CHAPTER 8
Symbolic Direct Product And Symbolic Product Of Fractional Factorial Designs

Summary

Symbolic Direct Product (SDP) and Symbolic Product (SP) are often used to construct orthogonal or non-orthogonal fractional factorial designs. The analysis is usually performed suiting to the particular component designs on ad-hoc basis.

In this Chapter, we have considered their analysis in full generality. It has been seen that, though the two products seem to be different there exists an equivalence relation between them in the sense that one can be obtained from the other in terms of pseudo-factors, suitably chosen.

We have presented the analysis of symbolic direct product design of a number of fractional factorial designs and have obtained that the estimates of the parameters of the new design can be expressed as a suitable (asterisk) product of the estimates of the parameters of the component designs. To analyse the symbolic product designs we have used the equivalence relation and thus translated the analysis to that of symbolic direct product designs by introducing pseudo-factors.
As a by-product, we have been able to isolate a class of orthogonal arrays for which the Rao's lower bound for the number of level combinations can be improved upon.

8.1 Introduction

Let for \( i = 1, 2 \) (throughout this chapter, we shall assume \( i = 1, 2 \) unless otherwise mentioned), \( E_i \) denote a \( m_1 \times s_{i1} \times \ldots \times s_{im_i} \) complete factorial experiment with the level combinations \( J(i) = (j_{i_1}(i), j_{i_2}(i), \ldots, j_{im_i}(i)) \), \( 0 \leq j_{k_i}(i) \leq (s_{ik_i} - 1) \), \( j_{k_i}(i) \) being a level of \( A_{ik_i} \), the \( k_i \)th factor of \( E_i \), \( 1 \leq k_i \leq m_i \). Also let \( T_i \) be a fraction of \( E_i \) with the following \( n_i \) level combinations.

\[
T_i = \begin{bmatrix}
  j_{i1}(1), \ldots, j_{im_i}(1) \\
  j_{i2}(1), \ldots, j_{2m_i}(1) \\
  \vdots \\
  j_{n_i1}(1), \ldots, j_{n_im_i}(1)
\end{bmatrix} \quad \begin{bmatrix}
  J_1(1) \\
  J_2(1) \\
  \vdots \\
  J_{n_i}(1)
\end{bmatrix} \quad \text{...(8.1.1)}
\]

A array \( T \) of dimension \( n_1 n_2 \times (m_1 + m_2) \) is defined to be the symbolic direct product (SDP) of \( T_1 \) and \( T_2 \).
(Raktoe, Hedayat and Federer 1981, page 187) where $T$ is given by

$$T = T_1 \times T_2 = \begin{bmatrix} J_1(1), & J_1(2) \\ J_1(1), & J_2(2) \\ & \cdots \cdots \\ J_1(1), & J_{n_2}(2) \\ & \cdots \cdots \\ J_{n_1}(1), & J_{n_2}(2) \end{bmatrix} = \left\{ J_{x_1}(1), J_{x_2}(2) \right\}, \ 1 \leq x_1 \leq n_1 \ldots \ldots \ldots \ldots (8.1.2)$$

$T$ can be considered as a fraction with $n_1 n_2$ level combinations from a $X s_{1k_1} X s_{2k_2}$ complete factorial experiment $E$ with the $(m_1 + m_2)$ factors $A_{1k_1}, 1 \leq k_1 \leq m_1$. It may be interesting to study the nature of $T$ as a factorial design when that of $T_1$ is known. The problem was indicated in Joiner, Raktoe and Federer (1980) and also in Raktoe, Hedayat and Federer (1981). Chakravorti (1956) was the first to use this type of product to orthogonal fractions of symmetrical factorials to generate orthogonal fractions of asymmetrical factorials.

$T_1$ can be assumed to be a completely randomised design constructed with the level combination of $E_1$ where the replication of the level combination is either zero or one. Thus the fraction $T$ can also be interpreted as the Kronecker product (Vartak 1955, 1963) of completely randomised designs $T_1$ and $T_2$. Kronecker product or its variants
have been studied by several authors, in particular, the analysis of Kronecker product of two equi-replicate and proper block designs has been considered in Gupta (1983). But the present set-up of the symbolic direct product of two fractions with replications zero or one for the treatments, where the general mean may also be included as an estimable effect is not a direct consequence of Gupta's method. Also the present analysis reveals some specific properties of T which will enable us to derive further results.

