In this chapter we introduce the notion of radical in \( \Gamma \)-semi-
group similar to those which have been developed in semigroup.

In [18] Ravisankar and Shukla studied radicals of \( \Gamma \)-rings.
Here we first introduce the notion of right \( \Gamma \)-S-act over a
\( \Gamma \)-semigroup and then develop the notion of radical of a
\( \Gamma \)-semigroup with the help of right \( \Gamma \)-S-act. We find that the
radical of a \( \Gamma \)-semigroup [26] is a quasi-regular \( \Gamma \)-ideal
which contains each quasi-regular right \( \Gamma \)-ideal of the
\( \Gamma \)-semigroup.

1. **RIGHT \( \Gamma \)-S-ACT**

In this section first we give some examples which motivate
us to study \( \Gamma \)-S-act.

**EXAMPLE 1.1.** Let \( A, B, C \) be three nonempty sets, \( S = T(B, C) \),
the set of all mappings from the set \( B \) into the set \( C \),
\( \Gamma = T(C, B) \) and \( M = T(A, C) \). If for \( f, g \in S \) and \( \alpha, \beta \in \Gamma \) we
define \( fg \) to denote the usual composition of mappings then
\( (S, \Gamma) \) is a \( \Gamma \)-semigroup. If \( x \in M \) and \( x\alpha f \) denotes the usual
mapping composition then we see that \( x\alpha f \in M \) and \( (x\alpha f)\beta g = 
x\alpha(f\beta g) \) for all \( f, g \in S, \alpha, \beta \in \Gamma \) and \( x \in M \).
EXAMPLE 1.2. Let $A$ and $B$ be two nonempty sets, $S = T(B, A)$, $\Gamma = T(A, B)$ where $T(A, B)$ has the usual meaning as in Example 1.1. For $f, g \in S$ and $\alpha \in \Gamma$ if $fag$ denotes the composition of mappings then $(S, \Gamma)$ is a $\Gamma$-semigroup. If we define a mapping from $A \times \Gamma \times S \to A$ by $(x, \alpha, f) \mapsto xaf$ where $xaf = (xa)f$, then we see that $xaf \in A$ and $(xaf)\beta g = x\alpha(f\beta g)$ for all $\alpha, \beta \in \Gamma$, $f, g \in S$ and $x \in A$.

EXAMPLE 1.3. Let $S$ be the set of all $m \times n$ matrices, $\Gamma$ be the set of all $n \times m$ matrices and $M$ be the set of all $r \times n$ matrices over the field of real numbers. Then $(S, \Gamma)$ is a $\Gamma$-semigroup if for $A, B \in S$, $\alpha \in \Gamma$ we define $A\alpha B$ to denote the usual matrix multiplication. Also for $x \in M$, $A, B \in S$, $\alpha, \beta \in \Gamma$, if $x\alpha A$ denotes the usual matrix multiplication then $x\alpha A \in M$ and $(x\alpha A)\beta B = x\alpha(AB)$.

Keeping the above examples in mind we define a right $\Gamma S$-act as follows.

**DEFINITION**: A right $\Gamma S$-act is a triple $(M, \Gamma, S)$, where $M$ is a nonempty set and $(S, \Gamma)$ is a $\Gamma$-semigroup, together with a mapping $(a, \alpha, s) \mapsto a\alpha s$ from $M \times \Gamma \times S$ into $M$ such that $(a\alpha s)\beta t = a\alpha(s\beta t)$ for all $a \in M$, $\alpha, \beta \in \Gamma$, $s, t \in S.$
The examples 1.1, 1.2 and 1.3 are examples of right \( \Gamma \)-\( S \)-act.

**DEFINITION**  
An element \( z \in M \) is called a zero element of the right \( \Gamma \)-\( S \)-act \((M, \Gamma, S)\) if \( z \alpha s = z \) for all \( \alpha \in \Gamma, s \in S \).

Now we give some examples of right \( \Gamma \)-\( S \)-act each of which contains a unique zero element.

**EXAMPLE 1.4.** Let \( A, B, C \) be three rings, \( S = \text{Hom}(B, C) \), the set of all homomorphisms from \( B \) into \( C \), \( \Gamma = \text{Hom}(C, B) \) and \( M = \text{Hom}(A, C) \). For \( f, g \in S \) and \( \alpha \in \Gamma \) we define \( f \alpha g \) to denote the usual mapping composition then \((S, \Gamma)\) is a \( \Gamma \)-semigroup. If \( x \in M \) and \( x \alpha f \) denotes the usual composition of mappings then \((M, \Gamma, S)\) is a right \( \Gamma \)-\( S \)-act and the zero homomorphism from the ring \( A \) into the ring \( C \) is the unique zero element of \((M, \Gamma, S)\).

