \text{as } n \to \infty \]

then the Fourier series of \( f(t) \) at \( t = x \) is summable \((N, p_n)\) to \( f(x) \), where \( \{p_n\} \) is a real non-negative and non-decreasing such that \( P_n \to \infty \), as \( n \to \infty \). He also proved.

THEOREM B. If the sequence \( \{p_n\} \) and \( \alpha(t) \) be same as in theorem A then if

\[ \overline{\psi}(t) = o[t \alpha(t)], \quad \text{as } t \to +0, \]

then the conjugate series (1.14.6) is summable \((N, p_n)\) to

\[ \frac{1}{2\pi} \int_{0}^{\pi} \psi(t) \cot \frac{t}{2} \, dt, \]

at every point where this integral exists.

In this paper, we extend the Theorem A and B to matrix summability. We establish the following theorems.

THEOREM 1 Let \( \|T\| \to (a_{nk}) \) be an infinite triangular matrix with \( a_{nk} \geq 0; \)
Let \( (a_{n,k})_{k=0}^{n} \) be a real non-negative and non-increasing sequence with respect to \( k \), if

\[
\Phi(t) = o\left(\frac{t}{\alpha(1/t)}\right) \quad \text{as } t \to +0,
\]

where \( \alpha(t) \) is non-negative and non-decreasing function of \( t \) such that

\[
\alpha(n) \to \infty \quad \text{as } n \to \infty
\]

and

\[
\int_{\frac{\delta}{t/n}}^{\frac{\delta}{t}} \frac{A_{n,n-2}}{t} \alpha(t) \, dt = O(1) \quad \text{as } n \to \infty
\]

then the Fourier series (1.14.2) at \( t = x \) is \( T \)-summable to \( f(x) \).

**THEOREM 2.** If the sequence \( (a_{n,k})_{k=0}^{n} \) and \( \alpha(t) \) be same as in Theorem 1, then if

\[
\overline{\psi}(t) = o\left[\frac{t}{\alpha(1/t)}\right], \quad \text{as } t \to +0,
\]

then the conjugate series (1.14.6) is \( T \)-summable to

\[
\frac{1}{2\pi} \int_{0}^{\infty} \overline{\psi}(t) \cot \frac{t}{2} \, dt,
\]

at every point where this integral exists.
1.19 ON THE \(||T||_1\) C_1 SUMMABILITY OF A SEQUENCE OF FOURIER SERIES

Let \(f(x)\) be a periodic with period \(2\pi\) and integrable in the sense of Lebesgue over an interval \((-\pi, \pi)\). The Fourier series and conjugate series is given by (1.14.2) and (1.14.6)

\[
f(t) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)
= \sum_{n=1}^{\infty} A_n(x),
\]

and conjugate series

\[
\sum_{n=1}^{\infty} (b_n \cos nx - a_n \sin nx) \equiv \sum_{n=1}^{\infty} B_n(x),
\]

\(\phi(t)\) and \(\psi(t)\) is also given by (1.16.3).

Mohanty and Nanda [28] (1954) proved the following theorem.

THEOREM A. If

\[
\psi(t) = \frac{1}{\log(1/t)} \text{ as } t \to 0
\]

\[a_n = O(n^{-p}), b_n = O(n^{-\delta}), 0 < \delta < 1,\]

then the sequence \(\{nB_n(x)\}\) is summable \((C,1)\) to the value \(\ell/\pi\).

From this result they have deduced a well known criterion, the Hardy
and Littlewood's [7] test for the convergence of the conjugate series

\[(1.14.6)\]

Varshney [53] (1959) improved Theorem A in the following form:

**THEOREM B.** If

\[
\int_0^t |\psi(u) \, du| = \psi_1(t) = o \left( \frac{t}{\log(1/t)} \right), \quad \text{as } t \to +0,
\]

then the sequence \(\{nB_n(x)\}\) is summable \((N, \frac{1}{n+1})_{C_1}\) to the value \(\ell/\pi\).

Various authors such as Sharma [40] (1970,1969), Lal [18] (1971), Dwivedi [4] (1971) have generalised the above theorem of Varshney in different directions. Recently Prasad [35] (1981) further extended the theorem of Varshney to \((N, p_n)_{C_1}\) summability of the sequence \(\{nB_n(x)\}\). He proved.