There is another type of product called symbolic product (SP), the particular cases of which were considered in Chapters 3 and 5. This was first used by Bush (1952b) on OA's to construct new orthogonal arrays from known solutions of them. We define it again for convenience of discussions and for comparability with the symbolic direct product.

Let \( m_1 = m_2 = m \) (say) in \( T_1 \)'s of (8.1.1). By the symbolic product of \( T_1 \) and \( T_2 \) we mean the array \( T_0 \) which is given by

\[
T_0 = T_1 \odot T_2 = \begin{bmatrix}
(J_1(1), J_1(2)) \\
(J_1(1), J_2(2)) \\
\vdots \\
(J_n(1), J_n(2)) \\
\end{bmatrix}
\]

\[
= \left\{(J_{r_1}(1), J_{r_2}(2)) \mid 1 \leq r_1 \leq n_1 \right\}
\]

\[ \ldots (8.1.3) \]
where the level combination \((J_1(1), J_2(2))\) in the 
\((r_1, r_2)\)th row of \(T_0\), we mean a \(m\)-component vector of 
levels whose elements are the ordered two-plets obtained 
by pairing the levels of \(A_{1k}\) in the level combination 
\(J_1(1)\) in \(T_1\) with the corresponding levels of \(A_{2k}\) in the 
level combination \(J_2(2)\) in \(T_2\), \(1 \leq k \leq m\). So \(T_0\) can be 
considered as a fraction of a \(m\)-factor factorial experi­
ment \(E_0\) with the \(k\)th factor \(A_k\) having \((s_{1k} s_{2k})\) levels, 
which are given by the ordered two-plets \((j_k(1), j_k(2))\), 
\(0 \leq j_k(i) \leq (s_{1k} - 1), 1 \leq k \leq m, 1 \leq i \leq 2\).

In Section 8.2, we have considered the analysis 
of \(T\) and in Section 8.3 we have established a similarity 
relationship between \(T\) and \(T_0\) by using pseudo-factors and 
hence its analysis can be performed from that of \(T\). In 
Section 8.4, we have introduced decomposability of the 
orthogonal arrays and established its close relation with 
symbolic product orthogonal arrays. For this class of 
decomposable orthogonal arrays, we have seen that Rao's 
inequality (1947, 1973) can be improved upon a lot.

8.2 Analysis of Symbolic Direct Product Designs

Let
\[
\theta_1 = \{ \theta_1(1), \ldots, \theta_m(1) \} / 0 \leq \theta_{k1} \leq s_{1k1} - 1
\]

and
be respectively the vectors of normalised factorial effects and the vector of treatment effects at the different level combinations (both arranged lexicographically) for the factorial experiment $E_i$ where $v_i = \prod_{k_i=1}^{m_i} s_{ik_i}$.

Then from (3.3.3) we can write

$$
t_i = P_i' \Theta_i \quad \ldots (8.2.1)
$$

where $P_i$ is $v_i \times v_i$ orthogonal matrix with the first row as $v_i^{-1/2}(1,1,\ldots,1)$. The vectors of treatment effects, the factorial effects and the relationship between them for $E_i$ can be written respectively as

$$
t = t_1 \otimes t_2, \Theta = \Theta_1 \otimes \Theta_2, \quad t = (P_1' \times P_2') \Theta \quad \ldots (8.2.2)
$$

where 'X' denotes the Kronecker product.

Let

$$
\Lambda_i = \left\{ \theta(a_{r1}^{(i)}, \ldots, a_{rm_1}^{(i)}) / 1 \leq r_i \leq w_i \right\} \quad \ldots (8.2.3)
$$

be the vector of factorial effects containing any set of $w_i$ effects (not necessarily all those effects belonging upto a certain order interactions, which is the usual practice in fractional designs) which are estimable from $T_i$. Other effects in $\Theta_i$ are assumed to be absent.
Let $Y_1$ be the vector of observations corresponding to the level combinations in $T_1$, i.e.,

$$Y_1 = \{y(T_1)^{j}\} = \{y(J_{T_1}^{r_1}(i)) / 1 \leq r_1 \leq n_1\} \quad \ldots(8.2.4)$$

where $y$ indicates observations on the level combinations.

