CHAPTER-IV

COMPLEMENTARY PERFECT TRIPLE CONNECTED DOMINATION NUMBER OF A GRAPH

In this chapter, we introduce a new domination parameter called complementary perfect triple connected domination number of a graph and investigate many new results of this parameter.

The concept of triple connected graphs was introduced by J. Paulraj Joseph etc. in [33]. The concept of connectedness plays an important role in any network. A variety of connectedness has been studied in the literature by considering the existence of a path between any two vertices.

In transportation networks, this enables a traveler to have a route from one city to any other city. If a traveler can finish some work enroute in any one of the third cities, then it will minimize money, distance, time, etc.

A communication network in which a transmitting node can send a message to two stations at one stretch will be more effective and economic. Such an optimization leads to the concept of triple connected graphs.

**Definition** A graph is said to be **triple connected** if any three vertices lie on a path in G.
Example All paths and cycles, complete graphs and wheels are some standard examples of triple connected graphs. Wheel graph, $K_{r,s}$ where $r, s \geq 2$ are also triple connected graphs.

The concept of triple connected domination number was introduced by G.Mahadevan etc. in [10].

Definition A set $S \subseteq V$ is a **triple connected dominating set** if $S$ is a dominating set of $G$ and the induced sub graph $<S>$ is triple connected. The **triple connected domination number** $\text{tc}(G)$ is the minimum cardinality taken over all triple connected dominating sets in $G$.

Example For the following graph $G$, $S = \{v_1, v_2, v_5\}$ forms a $\text{tc}$ - set of $G$. Hence $\text{tc}(G) = 3$.

The concept of complementary triple connected domination number was introduced by G.Mahadevan in [11].

Definition A set $S \subseteq V$ is a **complementary triple connected dominating set** if $S$ is a dominating set of $G$ and the induced sub graph $<V - S>$ is triple connected. The **complementary triple connected domination number** $\text{ctc}(G)$
is the minimum cardinality taken over all complementary triple connected dominating sets in $G$.

**Example** For the following graph $G$, $S = \{v_1, v_2\}$ is a ctc - set of $G_1$. Since $S$ is a dominating set and $<V - S> = P_4$ which is triple connected. Hence $\text{ctc}(G) = 2$.

Motivated by the above concept in this chapter we introduce a new concept called **Complementary perfect triple connected domination number**.

**Previous Results**

**Theorem A** [33] A tree $T$ is triple connected if and only if $T \cong P_n; n \geq 3$.

**Theorem B** [33] A connected graph $G$ is not triple connected if and only if there exists a $H$ - cut with $(G - H) \geq 3$ such that $V(H) N(C_i) = 1$ for at least three components $C_1, C_2, \text{ and } C_3$ of $G - H$.

**Complementary Perfect Triple Connected Domination Number**

**Definition 4.1** A subset $S$ of $V$ of a nontrivial graph $G$ is said to be **complementary perfect triple connected dominating set**, if $S$ is a triple connected dominating set and the subgraph induced by $<V - S>$ has a perfect matching.
The minimum cardinality taken over all complementary perfect triple connected dominating sets is called the **Complementary perfect triple connected domination number** and is denoted by $\text{cptc}$. Any triple connected dominating set with $\text{cptc}$ vertices is called a $\text{cptc}$-set of $G$.

**Example 4.2** For the graph $G_1$ in figure 4.1, $S = \{v_1, v_2, v_5\}$ forms a $\text{cptc}$-set of $G_1$. Hence $\text{cptc}(G_1) = 3$.

**Observation 4.3** Complementary perfect triple connected dominating set does not exists for all graphs and if exists, then $\text{cptc} \geq 3$.

**Example 4.4** For the graph $G_2$ in figure 4.2, any minimum dominating set must contain the supports and any connected dominating set containing these supports is not complementary perfect triple connected and hence $\text{cptc}$ does not exists.

**Remark 4.5** Throughout this paper we consider only connected graphs for which complementary perfect triple connected dominating set exists.
**Observation 4.6** The complement of the complementary perfect triple connected dominating set need not be a complementary perfect triple connected dominating set.

