In this chapter, we collect the basic definitions and theorems on graphs, which are needed for the subsequent chapters.

**Definition 2.1** A graph is a finite non-empty set of objects called vertices or nodes together with a set of unordered pairs of distinct vertices of G, called edges or lines. The vertex set and the edge set of G are denoted by $V(G)$ and $E(G)$ respectively. If $e = \{u, v\}$ is an edge, we write $e = uv$; we say that $e$ joins the vertices $u$ and $v$; $u$ and $v$ are adjacent vertices; $u$ and $v$ are incident with $e$. If two vertices are not joined, then we say that they are nonadjacent. If two distinct edges are incident with a common vertex, then they are said to be adjacent edges.

**Definition 2.2** The cardinality of the vertex set of a graph G is called the order of G and is denoted by ‘$n$ (or) $p$’. The cardinality of the edge set of G is called the size of G and is denoted by ‘$m$ (or) $q$’. A graph with $n$ (or) $p$ vertices and $m$ (or) $q$ edges is called a $(n, m)$ or $(p, q)$ - graph.

**Definition 2.3** Let $u$ and $v$ be (not necessarily distinct) vertices of a graph G. A **u-v walk** of G is a finite alternating sequence $w (u,v) : u = u_0e_1u_1e_2 \ldots e_nu_n = v$, of vertices and edges beginning with vertex $u$ and ending with vertex $v$ such that, $e_i = u_{i-1}u_i$, $i = 1, 2, 3, \ldots, n$. It is important to mention that the vertices need...
not be distinct and the same holds for the edges. The number \( n \) is called the \textbf{length} of the walk.

**Definition 2.4** The walk is said to be \textbf{open}, if \( u \) and \( v \) are distinct vertices, it is \textbf{closed} otherwise. A walk in which all the edges are distinct is called a \textbf{trail}.

**Definition 2.5** A walk in which all the vertices are distinct is called a \textbf{path}. A path on \( n \) vertices is denoted by \( P_n \).

**Definition 2.6** A \textbf{cycle} is a closed walk \( w(u, v) \) in which all the vertices are distinct except \( u = v \). A \textbf{cycle} on \( n \) vertices is denoted by \( C_n \).

**Definition 2.7** A graph \( G_1 \) is isomorphic to a graph \( G_2 \), if there exists a \textbf{bijection} from \( V(G_1) \) to \( V(G_2) \) such that \( uv \in E(G_1) \) if and only if \( (u)(v) \in E(G_2) \). In other words, two graphs \( G_1 \) and \( G_2 \) are \textbf{isomorphic} (written \( G_1 \cong G_2 \) or sometimes \( G_1 = G_2 \) and called equal) if there exists a one to one correspondence between their vertex sets, which preserves adjacency.

**Definition 2.8** A graph \( H \) is called a \textbf{subgraph} of \( G \), if \( V(H) \subseteq V(G) \) and \( E(H) \subseteq E(G) \).

**Definition 2.9** A \textbf{spanning subgraph} of \( G \) is a subgraph \( H \) with \( V(H) = V(G) \).

**Definition 2.10** Any sets of vertices of \( G \), the \textbf{induced subgraph} \(<S>\) is the maximal subgraph of \( G \) with vertex set \( S \). Thus two vertices of \( S \) are adjacent in \(<S>\) if and only if they are adjacent in \( G \).
Definition 2.11 The degree of vertex v in a graph G is the number of edges of G incident with v and is denoted by deg G(v) or deg(v) or d(v). The minimum and maximum degrees of vertices of G are denoted by d(G) and D(G) respectively.

Definition 2.12 A vertex of degree zero in G is called an isolated vertex.

Definition 2.13 A vertex of degree one is called a pendant vertex or an end vertex of G.

Definition 2.14 Any vertex that is adjacent to a pendant vertex is called a support.

Definition 2.15 The edge e = uv is called an isolated edge if deg e = 0 and pendant edge if either u or v is pendant vertex not both.

Definition 2.16 A graph G is regular of degree ‘r’ if and only if every vertex of G has degree r. Such graphs are called r-regular graphs. Any 3-regular graph is called a cubic graph.

Definition 2.17 A graph G is complete if every pair of its vertices is adjacent. A complete graph on n vertices is denoted by K_n.

Definition 2.18 A clique of a graph G is a maximal complete subgraph of G. The number of vertices in a clique of G is called the clique number of G and is denoted by (G).
Definition 2.19 A bipartite graph is a graph whose vertex $V(G)$ can be partitioned into two non-empty subsets $V_1$ and $V_2$ such that every edge of $G$ has one end in $V_1$ and other end in $V_2$; $(V_1, V_2)$ is called a bipartition of $G$.