The linear model for the design $T_1$ can be written as

$$E(Y_1) = X_1 \Lambda_1 \quad \ldots(8.2.5)$$

where the design matrix $X_1$ is obtained from $J_1$ in (8.2.1) by retaining only those rows corresponding to the level combinations $J_{T_1}^{r_1}(i), 1 \leq r_1 \leq n_1$, in $T_1$ and the columns corresponding to $w_1$ effects in $A_1$.

From (8.2.5) we get

$$\Lambda_1 = (X_1^T X_1)^{-1} X_1^T Y_1 \quad \ldots(8.2.6)$$

Next we define the following quantities

$$\bar{\Lambda} = \Lambda_1 \otimes \Lambda_2 = \{\theta(\alpha_{r_1}^{1}(1), \ldots, \alpha_{r_1}^{m_1}(1), \alpha_{r_1}^{r_2}(2), \ldots, \alpha_{r_2}^{m_2}(2) / 1 \leq r_1 \leq w_1, 1 \leq i \leq 2\}$$

$$X = X_1 \times X_2$$

$$Y = \{y(T)\} = \{y(T_1 \otimes T_2)\} \quad \ldots(8.2.7)$$

From (8.2.2), (8.2.5) and (8.2.7) the linear model for $Y$, when the effects in $\Theta$ other than those
in \( \overline{X} \) are negligible can be written as

\[ E(Y) = (X_1 \times X_2) \overline{X} \]  ...(8.2.8)

which gives the estimate of as

\[ \hat{\overline{X}} = \left[ \left\{ (X_1')^{-1}x_1' \right\} \times \left\{ (X_2')^{-1}x_2' \right\} \right] Y \]  ...(8.2.9)

Next we shall give an artificial expression

of \( \hat{\overline{X}} \) in terms of \( \hat{\overline{X}}_1 \) and \( \hat{\overline{X}}_2 \).

Write \( Y \) formally as

\[ Y = Y_1 \times Y_2 \]  ...(8.2.10)

where \( Y_1 \times Y_2 \) is to be understood as \( y(T_1 \times T_2) \) i.e the vector of observations on the level combinations obtained by taking direct product of \( T_1 \) and \( T_2 \). Then (8.2.5), (8.2.9), (8.2.10) and the definition of 'asterisk' product (Kurkijian and Zelen 1962) imply that

\[ \hat{\overline{X}} = \left[ (X_1')^{-1}x_1' Y_1 \right] * \left[ (X_2')^{-1}x_2' Y_2 \right] \]  ...(8.2.11)

\[ \Rightarrow \hat{\overline{X}} = \hat{\overline{X}}_1 * \hat{\overline{X}}_2 \]  ...(8.2.12)

This gives a formal way of writing the estimate

of the estimable parameter \( \overline{X} \) from \( T \) in terms of those from the component designs \( T_1 \) and \( T_2 \). When \( i \) exceeds 2, the extension is immediate and we get the following theorem.
Theorem 8.2.1

Let for $1 \leq i \leq p$, $\alpha_i$ be the vector of effects which is estimable from a fractional design (unblocked) of a $X^{s_1k_1}$ experiment $E_i$. Then the vector of effects $\alpha = \alpha_1 \times \ldots \times \alpha_p$ is estimable from the fractional design $T = T_1 \times \ldots \times T_p$ of a $X^{m_1s_1k_1 \times \ldots \times m_ps_pk_p}$ factorial experiment $E$. Again if $\hat{\alpha}_i$ be the best linear unbiased estimate (BLUE) of $\alpha_i$, then the BLUE of $\alpha$ can be formally written as

$$\hat{\alpha} = \hat{\alpha}_1 \times \hat{\alpha}_2 \times \ldots \times \hat{\alpha}_p$$

As an immediate consequence of Theorem 8.2.1, we get the following Corollaries.

Corollary 8.2.1

Let $\text{disp}(Y) = \sigma^2 I_n$, where $n = \prod_{i=1}^{p} n_i$, then

$$\text{Disp}(\hat{\alpha}) = \sigma^2 \left( D_1^{-1} X \ldots X D_p^{-1} \right)$$

where $D_i = (X_i^T X_i)$, $1 \leq i \leq p$. 
The Corollary follows from (8.2.9) and the standard results of Kronecker product of matrices.

Corollary 8.2.2

If $T_i$ be a saturated plan, then so is $T$.

