CHAPTER V

ON MATRIX SUMMABILITY OF A FOURIER SERIES AND ITS CONJUGATE SERIES

5.1 Let \( \sum u_n \) be a given infinite series with sequence of partial sums \( \{s_n\} \).

Let \( \Pi_\infty = (a_{n,k}) \) be an infinite triangular matrix with real constants. Then sequence-to-sequence transformation.

\[
(5.1.1) \quad t_n = \sum_{k=0}^{n} a_{n,k} s_k, \quad n=0,1,2, \ldots
\]

defines the T-transform of the sequence \( \{s_n\} \). The series \( \sum u_n \) is said to be T-summable to \( s \) if \( \lim_{n \to \infty} t_n = s \).

The necessary and sufficient conditions for T-method to be regular (i.e., \( \lim_{n \to \infty} s_n = s \) and \( \lim_{n \to \infty} t_n = s \)) are that

1. There is a constant \( A, \sum_{k=0}^{\infty} |a_{n,k}| < A \) for every \( n \); 

2. For every \( k, \lim_{n \to \infty} a_{n,k} = 0 \);
If matrix elements $a_{n,k} = 0$, for every $k > n$, then the matrix is called a triangular matrix. The matrix $T$-reduces to Nörlund matrix generated by the sequence of coefficients $\{p_n\}$ if

$$a_{n,k} = \begin{cases} p_{n-k} & \text{if } k \leq n \\ p_n & \text{if } k > n \end{cases}$$

where $P_n = \sum_{i=0}^{n} p_i \neq 0$

5.2. Let $f(t)$ be a periodic function with period $2\pi$, and integrable in the Lebesgue sense over $(-\pi, \pi)$. Let

$$\frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \equiv \sum_{n=1}^{\infty} A_n(t)$$

be the Fourier series of $f(t)$. The conjugate series of (5.2.1) is

$$\sum_{n=1}^{\infty} (b_n \cos nt - a_n \sin nt) \equiv \sum_{n=1}^{\infty} B_n(t)$$

We shall use the following notations:

$$\phi(x,t) \equiv \phi(x,t) = f(x+t) + f(x-t) - 2f(x),$$

$$\psi(t) \equiv \psi(x,t) = f(x+t) - f(x-t)$$
\[ \phi(t) = \int_{0}^{t} \phi(u) \, du, \quad \Psi(t) = \int_{0}^{t} \psi(u) \, du. \]

\[ N_n(t) = \frac{1}{2\pi} \sum_{k=0}^{n} \hat{a}_{n,k} \frac{\cos(k + \frac{1}{2})}{\sin t / 2}, \]

\[ \overline{N}_n(t) = \frac{1}{2\pi} \sum_{k=0}^{n} a_{n,k} \frac{\sin(k + \frac{1}{2})t}{\sin t / 2} \]

\[ A_n, r = \sum_{k} a_{n,k} \text{ and } \tau = \left[ \frac{1}{t} \right] \text{ an integral part of } \frac{1}{t}. \]

5.3. Iyengar [12] (1943) proved that, if

\[ \phi(t) = o \left[ \log \left( \frac{1}{t} \right)^{-1} \right] \]

as \( t \to 0 \), then the series (5.2.1) is summable by harmonic means to \( f(x) \).

Siddiqui [41], (1948) has improved the above result using stringent condition for harmonic means which was further improved recently by Pandey [33] 1983 for Nörlund means. He proved

**THEOREM A.** If

\[ \phi(t) = o[\alpha(t)], \quad \text{as } t \to +0 \]

Where \( \alpha(t) \) is a non-negative, non decreasing function of \( t \), such that
\[ (5.3.3) \quad \alpha\left( \frac{1}{n} \right) = o(1), \quad \text{as } n \to \infty \]

and

\[ (5.3.4) \quad \int_{1/n}^{n} \alpha(t) p_n \frac{dt}{t} = O\left( P_n \right), \quad \text{as } n \to \infty \]

then the Fourier series of \( f(t) \) at \( t=x \) is summable \( (N,p_n) \) to \( f(x) \), where \( \{p_n\} \) is a real non-negative and non-decreasing sequence such that \( P_n \to \infty, \ n \to \infty \).