Definition 2.20 Every vertex of $V_1$ is joined to every vertices of $V_2$, then $G$ is called a complete bipartite graph. The complete bipartite graph with bipartition $(V_1, V_2)$ such that $|V_1| = m$ and $|V_2| = n$ is denoted by $K_{m,n}$.

Definition 2.21 A star is a complete bipartite graph $K_{1,n}$.

Definition 2.22 A graph $G$ is said to be connected, if a path joins any two distinct vertices of $G$, otherwise $G$ is said to be disconnected.

Definition 2.23 A maximal connected subgraph of $G$ is called a component of $G$. Thus, a disconnected graph has at least two components. The number of components in a graph $G$ is denoted by $(G)$.

Definition 2.24 A graph is acyclic, if it has no cycles. A tree is a connected acyclic graph.

Definition 2.25 A spanning subgraph of $G$, which is a tree, is called a spanning tree of $G$.

Definition 2.26 A graph $G$ is unicyclic, if it is connected and contains exactly one cycle.
Definition 2.27 Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be any two graphs. The **union** of $G_1$ and $G_2$ is the graph $G = G_1 \sqcup G_2$ with vertex set $V = V_1 \cup V_2$ and edge set $E = E_1 \cup E_2$. The **join** of $G_1$ and $G_2$ is the graph $G = G_1 + G_2$ with vertex set $V = V_1 \cup V_2$ and edge set $E = E_1 \cup E_2 \cup \{uv : u \in V_1, v \in V_2\}$.

Definition 2.28 For $n \geq 4$, the **wheel** $W_n$ is defined to be the graph $K_1 + C_{n-1}$.

Definition 2.29 For any connected graph $G$, the graph with $n$-**components** each isomorphic to $G$ is written by $nG$.

Definition 2.30 The **corona** $G_1 \circ G_2$ is defined as the graph $G$ obtained by taking one copy of $G_1$ of order $p_1$ and $p_1$ copies of $G_2$ and then joining the $i^{th}$ vertex of $G_1$ to every vertex in the $i^{th}$ copy of $G_2$.

Definition 2.31 A vertex $v$ of a graph $G$ is called a **cut-vertex** of a graph $G$, if the removal of $v$ increases the number of components. Thus, if $v$ is a cut vertex of a connected graph $G$, then $G - v$ is disconnected.

Definition 2.32 An edge $e$ of a graph $G$ is called a **cut-edge or bridge** if the removal of $e$ increases the number of components.

Definition 2.33 A **block** of a graph is a maximal connected non-trivial subgraph without cut-vertices.

Theorem 2.34 Every non-trivial tree has at least two end vertices.

Definition 2.35 The **connectivity** $\gamma(G)$ of a graph $G$ is the minimum number of vertices whose removal results in a disconnected graph or $K_1$, the
trivial graph. Thus, the **connectivity** of a disconnected graph is zero, while the connectivity of a connected graph with a cut-vertex is 1.

**Definition 2.36** The **edge connectivity** \( \lambda(G) \) of a graph \( G \) is the minimum number of edges whose removal results in a disconnected or trivial graph.

**Theorem 2.37** For any graph \( G \), \( \lambda(G) \leq \delta(G) \leq \chi(G) \).

**Definition 2.38** A graph \( G \) is **\( n \)-connected** if \( \lambda(G) \geq n \) and **\( n \)-edge connected** if \( \lambda(G) \geq n \).

**Definition 2.39** Let \( G \) be a connected graph and let \( v \) be a vertex of \( G \). The **eccentricity** \( e_G(v) \) of \( v \) is the distance to a vertex farthest from \( v \). Thus, \( e_G(v) = \max\{d_G(u, v) : u \in V(G)\} \) where the distance \( d_G(u, v) \) between \( u \) and \( v \) is the minimum length of a path joining them.

**Definition 2.40** The minimum and maximum eccentricities are the **radius** and **diameter** of \( G \) denoted \( r(G) \) and \( \text{diam}(G) \) respectively.

**Definition 2.41** A vertex \( u \) is a **neighbour** of \( v \) in \( G \), if \( uv \) is an edge of \( G \), and \( u \neq v \). The set of all neighbours of \( v \) is the (open) neighbourhood of \( v \) or the neighbour set of \( v \) and is denoted by \( N(v) \); the set \( N[v] = N(v) \setminus \{v\} \) is the closed neighbourhood of \( v \) in \( G \).