**EXAMPLE 1.5.** In Example 1.2 if we take \( A, B \) to be rings, \( S = \text{Hom}(B, A), \Gamma = \text{Hom}(A, B) \). Then \((A, \Gamma, S)\) is a right \( \Gamma \)-\( S \)-act and \( 0_A \), the zero element of the ring \( A \) is the unique zero element of \((A, \Gamma, S)\).

**EXAMPLE 1.6.** The zero matrix is the unique zero element of the right \( \Gamma \)-\( S \)-act \((M, \Gamma, S)\) of Example 1.3.
Throughout this chapter we denote a right $\Gamma S$-act $(M, \Gamma, S)$ by $M$. Also we assume that every right $\Gamma S$-act $M$ to be considered in this chapter contains a unique zero element $z_M$ and we omit the term right before $\Gamma S$-act.

**DEFINITION**: Let $M$ be a $\Gamma S$-act. A nonempty subset $B$ of $M$ is called a $\Gamma S$-subact if $B \cap S \subseteq B$.

The subsets $\{z_M\}$ and $M$ of a $\Gamma S$-act $M$ are $\Gamma S$-subacts of $M$. These are called trivial $\Gamma S$-subacts. All other $\Gamma S$-subacts are termed as nontrivial $\Gamma S$-subacts.

**DEFINITION**: Let $M$ be a $\Gamma S$-act. An equivalence relation $\theta$ on $M$ is a $\Gamma S$-congruence if $(a, b) \in \theta$ implies $(a \alpha s, b \alpha s) \in \theta$ for all $a \in \Gamma$ and $s \in S$.

**DEFINITION**: A $\Gamma S$-homomorphism from a $\Gamma S$-act $M$ into a $\Gamma S$-act $M'$ is a mapping $\theta : M \to M'$ such that $(a \alpha s) \theta = (a \theta)s$ for all $a \in M$, $\alpha \in \Gamma$, $s \in S$. Moreover if $\theta$ is one to one and onto, then we call it $\Gamma S$-isomorphism.

**DEFINITION**: A $\Gamma S$-act $M$ is called (strictly) cyclic if there exists $m$ in $M$ and $\alpha$ in $\Gamma$ such that $(M \alpha m) M = M \alpha S \cup \{m\}$. Such an element $m$ is called a (strict) generator of $M$.
DEFINITION: A right congruence $\mathcal{R}$ on a $\Gamma$-semigroup $(S, \Gamma)$ is called maximal if $\mathcal{R}$ is not the universal relation on $S$ and if $\mathcal{T}$ is a right congruence on $(S, \Gamma)$ such that $\mathcal{R} \subseteq \mathcal{T}$ then either $\mathcal{R} = \mathcal{T}$ or $\mathcal{T}$ is the universal relation on $S$.

DEFINITION: A right congruence $\mathcal{R}$ on a $\Gamma$-semigroup $(S, \Gamma)$ is called modular if there exist $e$ in $S$ and $a$ in $\Gamma$ such that $(eaa, a) \in \mathcal{R}$ for all $a$ in $S$. $e$ is called left identity modulo $\mathcal{R}$ relative to $a$.

THEOREM 1.1. Every strictly cyclic $\Gamma S$-act is isomorphic to $S/\mathcal{R}$ as $\Gamma S$-act for some modular right congruence $\mathcal{R}$ on $(S, \Gamma)$. Conversely if $\mathcal{R}$ is a modular right congruence on $(S, \Gamma)$ then $S/\mathcal{R}$ is a strictly cyclic $\Gamma S$-act.

PROOF. Let $M$ be a strictly cyclic $\Gamma S$-act. Then there is some $m$ in $M$ and $a$ in $\Gamma$ such that $M = maS$. We define a relation $\mathcal{R}$ on $S$ by $\mathcal{R} = \{(a, b) \in S \times S : ma = mb\}$. Then it is easy to prove that $\mathcal{R}$ is a right congruence on $(S, \Gamma)$ and $S/\mathcal{R}$ is a $\Gamma S$-act, where $S/\mathcal{R} \times \Gamma x S \to S/\mathcal{R}$ is defined by $(s\mathcal{R}, \beta, t) \mapsto (s\beta t)\mathcal{R}$. That is $(s\mathcal{R})\beta t = (s\beta t)\mathcal{R}$. Since $m \in M = maS$, there exists $e$ in $S$ such that $mae = m$. Thus $maea = maa$ for all $a \in S$ which in turn implies that $(eaa, a) \in \mathcal{R}$ for all $a \in S$. Thus $e$ is a left identity modulo $\mathcal{R}$ relative to $a$. Therefore, $S/\mathcal{R}$ is a strictly cyclic $\Gamma S$-act.
to $a$. Hence $\varphi$ is modular right congruence. We define a map $\theta : M \to S/\varphi$ by $n\theta = a\varphi$ where $n = maa \in M$. If $n = maa = mab$ then $a\varphi = b\varphi$. So $\theta$ is well defined. Let $p \in M$, $\beta \in \Gamma$, $s \in S$ then $p\beta s \in M$. If $p = mat$ for some $t \in S$, then $(p\beta s)\theta = \beta (mat\theta s) = (t\varphi) (\beta s) = (p\theta) \beta s$. Thus $\theta$ is a $\Gamma S$-homomorphism. Also for $s \varphi \in S/\varphi$, $mas \in M$ and $(mas)\theta = s\varphi$. So $\theta$ is onto. Again $p\theta = n\theta$, where $p = maa$ and $n = mab$ for some $a, b \in S$, implies that $a\varphi = b\varphi$, that is $maa = mab$, so that $p = n$. Therefore $\theta$ is a $\Gamma S$-isomorphism of $M$ onto $S/\varphi$.