**THEOREM C.** Let \(p(u)\) be a monotonic decreasing and strictly positive for \(u \geq 0\). Let \(p_0 = p(n)\) and

\[(1.19.1) \quad P(u) = \int_0^u p(x) \, dx, \quad \text{such that } P(u) \to \infty \text{ as } u \to \infty\]

Let \(\alpha(t)\) be a positive non-decreasing function of \(t\). If

\[(1.19.2) \quad \int_0^t \psi(u) \, du = o \left( \frac{t}{\alpha(1/t)} \right) \text{ as } t \to +0,\]

then a sufficient condition that the sequence \(\{nB_n(x)\}\) be summable
Now we extend the above theorem to \( \|I\|_1 \) summability of the sequence \( \{nB_n(x)\} \).

We prove the following theorem.

**THEOREM.** If \( \|I\|_1 \equiv (a_{n,k}) \) be an infinite triangular matrix with \( a_{n,k} \neq 0 \) and \( a_{n,k} \) defined by.

\[
a_{n,k} = a_n(k), \quad a_n(u) \text{ being a strictly positive monotone non increasing function and}
\]

\[
(1.19.4) \quad A(n,n-u) = \int_{t=0}^{u} a_n(n-t) \, dt \to 1 \text{ as } n \to \infty \text{ for fixed } u \geq 0
\]

Let \( \alpha(t) \) a positive non-decreasing function of \( t \) if

\[
(1.19.5) \quad \Psi_1(t) = o\left( \frac{\alpha(1/t), t}{\log 1/t} \right) \text{ as } t \to +0
\]

then the sequence \( \{nB_n(x)\} \) be summable \( \|I\|_1 \) to the value \( \ell/\pi \) such that

\[
(1.19.6) \quad \frac{\alpha(n)}{\log n} \to 0 \text{ as } n \to \infty.
\]

We note that (1.19.5) and (1.19.2) are same while conditions (1.19.4)
and (1.19.6) in case of \((N,p_n)_C^1\) summability reduce to condition (1.19.1) and (1.19.3) respectively.

1.20. ON UNIFORM MATRIX SUMMABILITY OF A FOURIER SERIES:

Fourier series is given by (1.14.2)

\[
f(x) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)
\]

Saxena [37] (1965) has applied the concept of uniform summability for Fourier series by harmonic means. The object of present chapter to prove a theorem on uniform Matrix \((T)\) summability.

We shall prove the following theorem.

**THEOREM.** Let \(\langle a_{n,k} \rangle\) be an infinite triangular matrix such that the elements \(\langle a_{n,k} \rangle\) are non-negative and non-decreasing with \(k\) and if

\[
\Phi(t) = \int_0^t |\phi(u)| \, du = o\left(\frac{\epsilon(1/t)p_T}{\beta(p_T)}\right)
\]

Uniformly in a set \(E\) where \(\epsilon(t)\) and \(\beta(t)\) and \(\frac{\epsilon(t)}{\beta(t)}\) increasing monotonically with \(t\) and

\[
\epsilon(n) P_n = O\left[\beta(P_n)\right]
\]
as \( n \to \infty \) and \( \{p_n\} \) is a real non-negative and monotonically non-increasing sequence of coefficients such that \( P_n \to 0 \) as \( n \to \infty \), then the Fourier series (1.14.2) is summable \((T)\) uniformly in \( E \) to the sum \( f(x) \).

**1.21 ON UNIFORM GENERALISED NÖRLUND SUMMABILITY OF FOURIER SERIES**

The object of this chapter is to introduce the concept of uniform Nörlund summability which defines in section [1.11] as follows

\[(1.21.1) \quad \text{Let } u_0(x) + u_1(x) + \ldots \]

be any infinite series and

\[ U_i(x) = U_0(x) + U_1(x) + \ldots \]

Let \( \{R_n\} \) be a sequence of constants real or complex and let us write

\[ R_n = p_o q_{n-k} + \ldots \ldots p_n q_0 \]

If there exists a function \( U(x) \) such that

\[ \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} (U_{n-k}(x) - U) = o(1) \]

uniformly in a set \( E \) in which \( u = u(x) \) is bounded, then we shall say that the series (4.1) summable \((N,p_n,q_n)\) uniformly in \( E \) to the sum \( u \).