**Example 4.7** For the graph $G_3$ in figure 4.3, $S = \{v_1, v_2, v_3\}$ forms a complementary perfect triple connected dominating set of $G_3$. But the complement $V - S = \{v_4, v_5, v_6, v_7\}$ is not a complementary perfect triple connected dominating set.

![Figure 4.3](image)

**Observation 4.8** Every complementary perfect triple connected dominating set is a dominating set but not the converse.

**Example 4.9** For the graph $G_4$ in figure 4.4, $S = \{v_1\}$ is a dominating set, but not a complementary perfect triple connected dominating set.

![Figure 4.4](image)

**Observation 4.10** For any connected graph $G$, $(G) \leq cp(G) \leq cptc(G)$.
Example 4.11 For the graph $G_5$ in figure 4.5, $(G_5) = \{v_1\} = 1,$ 
\[ (G_5) = \{v_1, v_2\} = 2 \quad \text{and} \quad c_{ptc}(G_5) = \{v_1, v_2, v_3, v_4\} = 4. \] Hence $(G_5) \leq c_p(G_5) \leq c_{ptc}(G_5)$.

![Figure 4.5](image)

Theorem 4.12 If the induced subgraph of each connected dominating set of $G$ has more than two pendant vertices, and then $G$ does not contain a complementary perfect triple connected dominating set.

Proof This theorem follows from theorem A [33].

Example 4.13 For the graph $G_6$ in figure 4.6, $S = \{v_6, v_2, v_3, v_4\}$ is a minimum connected dominating set so that $c(G_6) = 4$. Here we notice that the induced subgraph of $S$ has three pendant vertices and hence $G$ does not have a complementary perfect triple connected dominating set.

![Figure 4.6](image)
Complementary perfect triple connected domination number for some standard graphs are given below.

1) For any cycle of order \( n \geq 5 \), \( \text{cptc} \ (C_n) = n - 2 \).

2) For any complete bipartite graph of order \( n \geq 5 \),
   \[ \text{cptc} \ (K_{p,q}) = 4, \]
   (where \( p, q \geq 2 \) and \( p + q = n \)).

3) For any complete graph of order \( n \geq 5 \),
   \[ \text{cptc} \ (K_n) = 4. \]

4) For any wheel of order \( n \geq 5 \), \( \text{cptc} \ (W_n) = 4 \).

Complementary perfect triple connected domination number for some special types of graphs are given below.

1) A Fan graph \( F_{p,q} \) is defined as the graph join \( K_p + P_q \), where \( K_p \) is the empty graph on \( p \) nodes and \( P_q \) is the path graph on \( q \) nodes. The case \( p = 1 \) corresponds to the usual fan graphs.

   For any Fan graph of order \( n \geq 5 \), \( \text{cptc} \ (F_{1,n-1}) = 4 \).
In figure 4.7 (a) $S = \{v_1, v_2, v_3\}$ is a complementary perfect triple connected dominating set. In figure 4.7 (b) $S = \{v_1, v_2, v_3, v_4\}$ is a complementary perfect triple connected dominating set.

2) The **Book** $B_n$ is the graph $S_n \circ P_m$ where $S_n$ is the star with $n+1$ vertices, as shown in figure 4.8.

$$\text{cptc}(B_3) = 4$$

**Figure 4.8**
For any Book graphs of order $n \geq 6$, $\text{cptc}(B_n) = 4$. Here $S = \{v_1, v_2, v_3, v_4\}$ is a complementary perfect triple connected dominating set.

3) The graph $C_m^{(t)}$ denote the one point union of $t$ cycles of length $m$. If $m = 3$, it is called Dutch $t$-windmill as shown in figure 4.9.

For any $C_m^{(t)}$ of order $n \geq 5$, $\text{cptc}(G) = 3$. In figure 4.9 (a) $S = \{v_1, v_2, v_3\}$ is a complementary perfect triple connected dominating set. In figure 4.9 (b) $S = \{v_1, v_2, v_3\}$ is a complementary perfect triple connected dominating set.
4) The **Bidiakis cube** is a 3-regular graph with 12 vertices and 18 edges as shown in figure 4.10.