Conversely let $\varphi$ be a modular right congruence on $(S, \Gamma)$ and let $e$ be a left identity modulo $\varphi$ relative to some $a \in \Gamma$. Then for any $a$ in $S$ $(e \varphi) a a = (e a a) \varphi = a \varphi$. Hence $e \varphi$ is a strict generator of $S/\varphi$. Thus $S/\varphi$ is a strictly cyclic $\Gamma S$-act.

**DEFINITION** : A $\Gamma S$-act $M$ is called irreducible if

(i) $M \cap S \neq \{z_M\}$

(ii) $M$ has only trivial $\Gamma S$-subacts.

**LEMMA 1.2**. Every irreducible $\Gamma S$-act is strictly cyclic.

**PROOF**. Let $M$ be an irreducible $\Gamma S$-act. If $z_M \neq a \in M$ then $a \cap S \neq \{z_M\}$. If possible let $a \cap S = \{z_M\}$. Let $B = \{b \in M : b \cap S = \{z_M\}\}$. Then $B$ is a $\Gamma S$-subact of $M$. Since
$z_M \not\in a \in B$, $B \not\in \{z_M\}$. Hence $B = M$. Therefore $M \triangleleft S = \{z_M\}$, which is a contradiction. Hence we must have $a \triangleleft S \not\in \{z_M\}$. So there exists $a \in \Gamma$ such that $a \alpha S \not\in \{z_M\}$. But $a \alpha S$ is a $\Gamma S$-subact of $M$. Hence $a \alpha S = M$. Therefore $M$ is strictly cyclic and every nonzero element is its strict generator.

**Theorem 1.3.** Every irreducible $\Gamma S$-act is isomorphic to $S/\varphi$ as a $\Gamma S$-act for some modular congruence $\varphi$ on $S$.

**Proof.** As every irreducible $\Gamma S$-act is strictly cyclic the above theorem follows from Theorem 1.1.

**Definition.** A $\Gamma S$-act $M$ is called totally irreducible if

1. $M \triangleleft S \not\in \{z_M\}$.
2. Identity relation and the universal relation are the only $\Gamma S$-congruences on $M$.

It is immediate that every totally irreducible $\Gamma S$-act is irreducible.

**Theorem 1.4.** Every totally irreducible $\Gamma S$-act is isomorphic to $S/\varphi$ as a $\Gamma S$-act for some maximal modular right congruence $\varphi$ on $(S, \Gamma)$. On the other hand for every such maximal modular right congruence $\varphi$ on $(S, \Gamma)$, $S/\varphi$ is a totally irreducible $\Gamma S$-act.
PROOF. Let $M$ be a totally irreducible $\Gamma S$-act. Then $M$ is also irreducible, hence strictly cyclic. So $M = a\mathbb{G}S$ for some $a(z_M) \in M$ and $a \in \Gamma$. Consequently $M$ is isomorphic to $S/\mathcal{P}$ where $\mathcal{P} = \{(s, t) \in S \times S : a s = a t\}$ is a modular right congruence on $(S, \Gamma)$. We now show that $\mathcal{P}$ is maximal. Suppose $\mathcal{R}$ be a right congruence on $(S, \Gamma)$ which properly contains $\mathcal{P}$. We define a relation $\gamma$ on $M$ by $\gamma = \{(b, c) \in M \times M : \text{each element of } b f \text{ is } \mathcal{R}\text{-related to each element of } c f\}$, where $f$ is a $\Gamma S$-isomorphism between $M$ and $S/\mathcal{P}$ defined by $m f = t \mathcal{P}$ when $m = a t s$ for some $t \in S$. It is immediate that $\gamma$ is a $\Gamma S$-congruence on $M$. Since $M$ is totally irreducible, either $\gamma$ is the identity relation or the universal relation on $M$. Now $\mathcal{R} \subset \mathcal{R}$, hence there exist $s, t \in S$ such that $s \not\subset t$ but $s \mathcal{R} t$. This shows that $s \mathcal{P} \not= t \mathcal{P}$. Hence $(a s) \mathcal{P} \not= (a t) \mathcal{P}$. This also implies that $a s \not= a t$ in $M$. But $s \not\subset t$ shows that every element of $(a s) \mathcal{P}$ is $\mathcal{R}$-related to every element of $(a t) \mathcal{P}$. Hence $a s \gamma a t$. Consequently $\gamma$ is the universal relation. This implies that $\mathcal{R}$ is the universal relation. Hence $\mathcal{P}$ is maximal. Conversely, suppose that $\mathcal{P}$ is a maximal modular right congruence on $(S, \Gamma)$. As $\mathcal{P}$ is modular, there exist $e \in S$ and $a \in \Gamma$ such that $(e a s, s) \in \mathcal{P}$ for all $s \in S$. Now $(e \mathcal{P}) a s = (e a s) \mathcal{P} = s \mathcal{P}$ for all $s$ in $S$. Hence $S/\mathcal{P} \subset S/\mathcal{P} \mathcal{R} S$. 
Thus \( S/\rho \Gamma S = S/\rho \neq \) zero, as \( \varnothing \) is not the universal relation.