Proof:

We note that when $\Lambda_i$ (containing $w_i$ effects) is estimable from $T_i$, $1 \leq i \leq p$, then $\Lambda = \Lambda_1 \times \ldots \times \Lambda_p$ is estimable from $T$ and it is obvious that $\Lambda$ contains $w = \prod_{i=1}^{p} w_i$ effects. This implies the Corollary.

Let $S_i$ be the class of designs with $n_i$ level combinations with $m_i$ factors allowing estimation of $\Lambda_i$ and $S$ be the class of designs obtained by taking the symbolic direct product of the designs in $S_i$, $i=1,2,\ldots,k$, i.e. if a design $T_i \in S_i$, then

$$S = S_1 \times \ldots \times S_k = \left\{ T/T_i=T_1 \times \ldots \times T_k, T_i \in S_i, 1 \leq i \leq k \right\}.$$

With this set-up, we get the following Corollary.

Corollary 8.2.3

If $T_i$ be $\phi_p$-optimal in $S_i$ (Kiefer 1974), $1 \leq i \leq k$, then $T$ is also $\phi_p$-optimal in $S$ for estimating $\Lambda$. 
Proof:

Let \( \text{disp} (Y_i) = \sigma_i^2 I_{n_i} \), so that

\[
\text{disp} (\hat{X}_i) = \sigma_i^2 \cdot D_i^{-1}
\]

where \( D_i = (X_i' X_i)^{-1} \), \( 1 \leq i \leq k \).

Also let \( \mu_{i1}, \mu_{i2}, \ldots, \mu_{iw_i} \) be the eigen-values of \( D_i \), then a design \( T_i \in \mathcal{S}_i \) will be said to be \( \mathcal{P}_p \)-optimal in \( \mathcal{S}_i \) if and only if it minimizes (Kiefer, 1974)

\[
\varphi_p(D_i) = \left[ \frac{1}{w_i} \sum_{j=1}^{w_i} \mu_{ij} \right]^{\frac{1}{p}}, \quad 0 < p < \infty
\]

The eigen-values of \( D = D_1 \times \cdots \times D_k \) are given by

\[
\prod_{i=1}^{k} \mu_{ij_i}, \quad 1 \leq j_i \leq w_i,
\]

and hence

\[
\varphi_p(D) = \left[ \left( \prod_{i=1}^{k} \mu_{ij_i} \right)^{-1} \sum_{j_1=1}^{w_1} \cdots \sum_{j_k=1}^{w_k} \left( \prod_{i=1}^{k} \mu_{ij_i} \right)^{-p} \right]^{\frac{1}{p}}, \quad 0 < p < \infty
\]

which is minimum if each of \( \left[ w_i^{-1} \sum_{j_1=1}^{w_i} \mu_{ij_i} \right]^{\frac{1}{p}}, \quad 1 \leq i \leq k \), is minimum. Hence the result.

8.3 Analysis of Symbolic Product Designs

From (8.1.2) and (8.1.3), we note that the SDP generates fractional designs for factorial experiments.
with increased number of factors, while symbolic product (which can be defined only when \( m_1 = m_2 \)) generates fractional experiments with the same number of factors but with increased number of levels for each factor.

Though the two products appear to be of different nature, it can be shown by the use of pseudo-factors that they are very closely related in the sense that one can be obtained from the other (when \( m_1 = m_2 \)) and there is an one to one correspondence between the level combinations generated by the two products.

Now by the use of pseudo-factors we shall establish a similarity relationship between the two products and also among the estimable effects.

From (8.1.2) and (8.1.3), the \( (r_1, r_2) \)th row of \( T \) and \( T_0 \) are respectively given by

\[
J_{r_1 r_2} = \begin{bmatrix} j_{r_1 1} (1), \ldots, j_{r_1 m} (1), j_{r_2 1} (2), \ldots, j_{r_2 m} (2) \end{bmatrix} \quad \ldots (8.3.1)
\]

\[
J_{r_1 r_2} (0) = \begin{bmatrix} (j_{r_1 1} (1), j_{r_2 1} (2)), \ldots, (j_{r_1 m} (1), j_{r_2 m} (2)) \end{bmatrix} \quad \ldots (8.3.2)
\]

