CHAPTER-3

ON \((Np_h^\infty)\) SUMMABILITY OF A FOURIER SERIES AND ITS CONJUGATE SERIES
ON (N, \( p_n^\alpha \)) SUMMABILITY OF A FOURIER SERIES AND CONJUGATE SERIES

3.1 If \( \sum_{n=0}^{\infty} a_n \) is a series, we shall use the notation

\[
S_n = \sum_{r=0}^{n} a_r
\]

Let \( \{p_n\} \) be a sequence with \( p_0 > 0 \) and \( p_n \geq 0 \) for \( n > 0 \) for \( \alpha \) real; we define

\[
P^\alpha_n = \sum_{r=0}^{\infty} p^\alpha_r
\]

where

\[
P_n = \sum_{r=0}^{n} p_r
\]

Definition [Cass 1969]: Nörlund summability \((N,p_n^\alpha)\) for \( \alpha > 0 \) and \( \sum_{r=0}^{\infty} a_r \) a series.

Let

\[
t_n^{\alpha} = \frac{1}{n^{\alpha}} \sum_{r=0}^{n} p_r^{\alpha} S_r
\]

If \( t_n^{\alpha} \to s \), as \( n \to \infty \),

we write

\[
\sum_{r=0}^{\infty} a_r = S(N,p_n^\alpha)
\]

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or

\[ S_n \rightarrow S\left(N, p^\alpha_n\right) \]

### 3.2

Let \( f(t) \) be a \( 2\pi \)-periodic and Lebesgue integrable function of \( t \) in the interval \((-\pi, \pi)\). Then the Fourier series of \( f(t) \) is given by

\[
\begin{align*}
& f(t) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \left( a_n \cos nt + b_n \sin nt \right) \\
& = \sum_{n=0}^{\infty} A_n(t)
\end{align*}
\]

The conjugate series of (3.2.1) is given by

\[
\sum_{n=1}^{\infty} \left( b_n \cos nt - a_n \sin nt \right) = \sum_{n=1}^{\infty} B_n(t)
\]

We write

\[
\begin{align*}
(3.2.3) \quad & \phi(t) = f(x + t) + f(x - t) - 2f(x) \\
(3.2.4) \quad & \psi(t) = f(x + t) - f(x - t) \\
(3.2.5) \quad & \Phi(t) = \int_0^t |\phi(u)| \, du, \\
(3.2.6) \quad & \Psi(t) = \int_0^t |\psi(u)| \, du, \\
(3.2.7) \quad & \tau = \left[ \frac{1}{t} \right], \text{ the integral part of } \frac{1}{t}. \\
(3.2.8) \quad & \bar{f}_n(x) = \frac{1}{2\pi} \int_0^\pi \psi(t) \cot \frac{t}{2} \, dt \\
\end{align*}
\]

and

\[
\begin{align*}
(3.2.9) \quad & \bar{f}(x) = \lim_{n \to \infty} \bar{f}_n(x), \\
\end{align*}
\]

when ever the latter exists.
3.3 In 1961 (Pati [37]) has studied the Norlund summability of the Fourier series (3.2.1). The object of the present chapter to develop the result of (Pati [37])

For generalized Norlund summability denoted \((N, p_n^\alpha)\) summability, for \(\alpha > -1\) by establishing the following two theorems.

**THEOREM 1.** For \(\alpha > -1\), Let \(\{p_n^\alpha\}\) be a non-negative monotonic non-increasing sequence of real constants such that its non-vanishing n-th partial sum \(P_n^\alpha\) given by (3.1.2) tends to infinity as \(n \to \infty\)

Let \(\lambda(t)\) and \(\mu(t)\) be two positive functions of \(t\) such that \(\lambda(t), \mu(t)\) and \(\frac{t\lambda(t)}{\mu(t)}\) increase monotonically with \(t\) and

\[
\lambda(n)p_n^\alpha = O\left[\mu(p_n^\alpha)\right], \quad \text{as } n \to \infty
\]

If

\[
\Phi(t) = \int_0^t |\phi(u)|du = o\left[\frac{\lambda(\frac{1}{t})p_{\tau}^\alpha}{\mu(p_{\tau}^\alpha)}\right],
\]

as \(t \to +\infty\), then the Fourier series of \(f(t)\) at \(t = x\) is summable \((N, p_n^\alpha)\) to \(f(x)\).

**THEOREM 2.** If \(\alpha > -1\), the sequence \(\{p_n^\alpha\}\) and the functions \(\lambda(t)\) and \(\mu(t)\) the same as theorem 1, then if
(3.3.3) \( \Psi(t) = \int_0^t |\psi(u)| \, du = o \left[ \frac{\lambda(t)}{\mu(P_t)} \right] \)

as \( t \to +\infty \), the conjugate series (2.2.2) is summable \((N, p^\alpha)\) to

\[
\frac{1}{2\pi} \int_0^\pi \psi(t) \left( \cot \frac{t}{2} \right) \, dt
\]

at every point where the integral exists.

3.4 We shall use the following lemmas in the proof of our theorems.

**LEMMA 1.** If \( \\{p_n\} \) is a non-negative and non-increasing sequence of constants, then for \( 0 < \alpha \leq b \leq \infty \), \( 0 < t \leq \pi \) and any \( n \).

\[
\left| \sum_{k=\alpha}^{b} p_k e^{i(n-k)t} \right| = O \left( P_t^{\alpha} \right)
\]

The proof of the lemma follows on the lines of (Mc Fadden [33])

**LEMMA 2.** If \( \\{p_n^{\alpha}\} \) is a non-negative and non-increasing sequence, then for \( \frac{1}{\alpha} < t \leq \delta < \pi \), we have

\[
|K_n(t)| = \left| \sum_{k=0}^{n} p_k^{\alpha} \frac{\sin \left( n - k + \frac{1}{2} \right) t}{\sin \frac{t}{2}} \right| = O \left[ \frac{P_t^{\alpha}}{t} \right]
\]

[65]
3.5 PROOF OF THEOREM 1.