Now let \( \Upsilon \) be a right \( \Gamma S \)-congruence on \( S/\rho \). Let

\[ \Upsilon = \{ (s,t) \in S \times S : \sigma \Upsilon t \in \rho \} \]

Then \( \Upsilon \) is a right congruence on \( (S,\Gamma) \) such that \( \Upsilon \) contains \( \varnothing \). Hence either \( \Upsilon = \varnothing \) or \( \Upsilon \) is the universal relation. Thus \( \Upsilon \) is either the identity relation or the universal relation. Consequently \( S/\rho \) is totally irreducible.

2. RADICAL OF \( \Gamma \)-SEMIGROUP

In this section we define annihilator of a \( \Gamma S \)-act and with its help we define radical of a \( \Gamma \)-semigroup. We find that the radical of a \( \Gamma \)-semigroup \( (S,\Gamma) \) is the intersection of all \( z\varnothing \) classes (\( z\varnothing \) being the zero element of the \( \Gamma \)-semigroup \( (S/\rho,\Gamma) \)) for all maximal modular right congruence \( \varnothing \) on \( (S,\Gamma) \). We also define right quasi-regular element, quasi-regular right \( \Gamma \)-ideal and nil right \( \Gamma \)-ideal of a \( \Gamma \)-semigroup \( (S,\Gamma) \). We find that the radical of a \( \Gamma \)-semigroup \( (S,\Gamma) \) is a quasi-regular \( \Gamma \)-ideal which contains every quasi-regular right \( \Gamma \)-ideal as well as every nil right \( \Gamma \)-ideal of \( (S,\Gamma) \). Finally we obtain that a radical of a \( \Gamma \)-ideal of a \( \Gamma \)-semigroup is the intersection of the \( \Gamma \)-ideal with the radical of the \( \Gamma \)-semigroup itself.
DEFINITION: Let $M$ be a $\Gamma S$-act. Then annihilator of $M$ denoted by $\text{Ann}_{S}(\Gamma)^{M}$ is defined as

$$\text{Ann}_{S}(\Gamma)^{M} = \{a \in S : M\Gamma a = \{z_{M}\}, \text{ } z_{M} \text{ being the zero element of } M\}.$$ 

It can be easily shown that if $\text{Ann}_{S}(\Gamma)^{M} \neq \emptyset$ then $\text{Ann}_{S}(\Gamma)^{M}$ is a $\Gamma$-ideal of $(S, \Gamma)$.

DEFINITION: Let $(S, \Gamma)$ be a $\Gamma$-semigroup. Then radical of $(S, \Gamma)$ denoted by $J_{\Gamma}(S)$ is defined as

$$J_{\Gamma}(S) = \cap \{\text{Ann}_{S}(\Gamma)^{M} : \text{ } M \text{ is totally irreducible } \Gamma S\text{-act}\}.$$ 

If the $\Gamma$-semigroup $(S, \Gamma)$ has no totally irreducible $\Gamma S$-act then $J_{\Gamma}(S) = S$.

THEOREM 2.1. For a $\Gamma$-semigroup $(S, \Gamma)$, $J_{\Gamma}(S) = \cap \{z_{\mathcal{C}} : \mathcal{C} \text{ is a maximal modular right congruence on } (S, \Gamma) \text{ and } z_{\mathcal{C}} \text{ is the zero element of the } \Gamma S\text{-act } S/\mathcal{C}\}$. [If no such $\mathcal{C}$ exists then we assume $\cap \{z_{\mathcal{C}}\} = S$].

PROOF. Let $s \in J_{\Gamma}(S)$, then $M \Gamma s = \{z_{M}\}$ for each totally irreducible $\Gamma S$-act of $M$. Now for each totally irreducible $\Gamma S$-act $M$ there exists a maximal modular right-congruence $\mathcal{C}$ on $(S, \Gamma)$ such that $S/\mathcal{C}$ is isomorphic to $M$ as $\Gamma S$-act.