He also proved:

**THEOREM B.** If the sequence \( \{p_n\} \) and \( \alpha(t) \) be same as in theorem A. Then if

\[ \Psi(t) = \alpha(t) \], \quad \text{as } t \to +0 \]

then the conjugate series (5.2.2) is summable \( (N,p_n) \) to

\[ \frac{1}{2\pi} \int_{0}^{\pi} \Psi(t) \cot \frac{t}{2} \ dt, \]

at every point where this integral exists.

In this note, we extend the theorem A and B to matrix summability. We establish the following theorem:

**THEOREM 1.** Let \( \|T\| \to (a_{nk}) \) be an infinite triangular matrix with \( a_{nk} \geq 0 \).
\[ A_{n,k} = \sum a_{n,k}, \text{ also } A_{n,0} = 1 \text{ for each } n \geq 0. \] Let \( \{a_{n,k}\}_{k=0}^{n} \) be a real non-negative and non-increasing sequence with respect to \( k \).

If

\[ \varphi(t) = o\left( \frac{t}{u(t/1)} \right), \text{ as } t \to +0. \]

Where \( \varphi(t) \) is non-negative and non-decreasing function of \( t \), such that

\[ \alpha(n) \to \infty, \text{ as } n \to \infty \]

and

\[ \int_{\ln n}^{n} \frac{n^{\alpha(n-1)}}{t} u(t) \, dt = o(1), \text{ as } n \to \infty \]

then the Fourier series at \( t = x \) is \( T \)-summable to \( f(x) \).

**THEOREM 2.** If the sequence \( \{a_{n,k}\}_{k=0}^{n} \) and \( \varphi(t) \) be same as in Theorem 1. then if

\[ \varphi(t) = o\left( \frac{t}{u(t/1)} \right), \text{ as } t \to +0. \]

Where \( \varphi(t) \) is a non-negative, non-decreasing function of \( t \) such that \( \alpha(n) \to \infty \) as \( n \to \infty \), then the conjugate series (5.2.2) is summable to
\[
\frac{1}{2\pi} \int_0^\infty \psi(t) \cot \frac{t}{2} \, dt,
\]
at every point where this integral exists.

5.4 We shall need the following lemmas to prove our theorem.

L E M M A 1. (Kishore and Hotta [15] 1971). If \((a_{nk})_{k=0}^n\) is non-negative and non-decreasing sequence with respect to \(k\), then for \(0 < a < d < \infty\), \(0 < t < \pi\), and for every \(n\) and \(\alpha\),

\[
\sum_{k=a}^b a_{n,k} e^{(n-k)t} \leq K A_{n,n-t}
\]

Where \(K\) is an absolute constant.

L E M M A 2. If \(0 < t < \frac{1}{n}\), then \(N_n(t) = o(n)\).


We have

\[
\left| N_n(t) \right| = \frac{1}{2\pi} \left| \sum_{r=0}^n a_{n,r} \frac{\sin((r+1/2)t)}{\sin(t/2)} \right|
\]

\[
= o\left( \sum_{r=0}^n (2r+1)a_{n,r} \right)
\]

\[
= o(2n+1) \sum_{r=0}^n a_{n,n-r}
\]

\[
= o(2n+1) = o(n) \text{, as } n \to \infty
\]
LEMMA 3. For $\frac{1}{n} \leq t < \pi$

$$|N_n(t)| = o \left[ \frac{(A_{n,1})}{t} \right].$$

PROOF OF THE LEMMAS: We have

$$|N_n(t)| = \frac{1}{2\pi} \left| \sum_{n=0}^{\infty} a_{n,n-1} \frac{\sin(n-r+1/2)t}{\sin(t/2)} \right|$$

$$= \frac{1}{2\pi|\sin t/2|} \left| \sum_{n=0}^{\infty} a_{n,n-1} \exp\{i(n-r+1/2)t\} \right|$$

$$= \frac{1}{\pi t} \left| \sum_{n=0}^{\infty} a_{n,n-1} \exp\{i(n-r)t\} \right|$$