![Figure 4.10](image)

For the Bidiakis cube graph $G$, $cptc(G) = 6$. Here $S = \{v_1, v_2, v_3, v_7, v_9, v_{11}\}$ is a complementary perfect triple connected dominating set.

5) The **Franklin graph** a 3-regular graph with 12 vertices and 18 edges as shown below in figure 4.11.

![Figure 4.11](image)
For the Franklin graph $G$, $\text{cptc}(G) = 6$. Here $S = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ is a complementary perfect triple connected dominating set.

6) The **Frucht graph** is a 3-regular graph with 12 vertices, 18 edges, and no nontrivial symmetries as shown below in figure 4.12. It was first described by Robert Frucht in 1939.

![Figure 4.12](image)

For the Frucht graph $G$, $\text{cptc}(G) = 6$. Here $S = \{v_3, v_4, v_8, v_9, v_{10}, v_{11}\}$ is a complementary perfect triple connected dominating set.

7) The **Dürer graph** is an undirected cubic graph with 12 vertices and 18 edges as shown below in figure 4.13. It is named after Albrecht Durer.

![Figure 4.13](image)
For the Dürer graph $G$, $\text{cptc}(G) = 6$. Here $S = \{v_1, v_4, v_5, v_6, v_7, v_{10}\}$ is a complementary perfect triple connected dominating set.

8) The **Wagner graph** is a 3-regular graph with 8 vertices and 12 edges, as shown in figure 4.14, named after Klaus Wagner. It is the 8-vertex Mobius ladder graph. Mobius ladder is a cubic circulant graph with an even number ‘$n$’ vertices, formed from an $n$-cycle by adding edges connecting opposite pairs of vertices in the cycle.

![Figure 4.14](image)

For the Wagner graph $G$, $\text{cptc}(G) = 4$. Here $S = \{v_1, v_2, v_7, v_8\}$ is a complementary perfect triple connected dominating set.

9) The **Triangular Snake** graph is obtained from a path $v_1,v_2,\ldots,v_n$ by joining $v_i$ and $v_{i+1}$ to a new vertex $w_i$ for $i = 1,2,\ldots,(n - 1)$ and denoted by $mC_3$ snake as shown in figure 4.15.
In figure 4.15 (a) $S = \{v_3, v_4, v_5\}$ is a complementary perfect triple connected dominating set. In figure 4.15 (b) $S = \{v_3, v_4, v_5, v_6, v_7\}$ is a complementary perfect triple connected dominating set. In general, $\text{cptc}(mC_3) = n - 4$.

10) The Herschel graph is a bipartite undirected graph with 11 vertices and 18 edges as shown in figure 4.16, the smallest non hamiltonian polyhedral graph. It is named after British astronomer Alexander Stewart Herschel.
For the Herschel graph $G$, $\text{cptc}(G) = 5$. Here $S = \{v_1, v_6, v_8, v_{10}, v_{11}\}$ is a complementary perfect triple connected dominating set.

11) Any cycle with a pendant edge attached at each vertex as shown in figure 4.17 is called **Crown graph** and is denoted by $C_n^+$.  

![Figure 4.17](image)

For the crown graph, $\text{cptc}$ does not exists.

12) Any path with a pendant edge attached at each vertex as shown in figure 4.18 is called **Hoffman tree** and is denoted by $P_n^+$. 

![Figure 4.18](image)

For the Hoffman tree, $\text{cptc}$ does not exists.

13) Any wheel with a pendant edge attached at each vertex as shown in figure 4.19 is called **Helm graph** and is denoted by $W_n^+$. 

![Figure 4.19](image)
For the Helm graph, \( \text{cptc} \) does not exists.

14) The **Bull graph** is a planar undirected graph with 5 vertices and 5 edges, in the form of a triangle with two disjoint pendant edges as shown in the figure 4.20.

For the bull graph, \( \text{cptc} \) does not exists.

15) The **Chvátal graph** is an undirected graph with 12 vertices and 24 edges as shown in figure 4.21, discovered by Václav Chvátal (1970).
For the Chvátal graph, \( \text{cptc}(G) = 4 \). Here \( S = \{v_1, v_2, v_3, v_4\} \) is a complementary perfect triple connected dominating set.