Hence $s \in J_{\Gamma}(S)$ if and only if $S/\mathcal{C} \Gamma s = \{z_{\mathcal{C}}\}$ where $z_{\mathcal{C}}$ is
the zero element of \( S/\rho \), for each maximal modular congruence \( \rho \) on \((S, \Gamma)\). This implies that \((t \rho)S = (t \alpha S) \rho = z \rho \) for each \( t \rho \in S/\rho \), \( \alpha \in \Gamma \). Since \( \rho \) is a modular right congruence there exist \( e \in S \) and \( \alpha \in \Gamma \) such that \((e \alpha s_1, s_1) \in \rho \) for all \( s_1 \in S \). Hence \( z \rho = (e \rho)S = (e \alpha S) \rho = s \rho \). From this it follows that \( s \in z \rho \) for all maximal modular right congruence \( \rho \) on \((S, \Gamma)\). Consequently \( J_\Gamma(S) \subseteq \{ z \rho : \rho \) is a maximal modular right congruence on \((S, \Gamma)\} \). Next let us suppose \( s \in \cap \{ z \rho : \rho \) is a maximal modular right congruence on \((S, \Gamma)\} \). If possible let \( M \) be a totally irreducible \( \Gamma S\)-act such that \( M \cap S \neq \{ z_M \} \). Hence there exist \( m \in M \), \( \alpha_1 \in \Gamma \) such that \( m \alpha_1 S \neq z_M \). Then \( (m \alpha_1 s) \Gamma S = M \). Hence there exist \( q \in S \) and \( \beta \in \Gamma \) such that \( m \alpha_1 s \beta q = m \). We define a relation \( \tau \) on \( S \) by \( \tau = \{(s_1, t_1) \in S \times S : m \alpha_1 s_1 = m \alpha_1 t_1 \} \). It can be shown that \( \tau \) is a right congruence on \((S, \Gamma)\), \( \tau \) is not the universal relation and \( s \beta q \) is a left identity modulo \( \tau \) relative to \( \alpha_1 \). Now \( m \alpha_1 S \cap S = M \) implies that \( m \alpha_1 S = M \). Then for each \( m_1 \in M \) there exists an element \( t \in S \) such that \( m_1 \alpha_1 t = m_1 \). We define a mapping \( \theta : M \to S/\tau \) by \( m_1 \theta = t \tau \). It can be shown that \( \theta \) is a \( \Gamma S\)-isomorphism. Now \( S/\tau \) is a totally irreducible \( \Gamma S\)-act with zero element \( z_M \). Since \( s \beta q \) is a left identity modulo \( \tau \) relative to \( \alpha_1 \) we have \((s \beta q) \tau \cap S = \).
But $s \in z \tau$, implies that $s \tau$ is the zero element of $S/\tau$. So $(s \tau)zq = s \tau$ that is $(s \beta q)\tau = s \tau = \text{zero element of } S/\tau$. So $s \beta q \in z_M \theta$. Therefore $S/\tau = (s \beta q)\tau \cap S = (z_M \theta)\tau \cap S = \{ z_M \theta \}$ which contradicts the fact that $\tau$ is not the universal relation. Therefore $mas = z_M$ for each $m$ in $M$ and each $a$ in $\Gamma$. Hence $M \cap S = \{ z_M \}$. Thus $s \in J_\tau (S)$.

Therefore $\cap \{ z \beta : \beta \text{ is a maximal modular right congruence on } (S, \Gamma) \} \subseteq J_\tau (S)$. Consequently we have $J_\tau (S) = \cap \{ z \beta : \beta \text{ is a maximal modular right congruence on } (S, \Gamma) \}$. 

DEFINITION: An element $e \in S$ of a $\Gamma$-semigroup $(S, \Gamma)$ is called right quasi-regular if there does not exist a maximal modular right congruence $\beta$ on $(S, \Gamma)$ such that $e$ is a left identity modulo $\beta$ relative to any $a \in \Gamma$.

DEFINITION: A right $\Gamma$-ideal of a $\Gamma$-semigroup $(S, \Gamma)$ is called quasi-regular if each element contained in it is a right quasi-regular element of $(S, \Gamma)$.

THEOREM 2.2: In a $\Gamma$-semigroup $(S, \Gamma)$ the radical $J_\Gamma (S)$ is a quasi-regular $\Gamma$-ideal which contains every quasi-regular right $\Gamma$-ideal of $(S, \Gamma)$.