In this chapter \( \{R_n\} \) to be a real non-negative monotonic non-increasing sequence such that \( R_n \to \infty \) as \( n \to \infty \) that the regularity
conditions (1.12.a) and (1.12b) are automatical satisfied.

In 1965 (Saxena) [38] established the following theorem

THEOREM A: If

\[ \phi(t) = o\left(\frac{t}{\log(1/t)}\right) \]

Uniformly in a set E in which s=s(x) is bounded as t→+0, then the series (1.14.2) is summable by harmonic means uniformly in E to the sum s.

In the present chapter we generalize theorem A by replacing the special sequence \( p_n = \frac{1}{n+1} \) by a more general sequence of coefficients. However Saxena [38] proved the following.

THEOREM 1: If \( \alpha(t) \) denotes a function of \( t \alpha(t) \) and \( t/\alpha(t) \) ultimately increases steadily with t

\[ \log n = O(\alpha(p_n)) \quad \text{as} \quad n \to \infty \]

\[ \phi(t) = o\left(\frac{t}{\alpha(p_n)}\right) \]

uniformly in a set E in which s=s(x) is bounded as t→ +0 to then the series (1.14.2) is summable \((N,p_n,q_n)\) uniformly in E to the sum's.

Our aim is to generalize above theorems for \((N,p_n,q_n)\) summability. However our theorem is as follows.

THEOREM: If \( \varepsilon(t) \) and \( \mu(t) \) be two positive function of \( t \) such that
\( \epsilon(t), \mu(t) \) and \( \frac{t \epsilon(t)}{\mu(t)} \) increases monotonically with \( t \) and

\[(1.21.5)\] \( \epsilon(n) \log n = O[\mu(R_n)] \) as \( n \to \infty \).

If

\[(1.21.6)\] \( \phi(t) = o\left[\frac{t \epsilon(1/t)}{\mu(R_t)}\right] \), as \( t \to +0 \),

uniformly in \( E \) in which \( s = s(x) \) is bounded, the series \( (1.14.2) \) is summable \( (N, p_n, q_n) \) uniformly in \( E \) to the sum \( s \).

**1.22 \((N, P)\) Summability of the Derived Series of Fourier Series.**

The derived series of a Fourier series \( (1.14.2) \) is given by

\[(1.22.1)\] \( \sum_{n=1}^{\infty} n (b_n \cos nt - a_n \sin nt) = \sum_{n=1}^{\infty} nB_n(t) \).

Let us write

\[\psi(t) = f(x+t) - f(x-t)\].

\[g(t) = \frac{\psi(t)}{4 \sin t / 2} - c,\]

where \( c \) is some function of \( x \) at \( t=x \).

\[G(t) = \int_0^t |g(u)| \, du\]

\[G^*(t) = \int_t^\infty \frac{|g(u)|}{u} \, du\]
Mohanty and Nanda [29] obtained the following result on the summability by logarithmic means of the derived series (1.22.1) of Fourier series (1.14.2)

**THEOREM A** If \( g(t) \in L(0,\pi) \) and

\[
\int_0^\pi \frac{|g(u)|}{u} \, du = o\left( \log \frac{1}{t} \right), \quad t \to 0,
\]

then the series (1.22.1) is summable \((R, \log n, 1)\) to the sum \( C \).

The object of the Chapter IX is to extend the above result for the summability \((N, p_n)\) (1.12.a) by proving the following.

**THEOREM:** Let \( \{p_n\} \) be a real non-negative, monotonic non-increasing sequence of constants such that

\[
(1.22.3) \quad \sum_{n=0}^{n} p_n = \lim_{n \to \infty} P_n, \quad \text{as } n \to \infty,
\]

\[
(1.22.4) \quad \sum_{m=1}^{n-1} \left| \Delta_m p_m \right| + np_n = O(1),
\]

\( n \to \infty \), if \( g(t) \in L(0,\pi) \) and

\[
(1.22.5) \quad \int_{0}^{1} \frac{|g(u)|}{u} \, du = o\left( \frac{\beta(1/t)P_\tau}{\log(1/t)} \right)
\]
(1.22.6) \( \left( \frac{B(n)}{\log n} \right) \to c(1) \)
then the (1.22.1) is summable \((\mathbb{N}, p_n)\) to the sum \(c\).

**1.23. STRONG \((\mathbb{N}, P_n)\) SUMMABILITY OF A FOURIER SERIES.**

Strong \((\mathbb{N}, p_n)\) summability has been defined in section (1.12b) and Fourier series given in (1.14.2).