16) The **Moser spindle** (also called the **Mosers' spindle** or **Moser graph**) is an undirected graph, named after mathematicians Leo Moser and his brother William, with seven vertices and eleven edges as shown in figure 4.22.

For the Moser spindle graph \( G \), \( \text{cptc}(G) = 3 \). Here \( S = \{v_1, v_2, v_3\} \) is a complementary perfect triple connected dominating set.
17) The **Fullerene graph** is the 3-regular planar graph with all faces of size 5 or 6 (including the external face). The smallest fullerene is the dodecahedral $C_{20}$ as shown in figure 4.23.

![Figure 4.23](image)

For the Fullerene graph, $\text{cptc}(G) = 12$. Here $S = \{v_1, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{17}\}$ is a complementary perfect triple connected dominating set.

18) The **diamond graph** is a planar undirected graph with 4 vertices and 5 edges as shown in figure 4.24. It consists of a complete graph $K_4$ minus one edge.

![Figure 4.24](image)

For the Diamond graph, $\text{cptc}$ does not exist.
19) The **Goldner–Harary graph** is a simple undirected graph with 11 vertices and 27 edges as shown in figure 4.25. It is named after A. Goldner and Frank Harary, who proved in 1975 that it was the smallest non-hamiltonian maximal planar graph.

![Figure 4.25](image)

**Figure 4.25**

For any Goldner–Harary graph, $c_{ptc}(G) = 5$. Here $S = \{v_1, v_4, v_5, v_6, v_{11}\}$ is a complementary perfect triple connected dominating set.

20) The **Grötzsch graph** is a triangle-free graph with 11 vertices, 20 edges, chromatic number 4, and crossing number 5 as shown in figure 4.26. It is named after German mathematician Herbert Grötzsch.
For the Grötzsch graph, $c_{ptc}(G) = 5$. Here $S = \{v_1, v_2, v_9, v_{10}, v_{11}\}$ is a complementary perfect triple connected dominating set.

21) The **Hoffman graph** is a 4 - regular graph with 16 vertices and 32 edges as shown in figure 4.27 discovered by Alan Hoffman [37] at the Mathematics Genealogy Project.
For the Hoffman graph, $\text{cptc} (G) = 6$. Here $S = \{v_1, v_2, v_6, v_8, v_{15}, v_{16}\}$ is a complementary perfect triple connected dominating set.

22) Markström's 24 - vertex cubic planar graph with no 4 or 8 - cycles as shown in figure 4.28 found in a computer search for counterexamples to the Erdős–Gyárfás conjecture.

![Figure 4.28](image)

For the Markstrom graph, $\text{cptc}(G) = 16$. Here $S = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_{10}, v_{13}, v_{14}, v_{17}, v_{18}, v_{19}, v_{20}, v_{21}, v_{23}\}$ is a complementary perfect triple connected dominating set.

23) The **Heawood graph** is an undirected graph 14 vertices and 21 edges as shown in the figure 4.29, named after Percy John Heawood.
For the Heawood graph, $c_{ptc}(G) = 8$. Here $S = \{v_1, v_2, v_3, v_6, v_7, v_9, v_{10}, v_{11}\}$ is a complementary perfect triple connected dominating set.

24) The **Möbius–Kantor graph** is a symmetric bipartite cubic graph with 16 vertices and 24 edges as shown in figure 4.30, named after August Ferdinand Möbius and Seligmann Kantor.
For the Mobius–Kantor graph, \( \text{cptc}(G) = 8 \). Here \( S = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\} \) is a complementary perfect triple connected dominating set.

25) The **Truncated Tetrahedron** is an Archimedean solid. It has 4 regular hexagonal faces, 4 regular triangular faces, 12 vertices and 18 edges as shown in figure 4.31. Archimedean solid means one of 13 possible solids whose faces are all regular polygons whose polyhedral angles are all equal.