We assume that the \( (s_{1k} s_{2k}) \) levels given by the ordered two-plets \( (j_k (1), j_k (2)) \) of the kth factor \( A_k \) of the \( X_{s_{1k} s_{2k}} \) factorial experiment \( E_0 \) can be replaced by the \( (s_{1k} s_{2k}) \) level combinations of the two pseudo-factors \( A_{1k} (0) \) and \( A_{2k} (0) \), \( 0 \leq j_k (i) \leq (s_{1k} - 1), 1 \leq k \leq m \).
Then given the combination $J_{r_1r_2}(0)$ in (8.3.2) of $T_0$, we can replace it by the level combination $J_{r_1r_2}$ of $T$ in terms of the $2m$ pseudo-factors $A_{ik}(0), 1 \leq i \leq 2, 1 \leq k \leq m$. Again, given the level combination $J_{r_1r_2}$ in (8.3.1) of $T$, we can replace it by the level combination $J_{r_1r_2}(0)$ of $T_0$ if we form ordered pairs of the levels of $A_{1k}$ and $A_{2k}$ and assume that the $(s_{1k}s_{2k})$ level combinations of these factors are the $(s_{1k}s_{2k})$ levels (given by the order two-plets) of a pseudo-factors $A_k(0), 1 \leq k \leq m$.

Therefore, we see that, by use of pseudo-factors, an one to one correspondence can be established among the level combinations in $T$ and $T_0$ and in that case, we call the fractions $T$ and $T_0$ to be similar and denote this by

$$T \sim T_0 \quad \ldots \quad (8.3.3)$$

Again, it is easy to verify that the same type of similarity relation holds among $\Theta$ and $\Theta_0$, and also among $\Lambda$ and $\Lambda_0$, i.e.

$$\Theta \sim \Theta_0; \quad \Lambda \sim \Lambda_0 \quad \ldots \quad (8.3.4)$$

where

$$\Theta_0 = \Theta_1 \otimes \Theta_2 = \{ \Theta_0 \ [(\alpha_1(1), \alpha_1(2)), \ldots, (\alpha_m(1), \alpha_m(2)) \} \quad \ldots \quad (8.3.5)$$
and
\[ \Lambda = \Lambda_1 \odot \Lambda_2 = \{ \Theta_0 (\alpha_{r_1}(1), \alpha_{r_2}(2)), \ldots, (\alpha_{r_m}(1), \alpha_{r_2m}(2)) \} \]

\[ \cdots \text{(8.3.6)} \]

So if we replace the \( m \) factors \( A_k \) by \( 2m \) pseudo-factors \( A_{1k}(0) \) and \( A_{2k}(0), 1 \leq k \leq m \) and thereby replace \( T_0 \) by \( T \), then the linear model \((8.2.8)\) holds good for \( T_0 \) as well (in terms of the pseudo-factors). Thus the BLUE of \( \Lambda_1 \otimes \Lambda_2 \) of the pseudo-factorial experiment can be obtained. Applying the similarity relation \((8.3.4)\), we can easily get the BLUE of \( \Lambda_1 \otimes \Lambda_2 = \Lambda_0 \), given in \((8.3.6)\), which is the vector of effects in terms of the original factors \( A_k, 1 \leq k \leq m \), i.e. to be precise, the conversion rule among the estimates of the pseudo-factorial effects, which is to be followed is

\[ \hat{\Theta} [\alpha_1(1), \ldots, \alpha_m(1), a_1(2), \ldots, a_m(2)] \]

\[ = \hat{\Theta}_0 [(\alpha_1(1), a_1(2)), \ldots, (\alpha_m(1), a_m(2))]. \]

So by the use of pseudo-factors, the analysis of \( T_0 \) has been translated into that of \( T \) and hence all the properties of the estimates obtained from \( T \) in Section 8.2, hold good for those obtained from \( T_0 \).
8.4 Lower Bound For The Number of Level Combinations

For A Class Of Orthogonal Arrays

Orthogonal arrays with variable number symbols (QAVS) in different rows were defined by Rao (1973) as a generalization of orthogonal arrays (OA) with the same number of symbols (Rao 1947) and lower bounds for the number of runs were also obtained. Below we define a class of orthogonal arrays (decomposable).

Definition 8.4.1

An QAVS \([n, m, \prod_{l=1}^{m} s_{1k} s_{2k}], d]\) = A, where

\[ n = n_1 n_2 \]

is said to be decomposable if the set of \(n\) rows of A can be partitioned into \(n_1\) sub-sets each of \(n_2\) rows such that (i) each sub-set is an QAVS \([n_2, m, \prod_{l=1}^{m} s_{2k}], d_2\], \(d_2 \geq d\) (ii) the QAVS's corresponding to different sub-sets are mutually isomorphic.