$$\leq \frac{1}{\pi t} \left| \sum_{n=0}^{\infty} a_{n,n-1} \exp\{i(n-r)t\} \right|$$

$$= o \left( \frac{(A_{n,n-1})}{t} \right), \text{ by Lemma 1}$$

LEMMA 4. If $\frac{1}{n} \leq t \leq \delta < \pi$, then

$$|\bar{N}_n(t)| = \frac{1}{2\pi} \left| \sum_{n=0}^{\infty} a_{n,n-1} \frac{\cos(n-r+1/2)t}{\sin t/2} \right|$$

$$= o \left[ \frac{(A_{n,n-1})}{t} \right].$$
The proof is similar to that of Lemma 3.

5.5. PROOF OF THE THEOREM 1: Let

\[ s_n(x) = \sum_{i=1}^{n} A_i(x) \]

then, we have

\[ s_n(x) - f(x) = \frac{1}{2\pi} \int_{0}^{\pi} \phi(t) \frac{\sin(n + 1/2)t}{\sin(t/2)} \, dt \]

using (5.1), we get

\[ t_n - f(x) = \sum_{r=0}^{n} a_{n,r} [s_i - f(x)] \]

\[ = \sum_{r=0}^{n} a_{n,r} \frac{1}{2\pi} \int_{0}^{\pi} \phi(t) \frac{\sin(r + 1/2)t}{\sin(t/2)} \, dt \]

\[ = \int_{0}^{\pi} \phi(t) \left[ \frac{1}{2\pi} \sum_{r=0}^{n} a_{n,r} \frac{\sin(r + 1/2)t}{\sin(t/2)} \right] \, dt \]

\[ = \int_{0}^{\pi} \phi(t) N_n(t) \, dt \] (J) say

In order to prove the theorem, we have to show that under our assumptions

\[ \int_{0}^{\pi} \phi(t) N_n(t) \, dt = o(1), \text{ as } n \to \infty. \]
We write, for $0 < \delta < \pi$

$$J = \int_0^\pi \phi(t) N_n(t) \, dt = \left( \int_0^{\alpha_n} + \int_{\alpha_n}^{\delta} + \int_{\delta}^\pi \right) \phi(t) N_n(t) \, dt$$

5.5.1

$$= J_1 + J_2 + J_3$$

Now, by Lemma 2.

$$J_1 = O \left[ \int_0^{\alpha_n} \phi(t) N_n(t) \, dt \right]$$

$$= O \left[ n \int_0^{\alpha_n} \left| \phi(t) \right| \, dt \right]$$

(5.5.2) $= o(1)$, as $n \to \infty$.

Again by Lemma 5,

$$J_2 = \int_{\alpha_n}^\delta \phi(t) N_n(t) \, dt$$

$$= O \left[ \int_{\alpha_n}^\delta \left| \phi(t) \right| A_n(t) \, dt \right]$$

$$= O \left[ \left\{ \left( \frac{1}{t} \right) \Phi(t) A_{n,\alpha_n} \right\} \frac{\alpha_n}{\alpha_n} \right] + O \left[ \int_{\alpha_n}^\delta \left\{ \phi(t) A_{n,\alpha_n} \right\} \left( \frac{1}{n^2} \right) \, dt \right]$$

$$+ O \left[ \int_{\alpha_n}^\delta \phi(t) \left( \frac{1}{t} \right) d\left( A_{n,\alpha_n} \right) \right]$$
\[
\alpha n^{1/2} \left( \frac{1}{\alpha(1/t)} \right) dt = o(1) \]

In view of (5.3.5), (5.3.6) and (5.3.7).