![Figure 4.31](image)

For the Truncated Tetrahedron, \( \text{cptc}(G) = 6 \). Here \( S = \{v_2, v_5, v_8, v_9, v_{10}, v_{11}\} \) is a complementary perfect triple connected dominating set.

26) The **Pappus graph** is a bipartite 3-regular undirected graph with 18 vertices and 27 edges as shown in figure 4.32 formed as the Levi graph of the Pappus configuration. It is named after Pappus of Alexandria, an ancient Greek mathematician.
For any Pappus graph, \( \text{cptc}(G) = 10 \). Here \( S = \{v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_16, v_{17}\} \) is a complementary perfect triple connected dominating set.

27) The Desargues graph is a distance-transitive cubic graph with 20 vertices and 30 edges as shown in figure 4.33. It is named after Gerard Desargues.
For any Desargues graph, $\text{cptc}(G) = 10$. Here $S = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}\}$ is a complementary perfect triple connected dominating set.

28) The **Tietze graph** is an undirected cubic graph with 12 vertices and 18 edges as shown in figure 4.34, formed by applying Y-transform to the Petersen graph and thereby replacing one of its vertices by a triangle. Y-transform means replacing Y subgraphs of a graph with the equivalent subgraph.

![Figure 4.34](image)

For the Tietze graph, $\text{cptc}(G) = 6$. Here $S = \{v_1, v_5, v_4, v_8, v_9, v_{10}\}$ is a complementary perfect triple connected dominating set.

29) The **Flower Snarks** are connected bridgeless cubic graphs with chromatic index 4. The name flower snarks sometimes used for $J_5$, a flower snark with 20
vertices and 30 edges. It is one of 6 snarks on 20 vertices as shown in figure 4.35.

![Diagram](image)

Figure 4.35

For the flower snark, \( \text{cptc}(G) = 12 \). Here \( S = \{v_3, v_4, v_5, v_7, v_8, v_9, v_{10}, v_{13}, v_{14}, v_{15}, v_{19}, v_{20}\} \) is a complementary perfect triple connected dominating set.

**Observation 4.14** If a spanning sub graph \( H \) of a graph \( G \) has a complementary perfect triple connected dominating set, and then \( G \) also has a complementary perfect triple connected dominating set.

**Example 4.15** For any graph \( G \) and \( H \) in figure 4.36, \( S = \{v_3, v_4, v_5\} \) is a complementary perfect triple connected dominating set and so \( \text{cptc}(G) = 3 \). For the spanning subgraph \( H \), \( S = \{v_3, v_4, v_5\} \) is a complementary perfect triple connected dominating set and so \( \text{cptc}(H) = 3 \).
Observation 4.16 Let $G$ be a connected graph and $H$ be a spanning subgraph of $G$. If $H$ has a complementary perfect triple connected dominating set, then $\text{cptc} (G) \leq \text{cptc} (H)$ and the bound is sharp.

Example 4.17 For the graph $G$ in figure 4.37, $S = \{v_1, v_2, v_7, v_8\}$ is a complementary perfect triple connected dominating set and so $\text{cptc} (G) = 4$. For the spanning subgraph $H$ in figure 4.37 of $G$, $S = \{v_1, v_2, v_3, v_6, v_7, v_8\}$ is a complementary perfect triple connected dominating set and so $\text{cptc} (H) = 6$.

Theorem 4.18 For any connected graph $G$ with $n \geq 5$, we have $3 \leq \text{cptc}(G) \leq n - 2$ and the bounds are sharp.

Proof The lower and upper bounds trivially follows from definition 4.1. For $C_5$, the lower bound is attained and for $C_9$ the upper bound is attained.
Theorem 4.19 For a connected graph G with 5 vertices, $\text{cptc}(G) = n - 2$ if and only if G is isomorphic to $C_5$, $W_5$, $K_5$, $K_{2,3}$, $C_3^{(2)}$, $K_5 - \{e\}$, $K_4(P_2)$, $C_4(P_2)$, $C_3(P_3)$, $C_3(2P_2)$ or any one of the graphs shown in Figure 4.38.