PROOF: Let $n$ be any element of $J_\Gamma (S)$, $s$ be in $S$ and $a$ be
If $M$ is a totally irreducible $\Gamma S$-act, then $M \Gamma (\text{son}) = (M \Gamma \text{s}) \text{an} \leq M \text{an} = \{z_M\}$, where $z_M$ is the zero element of $M$. Also $M \Gamma (\text{nas}) = (M \Gamma \text{n}) \text{as} = \{z_M \text{as}\} = \{z_M\}$. Thus $J_\Gamma (S)$ is a $\Gamma$-ideal of $(S, \Gamma)$. Suppose that $e$ belongs to $J_\Gamma (S)$ and $f$ is a maximal modular right congruence on $(S, \Gamma)$ such that $e$ is a left identity modulo $f$ relative to some $\alpha \in \Gamma$. Now $e$ is contained in the zero of $S/f$, which contradicts the fact that $f$ is not the universal relation. Therefore $J_\Gamma (S)$ is quasi-regular. Suppose that $T$ is a quasi-regular right $\Gamma$-ideal of $(S, \Gamma)$, $t$ belongs to $T$ and $M$ is a totally irreducible $\Gamma S$-act such that $M \Gamma t \neq \{z_M\}$. So there exist $m$ in $M$ and $s$ in $\Gamma$ such that $m \alpha t \neq z_M$. Thus $(m \alpha t) \Gamma S = M$. Hence there exist $s$ in $S$ and $\beta$ in $\Gamma$ such that $m \alpha t \beta s = M$. We now define a relation $\gamma$ on $S$ by $\gamma = \{(u, v) \in S \times S : m \alpha u = m \alpha v\}$. It is immediate that $\gamma$ is a right congruence on $(S, \Gamma)$ and $M$ is isomorphic to $S/\gamma$ as $\Gamma S$-act under the isomorphism $\theta : M \rightarrow S/\gamma$ defined by $m_1 \theta = r_\gamma$ where $m_1 = m \alpha r$ for some $r \in S$. Therefore $S/\gamma$ is a totally irreducible $\Gamma S$-act, consequently $\gamma$ is maximal. Also $(t \beta s)$ is a left identity modulo $\gamma$ relative to $\alpha$. Thus $(t \beta s)$ is not right quasi-regular element of $(S, \Gamma)$, which is a contradiction. Therefore if $M$ is a totally irreducible $\Gamma S$-act then $M \Gamma t = \{z_M\}$. 
Thus \( t \) is in \( \mathbb{J}_P(S) \). So \( \mathbb{J}_P(S) \) contains every quasi-regular right \( \Gamma \)-ideal of \( (S, \Gamma) \).

**DEFINITION**: Let \( o \) be the zero element of the \( \Gamma \)-semigroup \( (S, \Gamma) \). Then an element \( s \) in \( S \) is called nilpotent if for each \( a \) in \( \Gamma \) there is a positive integer \( n \) such that \( (sa)^n s = o \).

**DEFINITION**: A right \( \Gamma \)-ideal of a \( \Gamma \)-semigroup \( (S, \Gamma) \) is called nil if each of its element is nilpotent.

**THEOREM 2.3**: If \( s \) is a nilpotent element of a \( \Gamma \)-semigroup \( (S, \Gamma) \) then \( s \) is a right quasi-regular element of \( (S, \Gamma) \).

**PROOF**. Suppose that \( s \) is nilpotent but not right-quasi regular element of \( S \). Let \( \mathcal{F} \) be a maximal modular right congruence on \( (S, \Gamma) \) such that \( s \) is a left identity modulo \( \mathcal{F} \) relative to some \( a \in \Gamma \). For each positive integer \( n \), \( (sa)^n s, s \) \( \in \mathcal{F} \). Hence \( (s, o) \in \mathcal{F} \). Since \( s \) is a left identity modulo \( \mathcal{F} \) relative to \( a \), it follows that \( (t, o) \in \mathcal{F} \) for all \( t \in S \), which contradicts the fact that \( \mathcal{F} \) is not the universal relation. Therefore \( s \) is a right quasi-regular element of \( (S, \Gamma) \).

**COROLLARY 2.4**: In a \( \Gamma \)-semigroup \( (S, \Gamma) \) the radical \( \mathbb{J}_P(S) \) contains each nil right \( \Gamma \)-ideal of \( (S, \Gamma) \).
THEOREM 2.5. If $T$ is a $\Gamma$-ideal of $(S, \Gamma)$ and $t$ belongs to $T$ then $t$ is a right quasi-regular element of $(S, \Gamma)$ if and only if $t$ is a right quasi-regular element of $(T, \Gamma)$.

**Proof.** Let $\mathcal{P}$ be a maximal modular right congruence on $(T, \Gamma)$ and $t$ be a left identity modulo $\mathcal{P}$ relative to some $a \in \Gamma$.