**Definition 2.42** The **open neighbourhood** \( N(S) \) of a set \( S \) of vertices is the set of all vertices adjacent to the vertices in \( S \).

**Definition 2.43** \( N[S] = N(S) \setminus \{v\} \) is called the **closed neighbourhood** of \( S \).
Definition 2.44 A **subdivision** of an edge $e = uv$ of a graph $G$ is the replacement of the edge $e$ by a path $(u, w, v)$. The graph obtained from $G$ by subdividing each edge of $G$ exactly once is called the **subdivision graph** of $G$ denoted by $S(G)$.

Definition 2.45 A vertex and an edge are said to **cover** each other if they are incident. A set of vertices which cover all the edges of a graph is called a **cover** of $G$. The smallest number of vertices in any cover for $G$ is called its **covering number** and is denoted by $\theta_0$.

Definition 2.46 A set $S$ of vertices (edges) in a graph $G$ is said to be an **independent (edge independent)** set if no two vertices (edges) in $S$ are adjacent in $G$.

Definition 2.47 $S$ is called a **maximal** independent (edge independent) set provided it is not a proper subset of some other independent (edge independent) set, The maximal cardinality of an independent (edge independent) set of $G$ is called the **independence (edge independence)** number of $G$ and is denoted by $\theta_0(1)$.

Definition 2.48 Any set $M$ of independent lines of a graph $G$ is called a **matching** of $G$.

Definition 2.49 If $uv \in M$, we say that $u$ and $v$ are **matched** under $M$. We say that, the points $u$ and $v$ are $M$ – **saturated**. A matching $M$ is called a **perfect matching** if every point of $G$ is $M$ – saturated.
Theorem 2.50 For any non-trivial connected graph G, \( 0 + 0 = p = 1 + 1 \).

Definition 2.51 For any real number \( x \), \( \lfloor x \rfloor \) denotes the largest integer less than or equal to \( x \) and \( \lceil x \rceil \) denotes the smallest integer greater than or equal to \( x \).

Definition 2.52 A set \( S \subseteq V(G) \) is said to be a **dominating set** in \( G \), if every vertex in \( V - S \) is adjacent to some vertex in \( S \). The **domination number** of \( G \) is the minimum cardinality taken over all dominating sets in \( G \) and is denoted by \( \gamma(G) \).

Definition 2.53 A dominating set \( D \) of a graph \( G \) is said to be **independent dominating set** if no two vertices in \( D \) are adjacent. The **independent domination number** is the minimum cardinality taken over all independent dominating sets of \( G \) and is denoted by \( \gamma_i(G) \) or \( i(G) \).

Definition 2.54 A dominating Set \( D \) is a **total dominating set** if the induced subgraph \( D \) has no isolated vertices. The **total domination number** \( \gamma_t(G) \) of a graph \( G \) is the minimum cardinality of a total dominating set.

Definition 2.55 A dominating set \( D \) is a **connected dominating set** if the induced subgraph \( D \) is connected. The **connected domination number** \( \gamma_c(G) \) of a graph \( G \) is the minimum cardinality of a connected dominating set.

Definition 2.56 A Set \( S \subseteq V \) is a **complementary connected dominating set**, if \( S \) is a dominating set of \( G \) and the induced subgraph \( V - S \) is connected.
The **complementary connected domination number** \( cc(G) \) is the minimum cardinality taken over all complementary connected dominating sets in \( G \).

**Theorem 2.57** For any graph \( G \), \( cc(G) \leq n \).

**Theorem 2.58** For any graph \( G \), \( cc(G) = n - 1 \) if and only if \( G \) is a star.

If \( G \) is not a star, then \( cc(G) = n - 2, (n \geq 3) \).

**Theorem 2.59** For any graph \( G \), \( (G) \leq cc(G) + 1 \).

**Theorem 2.60** [Due to Brook] If \( G \) is neither a complete graph nor an odd cycle, then \( (G) = (G) \).

**Theorem 2.61** If \( G \) is a graph of order \( p \), with maximum degree \( \Delta \), then \( p / (\Delta + 1) \).

**Definition 2.62** A Colouring of a graph is an assignment of colours to its vertices so that no two adjacent vertices have the same color. An \( n \)–colouring of a graph \( G \) uses \( n \) colours. The **Chromatic number** is defined to be the minimum \( n \) for which \( G \) has an \( n \)-colouring.