Again the following theorem can be proved very easily.

Theorem 8.4.1

An QAVS \([n, m, \prod_{l=1}^{m} s_{1k} s_{2k}], d]\) is decomposable if and only if the array is a symbolic product of two QAVS's \(A_i [n_i, m, \prod_{l=1}^{m} s_{1k}], d_i]\), \(i = 1, 2\), \(d = \min (d_1, d_2)\).
Theorem 8.4.2

For the class of decomposable QAVS's, the lower bound for \( n \) is given by

\[ n \geq \max (n_1w_2, n_2w_1) \geq w_1w_2 \]  

where \( w_1 \) = number of estimable effects from \( A_1 \) and \( w_1w_2 \) = number of estimable effects from \( A \).

We write \( s_k = s_{1k} s_{2k}, 1 \leq k \leq m \). Then for the OAVS \( [n, m, X s_k, d] = A \), the Rao's lower bound (1973) for \( n \) is given by

\[ n \geq 1 + P_1 + \ldots + P_t \quad \text{when } d = 2t \]
\[ n \geq 1 + P_1 + \ldots + P_t + Q \quad \text{when } d = 2t+1 \]  

where

\[ P_r = \sum_{1 \leq i_1 < \ldots < i_r \leq m} (s_{i_1} - 1) \ldots (s_{i_r} - 1), 1 \leq r \leq t \]  

\[ Q = \max Q_i, Q_i=(s_{i_1} - 1) \sum_{1 \leq i_1 < \ldots < i_t \leq m} (s_{i_1} - 1) \ldots (s_{i_t} - 1), \quad i \neq i_1, \ldots, i_t \]  

The right hand side of (8.4.2) actually gives the number of estimable effects from the OAVS \( A \). We shall show that the number estimable effects \( w \) from \( A \) does not exceed the number of estimable effects \( w_1w_2 \) when \( A \) is decomposable. For simplicity, we assume that
A = OA \[n \times m \times s \times 2\] is decomposable into OA's

\[A_i = OA \[n_i \times m \times s_i \times 2\], i = 1, 2.\]

\(A_1\) being an orthogonal array of strength two, from the standard results (cf Rao 1947) all the main-effects and the mean estimable, i.e \(\Theta(0, \ldots, 0), \Theta(\alpha_1, 0, \ldots, 0), \Theta(0, \alpha_1, 0 \ldots 0), \ldots, \Theta(0, 0, \ldots, \alpha_1), \alpha_1 = 1, 2, \ldots, s_i - 1, 1 \leq i \leq 2,\)

are estimable. Then

\[w_1 = \sum_{j=0}^{1} \binom{m}{j}(s_i - 1)^j, 1 \leq i \leq 2 \quad \ldots (8.4.5)\]

Now \(A\) being decomposable, it can be expressed as the symbolic product of \(A_1\) and \(A_2\) and from (8.3.7), \(w_1 w_2\) effects given by

\[w = \sum_{j=0}^{1} \binom{m}{j}(s_1 s_2 - 1) \quad \ldots (8.4.7)\]

are estimable, which include the mean, all main-effects and some two-factor interactions of the \((s_1 s_2)^m\) factorial experiment.

But from (8.4.2) we get

\[w_1 w_2 \geq w \quad \ldots (8.4.8)\]
which implies that the bound given in (8.4.1) is sharper than that given in (8.4.2) when \( d_1 = d_2 = d = 2 \).

This can be proved to be true for any general strength and also for the asymmetric case. This is some sort of analogous attempt as of improving Fisher's inequality for the case of resolvable BIBD's (Bose 1942).