Lastly by virtue of R-L theorem and the regularity of the method of summation, we have

\[
J_n = \int_{0}^{1} \phi(t) N_n(t) dt
\]

= o \int_{0}^{1} \phi(t) N_n(t) dt
Hence, on collecting (5.5.2), (5.5.3) and (5.5.4) we have

\[ J = o(1) \]

Which completes the proof of Theorem 1

5.6. PROOF OF THE THEOREM 2:

Let \( S_n(x) \) denote the \( n \)-th partial sum of the series \( \sum B_n(x) \). Then we have

\[
S_n(x) = \frac{1}{2\pi} \int_0^x \psi(t) \frac{\cos t/2 - \cos(n + 1/2)t}{\sin t/2} dt
\]

For \( \sum B_n(x) \), making use of the formula (5.1) we get

\[
in - \left( \frac{1}{2\pi} \right) \int_0^x \psi(t) \cot \frac{t}{2} dt
\]

\[
= \sum_{r=0}^n a_{n,r} \bar{S}_r(x) - \frac{1}{2\pi} \int_0^x \psi(t) \cot \frac{t}{2} dt
\]

\[
= \int_0^x \psi(t) \left\{ \left( \frac{1}{2\pi} \right) \sum_{r=0}^n a_{n,r} \frac{\cos(r + 1/2)t}{\sin t/2} \right\} dt.
\]

\[
= - \frac{1}{2\pi} \int_0^x \psi(t) \hat{N}_n(t) dt \quad (= H \text{ say})
\]
To prove the Theorem 2, we have to show that, under our assumptions.

\[ \int_0^n \psi(t) N_n(t) \, dt = o(1), \text{ as } n \to \infty. \]

For \(0 < \delta < \pi\), we have.

\[ H = \frac{1}{2\pi} \int_0^n \psi(t) N_n(t) \, dt = \frac{1}{2\pi} \left( \int_0^{\delta n} + \int_{\delta n}^n + \int_n^\infty \right) \psi(t) N_n(t) \, dt \]

\[ = H_1 + H_2 + H_3, \text{ say} \]

since the conjugate function exists, there for

\[ \left( \frac{1}{2\pi} \right) \int_0^{\delta n} \psi(t) \cot \left( \frac{t}{2} \right) \, dt = o(1) \]

also

\[ \frac{1}{2\pi} \sum_{r=0}^n a_{n,n-r} \frac{\cos \frac{1}{2} \left( t - \cos \left( n - r + \frac{1}{2} \right) t \right)}{\sin \left( t / 2 \right)} \]

\[ = \frac{1}{2\pi} \sum_{r=0}^n a_{n,n-r} \sum_{k=0}^{n-r} 2 \sin kt \]

\[ = o \left[ \sum_{r=0}^n a_{n,n-r} \sum_{k=0}^{n-r} \left| \sin kt \right| \right] = o \left[ \sum_{r=0}^n a_{n,n-r} \left( n-r \right) \right] \]

\[ = o(n) \text{ for } 0 \leq t \leq \pi. \]

Therefore
\[ H_1 = \int_0^\infty \psi(t) \tilde{N}_n(t) \, dt \]

\[ = \int_0^\infty \psi(t) \sum_{r=0}^n a_{n,n-r} \frac{\cos(n-r+1/2)t}{\sin(t/2)} \, dt \]

\[ = -\int_0^\infty \psi(t) \sum_{r=0}^n a_{n,n-r} \frac{\cos t/2 - \cos(n-r+1/2)t}{\sin(t/2)} \]

\[ + \left( \frac{1}{2\pi} \right) \sum_{r=0}^n a_{n,n-r} \int_0^\infty \psi(t) \cot \frac{t}{2} \, dt \]

\[ = o \left\{ n \int_0^\infty |\psi(t)| \, dt \right\} + o(1) \]

\[ = o \left[ n \psi \left( \frac{1}{n} \right) \right] + o(1) \]

(5.6.1) \[ = o(1) \text{ as } n \to \infty \]

Now for \( \frac{1}{n} \leq t \leq \delta \)

(5.6.2) \[ H_2 = o \left[ \int_{1/n}^\delta |\psi(t)| \left| \tilde{N}_n(t) \right| \, dt \right] \]

\[ = o \left[ \int_{1/n}^\delta |\psi(t)| \frac{A_{n,0-1}}{t} \, dt \right] \]

\[ = o(1) \text{ as in } J_2 \]
also

(5.6.3) \[ H_3 = o(1) \]

In view of R-L theorem and the regularity of the method of summation, hence on collecting (5.6.1), (5.6.2) and (5.6.3), we get

\[ H = o(1). \]

Which completes the proof of the Theorem 2.

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