Luan [24] obtained a well-known result for the summability of Fourier series and related series and sequences by logarithmic mean. The theorem due to Luan [24] has been further generalized by Ganguly [6] by considering general summability method \((\mathbb{N}, p_n)\).

Here, in chapter X, we [51] have studied strong \((\mathbb{N}, p_n)\) summability (1.12.1) of a Fourier series (1.14.2) by establishing the following:

**THEOREM.** Let \(\{p_n\}\) be a non-negative, monotonic non-increasing sequence of constants such that its non-vanishing \(n\)-th partial sum \(P_n \to \infty\) as \(n \to \infty\) and \(\{e_n\}\) be a suitable sequence of constants \(\pm 1\) such that

\[
(1.23.1) \quad \sum_{m=1}^{n-1} |\Delta_m p_m e_m| + n p_n e_n = O(1).
\]

as \(n \to \infty\). If
where \(\beta(n)\to\infty\) as \(n\to\infty\)

\[
\sum_{m=0}^{n} p_m \left| \sigma_m(x) - f(x) \right| = o(P_n)
\]

as \(n\to\infty\), where \(\sigma_m(x)\) is the \(m\)-th partial sum of the series (1.14.2).

It is worth noticing that the result of our theorem follows under very general condition

**1.24 DEGREE OF APPROXIMATION TO A FOURIER BY TRIANGULAR MATRIX OF ITS DERIVED FOURIER SERIES**

Let \(f(t)\) be a \(2\pi\)-periodic and Lebesgue integrable function of \(t\) in the interval \((-\pi,\pi)\). Then the Fourier series of \(f(t)\) is given as in (1.14.2) and its derived series is given as in (1.14.7), which is obtained by differentiating the series (1.14.2) term by term.

We write

\[
\begin{align*}
\phi(t) &= f(x+t) + f(x-t) - 2f(x) \\
h(t) &= f(x+t) - f(x-t)
\end{align*}
\]
and

\[ g(t) = f(x+t) - f(x-t) - 2t f'(x). \]

Where \( f'(x) \) is the first derivative of \( f(t) \) at \( t=x \).

Siddiqui [42] has proved the following theorem on the degree of approximation to a function [1.13b] by Cesaro means (14) of its Fourier series.

**THEOREM A.** Let \( 0<k<1 \) and \( 0<\delta<\pi \) if \( x \) is a point such that

\[ t(x) < \delta \]

(1.24.1) \[ \int_0^t |d \phi(u)| \leq A\psi(t). \]

Where \( 0 \leq t \leq \delta \) then

\[ \sigma_n^k(x) - f(x) = O[ \psi(\frac{1}{n}) + O(n^{-k}) ], \]

where \( \sigma_n^k(x) \) is the Cesaro mean of order-\( k \) of the Fourier series.

(1.14.2) and \( \psi(t) \) is a positive increasing function such that

\[ \int_{1/\psi(n)}^{\infty} \frac{\psi(t)}{t^2} \, dt = O \left[ n \psi(1/n) \right], \, n \to \infty \]

Here in this chapter. We propose to establish a theorem on the degree of approximation to a function [1.13] by triangular matrix of the derived Fourier series [1.14.7]. It is well-known [56] that the derived series of a Fourier series is not, in general, a
Fourier series. It is only when a function is an integral that its derived series is a Fourier series. It is also important to note that the triangular matrix summability includes the Cesaro summability (1.4) as its special case. Our theorem reads as follows:

**THEOREM** Let $t_n^1$ denote the ($\wedge$) mean of the series (iBl) and $\psi(t)$ be a positive increasing function satisfying the condition (1.24.3).

Let $0<k<1$ and $0<\delta \leq \pi$. If $x$ is a point such that

\[(1.24.4) \int_0^t |dg(u)| = O[\psi(t)], \quad 0 \leq t \leq \delta\]

\[(1.24.5) t'_n - f'(x) = O[\psi(1/n)] \text{ as } n \to \infty\]

where $t'_n$ denote the ($\wedge$) means of the series (1.13.1b) and $\psi(t)$ is a positive increasing function such that

\[\int_{t/n}^\delta \frac{\psi(t)}{t} \, dt = O[\psi(1/n)], \text{ as } n \to \infty\]