We also get a mathematical inequality from (8.4.8) where \( w_i \) and \( w \) are taken from the right hand side of (8.4.2) for different \( d \)'s. For simplicity if \( d_1 = d_2 = d = 2t \), (8.4.8) gives

\[
\sum_{j=0}^{t} \binom{m}{j}(s_1-1)^j \sum_{j=0}^{t} \binom{m}{j}(s_2-1)^j \geq \sum_{j=0}^{t} \binom{m}{j}(s_1s_2-1)^j
\]

\[\ldots(8.4.9)\]

8.5 Examples

Example 8.5.1

Let \( T_i, i = 1,2, \) be an orthogonal array \([n_i, m_i, s_i, g_i + t_i]\), so that it gives a \((g_i, t_i)\) plan (i.e all \( g_i \) and less factor effects are estimable, when \( t_i + 1 \) and higher factor effects are negligible, \( t_i \geq g_i \)) of a symmetrical \( s_i \) factorial experiment. Also let

\[\Lambda_i' = (\Lambda_{i1}', \Lambda_{i2}')\]

\[\ldots(8.5.1)\]
where $\overline{X}_{11}$ is the vector of all $g_1$ and less factor effects and $\overline{X}_{12}$ is the vector of all $g_1 + 1, g_1 + 2, \ldots, t_1$ factor effects. All other effects are assumed to be negligible. $X_1$ can be similarly partitioned as

$$X_1 = (x_{11}, x_{12}) \ldots (8.5.2)$$

where the columns of $X_{1k}$ correspond to the elements in $\overline{X}_{1k}, 1 \leq k \leq 2$. We write the following vectors and matrices

$$\overline{X}' = (\overline{X}'_{11} \otimes \overline{X}'_{21}, \overline{X}'_{11} \otimes \overline{X}'_{22}, \overline{X}'_{12} \otimes \overline{X}'_{21}, \overline{X}'_{12} \otimes \overline{X}'_{22})$$

$$x' = (x_{11}' \times x_{21}', x_{11}' \times x_{22}', x_{12}' \times x_{21}', x_{12}' \times x_{22}') \ldots (8.5.3)$$

$$D_1 = X_1 x_1 = \begin{bmatrix} D_{11}(i), & D_{12}(i) \\ D_{21}(i), & D_{22}(i) \end{bmatrix} \ldots (8.5.5)$$

where

$$D_{kk'}(i) = x_{ik} x_{ik'}, 1 \leq k, k' \leq 2 \ldots (8.5.6)$$

and $T_1$ being an orthogonal $(g_1, t_1)$ plan, it is known that

$$D_{12}(i) = D_{21}(i) = 0, D_{11}(i) = \text{Diag} \left( \frac{n_1}{v_1}, \ldots, \frac{n_1}{v_1} \right) \ldots (8.5.7)$$

$D_{22}(i)$ is not necessarily non-singular and $v_i = s_i'$. 
Then from (8.5.3) - (8.5.7) and from the results of Section 8.2, we get that
\[ D \cdot \mathcal{X} = X' (Y_1 \boxtimes Y_2) \] \tag{8.5.8}
where
\[ D = \text{Diag} \left( D_{11}(1) x D_{11}(2), D_{12}(1) x D_{12}(2), D_{21}(1) x D_{21}(2), D_{22}(1) x D_{22}(2) \right) \] \tag{8.5.9}

and for \( 1 \leq k, k' \leq 2 \), \( D_{kk}(1) x D_{k'k'}(2) \), \((k, k') \neq (1, 1)\),
are not-necessarily non-singular. So from (8.5.8) - (8.5.10) it is obvious that the parameters in \( \mathcal{X}_{11} \boxtimes \mathcal{X}_{22} \) are orthogonally estimable from \( T \) are also variance balanced (both elementwise) in the presence of the nuisance parameters \( \mathcal{X}'_{1k} \boxtimes \mathcal{X}'_{2k'} \), \((k, k') \neq (1, 1)\). Thus \( T \) gives an orthogonal balanced plan for an \( s_1^{m_1} x s_2^{m_2} \) factorial experiment. For \( 1 \leq i \leq p \), the extension is obvious.

Remark 8.5.1

The above example shows how orthogonal fractional plans for asymmetrical factorials of the type \( s_1^{m_1} \times \ldots \times s_p^{m_p} \) can be constructed from symmetrical factorials of...
the type $s_i$, $1 \leq i \leq p$. This was first considered by Chakravorti (1956) and here we have shown how the results can be obtained through a systematic analysis of the product. It is also to be noted that $T$ is an OAVS \( [n, m, x s_i, d] \) where $n = \prod_{l=1}^{P} n_l$, $m = \sum_{l=1}^{P} m_l$, $d = \min \{d_1 + t_1\}$.

Example 8.5.2

Let $T_1$ and $T_2$ be an orthogonal array (OA) and a balanced array (BA) with parameters $(n_1, m_1, 2, 2)$ and $(n_2, m_2, 2, 2)$ respectively.