![Figure 4.38](image)

Proof Suppose G is isomorphic to $C_5$, $W_5$, $K_5$, $K_{2,3}$, $F_2$, $K_5 - \{e\}$, $K_4(P_2)$, $C_4(P_2)$, $C_3(P_3)$, $C_3(2P_2)$ or any one of the graphs $H_1$ to $H_7$ given in figure 4.38, then clearly $\text{cptc}(G) = n - 2$. Conversely, let G be a connected graph with 5 vertices and $\text{cptc}(G) = 3$. Let $S = \{x, y, z\}$ be a $\text{cptc}$ - set, then clearly $<S> = P_3$ or $C_3$.

Let $V - S = V(G) - V(S) = \{u, v\}$, then $<V - S> = K_2 = uv$.

Case (i) $<S> = P_3 = xyz$.

Since G is connected, there exists a vertex say x (or y, z) in $P_3$ is adjacent to u (or v) in $K_2$, then $\text{cptc}$ - set of G does not exists. But on increasing the degrees of the vertices of S, let x be adjacent to u and z be adjacent to v.

If $d(x) = d(y) = d(z) = 2$, then G $C_5$. Now by increasing the degrees of the vertices, by the above argument, we have G $K_5$, $K_5 - \{e\}$, $K_4(P_2)$, $C_4(P_2)$,
C₃(P₃) or any one of the graphs H₁ to H₇ given in figure 4.38. Since G is connected, there exists a vertex say y in P₃ is adjacent to u (or v) in K₂, then cptc - set of G does not exists. But on increasing the degrees of the vertices of S, let y be adjacent to v, x be adjacent to u and v and z be adjacent to u and v. If d(x) = 3, d(y) = 3, d(z) = 3, then G W₅. Now by increasing the degrees of the vertices, by the above argument, we have G ∈ K₂,₃, C₃(2P₂). In all the other cases, no new graph exists.

**Case (ii) <S> = C₃ = xyzx.**

Since G is connected, there exists a vertex say x (or y, z) in C₃ is adjacent to u (or v) in K₂, then S = {x, u, v} forms a cptc - set of G, so that

cptc (G) = p – 2. If d(x) = 3, d(y) = d(z) = 2, then G ∈ C₃(P₃). If d(x) = 4, d(y) = d(z) = 2, then G ∈ C₃(2). In all the other cases, no new graph exists.

**Nordhaus – Gaddum Type result:**

**Theorem 4.20** Let G be a graph such that G and G have no isolates of order p ≥ 5, then (i) \(\text{cptc} (G) + \text{cptc} (\overline{G}) \leq 2(n – 2)\)

(ii) \(\text{cptc} (G) \cdot \text{cptc} (\overline{G}) \leq (n – 2)^2\) and the bounds are sharp.

**Proof** The bounds directly follow from theorem 4.18. For the cycle C₅, cptc (G) + cptc (\overline{G}) = 2(n – 2) and cptc (G) \cdot cptc (\overline{G}) ≤ (n – 2)^2.

**Theorem 4.21** \(\gamma_{\text{cptc}} (G) \geq n / (\Delta + 1)\)

**Proof** Every vertex in V - S contributes one to degree sum of vertices of S. So \(|V - S| \sum_{u \in S} d(u)\) where S is a complementary perfect triple connected
dominating set. So \(|V - S| \geq \gamma_{cptc} \Delta\) which implies \((|V| - |S|) \geq \gamma_{cptc} \Delta\). Therefore \(n - \gamma_{cptc} \geq \gamma_{cptc} \Delta\), which implies \(\gamma_{cptc} (\Delta + 1) \geq n\). Hence \(\gamma_{cptc} \geq \frac{n}{(\Delta + 1)}\).

**Theorem 4.22** Any complementary perfect triple connected dominating set of \(G\) must contains all the pendant vertices of \(G\).

**Proof** Let \(S\) be any complementary perfect triple connected dominating set of \(G\). Let \(v\) be a pendant vertex with support say \(u\). If \(v\) does not belong to \(S\), then \(u\) must be in \(S\), which is a contradiction \(S\) is a complementary perfect triple connected dominating set of \(G\). Since \(v\) is a pendant vertex, so \(v\) belongs to \(S\).