We now define a relation $\gamma$ on $S$ by

$$\gamma = \{(r,s) \in S \times S : (t \alpha r \beta p, t \alpha s \beta p) \in \mathcal{P} \text{ for all } p \in S \text{ and } \beta \in \Gamma\}.$$  

It is immediate that $\gamma$ is a right congruence on $(S, \Gamma)$ and $t$ is a left identity modulo $\gamma$ relative to $a$. Let $z_{\mathcal{P}}$ be the zero element of $T/\mathcal{P}$. If $\gamma$ is universal then $(t,z) \in \gamma$. So $(t \alpha r \beta p, t \alpha z \beta p) \in \mathcal{P}$ or $(t \alpha p, z \alpha p) \in \mathcal{P}$ or $(p, z \alpha p) \in \mathcal{P}$. Thus $p \mathcal{P} = (z \alpha p) \mathcal{P} = (z \mathcal{P}) \alpha p = z \mathcal{P}$. Therefore $(p,z) \in \mathcal{P}$ for all $p \in T$. So $\mathcal{P}$ is universal which is a contradiction. Hence $\gamma$ is not universal relation.

Let $\mathcal{P}$ be a maximal right congruence on $(S, \Gamma)$ containing $\gamma$ then $t$ is a left identity modulo $\mathcal{P}$ relative to $a$. So $t$ is not a right quasi-regular element of $(S, \Gamma)$. Therefore each element of $T$ which is right quasi-regular element of $(S, \Gamma)$ is a right quasi-regular element of $(T, \Gamma)$. Conversely suppose that $\mathcal{P}$ is a maximal modular right congruence on $(S, \Gamma)$ and $t$ is a left identity modulo $\mathcal{P}$ relative to some $a \in \Gamma$. If $r,s$ are any two elements of $T$, let $(r,s) \in \gamma$ if and
only if \((r, s) \in \varrho\). Then \(\Upsilon\) is a right congruence on \(T\) with \(t\) as a left identity modulo \(\Upsilon\) relative to \(a\) in \(\Gamma\). Also \(\Upsilon\) is not universal relation. Let \(\Upsilon\) be a maximal right congruence on \((T, \Gamma)\) which contains \(\Upsilon\), then \(t\) is a left identity modulo \(\Upsilon\) relative to \(a\). Hence each element of \(T\) which is right quasi-regular element of \((T, \Gamma)\) is also a right quasi-regular element of \((S, \Gamma)\).

**Lemma 2.6.** If \((S, \Gamma)\) and \((T, \Gamma')\) are two \(\Gamma\)-semigroups and \((\theta, \phi)\) be a homomorphism from \((S, \Gamma)\) onto \((T, \Gamma')\) then 
\[
[J_{\Gamma}(S)]_{\theta} \subseteq J_{\Gamma'}(T).
\]

**Proof.** Suppose that \(r \in J_{\Gamma}(S)\) and \(r\theta\) is not a right quasi-regular element of \(T\). Let \(\varrho\) be a maximal modular right congruence on \((T, \Gamma')\) such that \(r\theta\) is a left identity modulo \(\varrho\) relative to some \(a'\) in \(\Gamma'\). We now define
\[
\Upsilon = \{(s, p) \in S \times S : s\theta \varrho p \theta\}. \quad \text{Then} \quad \Upsilon \quad \text{is a right congruence on} \quad (S, \Gamma), \quad \text{which is not universal.} \quad \text{Now if} \quad s \in S \quad \text{and} \quad a \in \phi^{-1}(a') \quad \text{then} \quad (r \alpha s) \theta = (r \theta) (a \phi) (s \theta) = (r \theta) a' (s \theta). \quad \text{But} \quad (r \theta a' s \theta, s \theta) \in \varrho. \quad \text{That is} \quad ((r \alpha s) \theta, s \theta) \in \varrho. \quad \text{Thus} \quad (r \alpha s, s) \in \Upsilon \quad \text{for all} \quad s \in S \quad \text{so that} \quad r \quad \text{is a left identity modulo} \quad \Upsilon \quad \text{relative to} \quad a. \quad \text{Let} \quad \Upsilon' \quad \text{be a maximal modular right congruence on} \quad (S, \Gamma) \quad \text{which contains} \quad \Upsilon. \quad \text{Then} \quad r \quad \text{is also left identity modulo} \quad \Upsilon' \quad \text{relative to} \quad a.
This is a contradiction to the fact that $J_{\Gamma}(S)$ is a quasi-regular $\Gamma$-ideal. Thus if $r$ belongs to $J_{\Gamma}(S)$ then $r\theta$ is a right quasi-regular element of $(T, \Gamma')$. Thus $\{r\theta\} \cup (r\theta)\Gamma' T$ is a quasi-regular right $\Gamma$-ideal of $(T, \Gamma')$ and so $\{r\theta\} \cup (r\theta)\Gamma' T \subseteq J_{\Gamma}(T)$. Therefore $[J_{\Gamma}(S)] \theta \subseteq J_{\Gamma}(T)$.