Also let

\[ \mathbf{\bar{A}}_1' = [\Theta_1(0, \ldots, 0), \Theta_1(1, 0, \ldots, 0), \ldots, \Theta_1(0, \ldots, 0, 1)] \]
and

\[ \mathbf{\bar{A}}_2' = [\Theta_2(0, \ldots, 0), \Theta_2(1, 0, \ldots, 0), \ldots, \Theta_2(0, \ldots, 0, 1)] \]

be estimable from $T_1$ and $T_2$ respectively. Then from the fraction $T = T_1 \otimes T_2$, the effects in $\mathbf{\bar{A}} = \mathbf{\bar{A}}_1 \otimes \mathbf{\bar{A}}_2$ are estimable, where

\[ \mathbf{\bar{A}}' = [\Theta(0, 0, \ldots, 0, 0), \Theta(0, 0, 1, 0, \ldots, 0), \ldots, \Theta(0, \ldots, 1, 0, \ldots, 0, 1)] \]

\[ = [(\mu, A_{21}, \ldots, A_{2m_2}), (A_{11}, A_{11} A_{21}, \ldots, A_{11} A_{2m_2}), \ldots, (A_{1m_1}, A_{1m_1} A_{21}, \ldots, A_{1m_1} A_{2m_2})]. \]
\[ = 160 = \]

\( \mu \) is general mean, \( A_{1k_1} \) is the main-effect of the \( k_1 \)th factor in \( T_1 \), \( A_{1k_1} A_{2k_2} \) is the interaction of \( k_1 \)th factor in \( T_1 \) and the \( k_2 \)th factor in \( T_2 \). If \( \text{disp}(Y_1) = \sigma^2 I_{n_1} \), then it is known that

\[ \text{Disp} \left( \frac{\hat{\mu}}{n_1} \right) = \sigma^2 n_1^{-1} \cdot v_1 \cdot I_{(m_1+1)}, \]

\[ \text{Disp} \left( \frac{\hat{\mu}}{n_2} \right) = \sigma^2 B, \quad \cdots (8.5.11) \]

where

\[ B = \begin{bmatrix} a & b \cdot 1_2^T \\ b \cdot 1_2 & (a-c) I_2 + E_2 \end{bmatrix} \quad \cdots (8.5.12) \]

\( v_1 = 2^{m_1} \), \( E_2 = m_2 \times m_2 \) matrix with all elements unity.

\( I_2 = m_2 \times m_2 \) identity matrix.

\( 1_2 = (1, 1, \ldots, 1)^{1 \times m_2} \).

\( a, b, \) and \( c \) are some constants depending on the index parameters of \( T_2 \) (cf. Srivastava 1970).

Then Corollary (8.2.1) implies that

\[ \text{Disp} \left( \frac{\hat{\mu}}{n_1} \right) = n_1^{-1} \cdot v_1 \cdot \sigma^2 \cdot \text{diag} (B, B, \ldots, B), \quad \cdots (8.5.13) \]

where

\[ \text{Disp} (Y) = \sigma^2 I_{n_1 n_2}. \]

**Example 8.5.2**

Let \( T_0 = T_2 \odot T_1 \), where \( T_1 \) and \( T_2 \) are the same as in Example 8.5.1 with \( m_1 = m_2 = m \) (say) and
\( \Delta_i = [\Phi_i(0, \ldots, 0), \Phi_i(1, 0, \ldots, 0), \ldots, \Phi_i(0, 0, \ldots, 1)] \),
i = 1, 2. Then from (8.3.7), the estimable effect \( \Delta_0 \) from \( T_0 \) is given by

\[ \Delta_0' = [\Phi_0((0, 0), \ldots, (0, 0)), \ldots, \Phi_0((0, 0), \ldots, (1, 1))] \]

and the dispersion matrix of \( \Delta_1 \) is given by

\[
\text{Disp}(\Delta_0') = \sigma^2 n_1 \cdot V_1
\]

\[
= \begin{bmatrix}
a.I, & b.I, & b.I, & \ldots, & b.I \\
b.I, & a.I, & c.I, & \ldots, & c.I \\
\vdots & & & & \\
b.I, & c.I, & c.I, & \ldots, & a.I
\end{bmatrix}
\]

\( \ldots (8.5.14) \)

where \( I \) is an identity matrix of order \((m+1)\).