**Observation 4.23** There exists a graph for which \(\gamma_{cp}(G) = \gamma_{tc}(G) = \gamma_{cptc}(G)\) is given below.

For the graph \(H\) in figure 4.39, \(S = \{v_1, v_3, v_7\}\) is a triple connected dominating set complementary perfect and complementary perfect triple connected dominating set. Hence \(\gamma_{cp}(G) = \gamma_{tc}(G) = \gamma_{cptc}(G) = 3\).
Complementary Perfect Triple Connected Domination Number and Other Graph Theoretical Parameters

Let us now discuss the relationship of complementary perfect triple connected domination number with connectivity, maximum degree and chromatic number.

**Theorem 4.24** For any connected graph $G$ with $n \geq 5$ vertices,

$$
cptc(G) + \kappa(G) \leq 2n - 3$$

and the bound is sharp if and only if $G \cong K_5, K_6$.

**Proof** Let $G$ be a connected graph with $n \geq 5$ vertices. We know that $
 \kappa(G) \leq n - 1$ and by theorem 4.18, $c_{ptc}(G) \leq n - 2$. Hence $c_{ptc}(G) + \kappa(G) \leq 2n - 3$. Suppose $G$ is isomorphic to $K_5, K_6$. Then clearly $c_{ptc}(G) + \kappa(G) = 2n - 3$. Conversely, let $c_{ptc}(G) + \kappa(G) = 2n - 3$. This is possible only if $c_{ptc}(G) = n - 2$ and $\kappa(G) = n - 1$. But $\kappa(G) = n - 1$, and so $G \cong K_n$.

**Case (i)** If $n$ is odd, then $c_{ptc}(G) = 3 = n - 2$, so that $n = 5$. Hence $G \cong K_5$.

**Case (ii)** If $n$ is even, then $c_{ptc}(G) = 4 = n - 2$, so that $n = 6$. Hence $G \cong K_6$.

**Theorem 4.25** For any connected graph $G$ with $n \geq 5$ vertices,

$$
cptc(G) + \chi(G) \leq 2n - 2$$

and the bound is sharp if and only if $G \cong K_5, K_6$.

**Proof** Let $G$ be a connected graph with $n \geq 5$ vertices. We know that $\chi(G) \leq n$ and by theorem 4.18, $c_{ptc}(G) \leq n - 2$. Hence $c_{ptc}(G) + \chi(G) \leq 2n - 2$. Suppose $G$ is isomorphic to $K_5, K_6$. Then clearly $c_{ptc}(G) + \chi(G) = 2n - 2$. Conversely, let $c_{ptc}(G) + \chi(G) = 2n - 2$. This is possible only if $c_{ptc}(G) = n - 2$ and $\chi(G) = n$. But $\chi(G) = n$, and so $G$ is isomorphic to $K_n$. 

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Case (i) If $n$ is odd, then $\text{cptc}(G) = 3 = n - 2$, so that $n = 5$. Hence $G \cong K_5$.

Case (ii) If $n$ is even, then $\text{cptc}(G) = 4 = n - 2$, so that $n = 6$. Hence $G \cong K_6$.

**Theorem 4.26** For any connected graph $G$ with $n \geq 5$ vertices, $\text{cptc}(G) + \chi(G) \leq 2n - 3$ and the bound is sharp, if $G$ is isomorphic to $W_5, K_5, C_3(2), K_5 - \{e\}, K_4(P_2), C_3(2P_2)$ or any one of the graphs shown in figure 4.40.

Figure 4.40
Proof Let $G$ be a connected graph with $n \geq 5$ vertices. We know that, 
\[(G) \leq n - 1\] and by theorem 4.18, \[\text{cptc}(G) \leq n - 2.\] Hence \[\text{cptc}(G) + (G) \leq 2n - 3.\] Let $G$ be isomorphic to $W_5$, $K_5$, $C_3(2)$, $K_5 - \{e\}$, $K_4(P_2)$, $C_3(2P_2)$ or any one of the graphs $G_1$ to $G_4$ given in figure 4.40, then clearly \[\text{cptc}(G) + (G) = 2n - 3.\]