**NOTE:** Let $(S, \Gamma, \mu)$ be a $\Gamma$-semigroup and $I$ be a $\Gamma$-ideal of $(S, \Gamma, \mu)$. Let $S/I = \{I, x : x \in S - I\}$.

Now $(S/I, \Gamma, \mu')$ is a $\Gamma$-semigroup if we define

$$(x, a, y)\mu' = \begin{cases} (x, a, y)\mu & \text{if } (x, a, y)\mu \in S - I \\ I & \text{otherwise} \end{cases}$$

and $(I, a, I)\mu' = (I, a, y)\mu' = (x, a, I)\mu' = I$ for all $x, y \in S - I$ and $a \in \Gamma$. We denote this $\Gamma$-semigroup simply by $(S/I, \Gamma)$ or $(\tilde{S}, \Gamma)$. Also if $(M, \Gamma, S)$ is a $\Gamma S$-act then we define a mapping $(M, \Gamma, \tilde{S}) \rightarrow M$ by $(m, a, \tilde{s}) \rightarrow m \tilde{s} = \begin{cases} m \tilde{s} & \text{if } s \notin I \\ z_{M} & \text{if } s = I \end{cases}$

Then it is easy to see that $(M, \Gamma, \tilde{S})$ is a $\Gamma S$-act.

**THEOREM 2.7.** If $T$ is a $\Gamma$-ideal of a $\Gamma$-semigroup $(S, \Gamma)$ then $J_{\Gamma}(T) = J_{\Gamma}(S) \cap T$.

**PROOF.** It is immediate that $J_{\Gamma}[S/J_{\Gamma}(S)]$ is zero. Because every totally irreducible $\Gamma S$-act is a totally irreducible $\Gamma S/J_{\Gamma}(S)$-act and conversely. Let $Q = (T \cup J_{\Gamma}(S))/J_{\Gamma}(S)$. 


Since \([J_p(Q) \cap Q \subseteq J_p(Q)],\) by Theorem 2.2 and 2.5 each element of \([J_p(Q) \cap Q \subseteq J_p(Q)],\) is a right quasi-regular element of \((Q, \Gamma)\) and of \((S \cap J_p(S), \Gamma)\). Thus \(J_p(Q) \cap Q \subseteq J_p[S \cap J_p(S)] = \text{zero.}\)

Let \(P = \{p : p \text{ is an element of } TyJ_p(S) \text{ such that } P/J_p(S) \subseteq J_p(Q)\}.\) Since \(P/J_p(S) \subseteq J_p(Q)\), \(P/J_p(S) = \text{zero}\), by Corollary 2.4. \(P/J_p(S) \subseteq J_p(Q)\). Thus \(J_p(Q) = P/J_p(S)\). If \(R = \{r : r \in S \text{ such that } [r/J_p(S)] \cap Q = \text{zero}\},\) then \(R/J_p(S)\) is a \(\Gamma\)-ideal of \((S/J_p(S), \Gamma)\) and \(J_p(Q) = [R/J_p(S)] \cap Q\). Thus \(J_p(Q)\) is a \(\Gamma\)-ideal of \((S/J_p(S), \Gamma)\). Hence \(J_p(Q)\) is a quasi-regular \(\Gamma\)-ideal of \((S/J_p(S), \Gamma)\). Hence \(J_p(Q)\) is zero. Let \((\theta, I)\) be the natural homomorphism from \((T \cup J_p(S), \Gamma)\) onto \((Q, \Gamma)\), defined by \(t\theta = t/J_p(S)\) for all \(t \in T \cup J_p(S)\) and \(I\) is the identity mapping on \(\Gamma\). Then by Lemma 2.6 \(J_p(T \cup J_p(S)) \subseteq J_p(Q) = J_p(S)/J_p(S)\) and thus \(J_p(T \cup J_p(S)) \subseteq J_p(S)\). By Theorem 2.2 and 2.5 \(J_p(T)\) is a quasi-regular \(\Gamma\)-ideal of \((T \cup J_p(S), \Gamma)\). Hence \(J_p(T) \subseteq J_p(T \cup J_p(S)) \subseteq J_p(S)\). Therefore \(J_p(T) \subseteq J_p(S) \cap T\) \(\cdots \cdots \cdots (A)\).

Since \(J_p(S) \cap T\) is a \(\Gamma\)-ideal of \((S, \Gamma)\) each element of \(J_p(S) \cap T\) is a right quasi-regular element of \((J_p(S) \cap T, \Gamma)\). Thus \(J_p(S) \cap T \subseteq J_p[(J_p(S) \cap T)]\). Since \(J_p(S) \cap T\) is a \(\Gamma\)-ideal of the \(\Gamma\)-semigroup \((T, \Gamma)\), we have by \((A)\) \(J_p[(J_p(S) \cap T)] \subseteq J_p(T) \cap [J_p(S) \cap T]\).
Therefore $J_p(S) \cap T = J_p(T)$.

This completes the proof of the theorem.