Let $\delta$ be a fixed positive number less than $\frac{1}{2}$. If $\sigma_n(x)$ denote the $(n+1)^{th}$ partial sum of the series (3.2.1), then it is well known that

$$\sigma_n(x) - f(x) = \frac{1}{2\pi} \int_0^\pi \phi(t) \frac{\sin \left( n + \frac{1}{2} \right) t}{\sin \frac{1}{2} t} dt$$

Hence, if $T_n^\alpha(x)$ denotes the $(N, p_n^\alpha)$ mean of the sequence $\{\sigma_n(x)\}$, then by application of (3.1.3), we have

$$T_n^\alpha(x) - f(x) = \frac{1}{p_n^\alpha} \sum_{v=0}^n p_v^\alpha \{S_{n-v}(x) - f(x)\}$$

$$= \frac{1}{2\pi p_n^\alpha} \int_0^\pi \phi(t) \left\{ \sum_{v=0}^n p_v^\alpha \frac{\sin \left( n - v + \frac{1}{2} \right) t}{\sin \frac{1}{2} t} \right\} dt$$

$$= \int_0^\pi \phi(t) N_n(t) dt, \text{ say,}$$

where

$$N_n(t) = \frac{1}{2\pi p_n^\alpha} \sum_{v=0}^n p_v^\alpha \frac{\sin \left( n - v + \frac{1}{2} \right) t}{\sin \frac{1}{2} t}$$
Let us write

\[ T_n^\alpha (x) - f(x) = \left\{ \int_0^{\frac{1}{n}} + \int_{\frac{1}{n}}^{\frac{\delta}{n}} + \int_\delta^\pi \right\} \phi(t)N_n(t)dt \]

(3.5.1) \[ = I_1 + I_2 + I_3, \quad \text{say} \]

In order to prove our theorem, we have to show that

\[ T_n^\alpha (x) - f(x) = o(1), \quad \text{as} \quad n \to \infty \]

(3.5.2) Now, it is easy to see that

\[ |N_n(t)| = O(n) \]

(3.5.3) as \( n \to \infty, \quad 0 < t \leq \frac{1}{n} \), and making use of lemma 2, we have

\[ \frac{1}{n} < t \leq \delta < \pi, \]

(3.5.4) \[ |N_n(t)| = O\left[ \frac{P_n^\alpha \lambda(n)p_n^\alpha}{\mu(p_n^\alpha)} \right] \]

Let us consider first \( I_1 \). By the application of (3.3.2) and (3.5.3), we have

\[ |I_1| = \left| \int_0^{\frac{1}{n}} \phi(t)N_n(t)dt \right| \]

\[ \leq \int_0^{\frac{1}{n}} |\phi(t)||N_n(t)|dt \]

\[ = O(n) \cdot O\left[ \frac{\lambda(n)p_n^\alpha}{\mu(p_n^\alpha)} \right], \quad \text{as} \quad n \to \infty \]
\[ = o\left[ \frac{\lambda(n)P_n^\alpha}{\mu(P_n^\alpha)} \right], \text{ since } np_n^\alpha \leq P_n^\alpha \]

(3.5.5) \( = o(1) \), by (3.3.1) as \( n \to \infty \)

Next, consider \( I_2 \), we have by application of (3.3.1), (3.3.2) and (3.5.4),

\[ |I_2| = O\left[ \int_{j_n^\delta}^{\delta} \phi(t) |N_n(t)| dt \right] \]

\[ = O\left[ \frac{1}{P_n^\alpha} \right]^{\delta} \phi(t) \frac{P_n^\alpha}{t} dt \]

\[ = O\left[ \frac{1}{P_n^\alpha} \right] \int_{j_n^\delta}^{\delta} \Phi(t) \left( \frac{P_n^\alpha}{t} \right) dt \]

\[ = o[1] + o\left[ \frac{\lambda(n)P_n^\alpha}{P_n^\alpha \mu(P_n^\alpha)} \right] + o\left[ \frac{\lambda(n)P_n^\alpha}{P_n^\alpha \mu(P_n^\alpha)} \right] \int_{j_n^\delta}^{\delta} \frac{P_n^\alpha}{t} dt \]

\[ = o[1] + o\left[ \frac{\lambda(n)P_n^\alpha}{P_n^\alpha \mu(P_n^\alpha)} \right] + o\left[ \frac{\lambda(n)P_n^\alpha}{P_n^\alpha \mu(P_n^\alpha)} \right] \times O(nP_n^\alpha) \]

\[ = o[1] + o(1) + o\left[ \frac{\lambda(n)np_n^\alpha}{\mu(P_n^\alpha)} \right] \]

[68]
Lastly, we have

$$|I_3| = \int_0^\pi \phi(t)N_n(t)dt$$

(3.5.7) = o (1) , as \( n \to \infty \)

By using Riemann-Lebesgue theorem and the regularity of the method of summation.

Hence from (3.5.5), (3.5.6) and (3.5.7), we see that

$$T_n^\alpha(x) - f(x) = o(1), \quad \text{as} \quad n \to \infty,$$

This completes the proof of theorem 1.

**PROOF OF THEOREM 2.** Let \( \sigma_n(x) \) denote the \((n + 1)^{th}\) partial sum of the series (3.3.2). Then we have

$$\overline{\sigma_n(x)} = \frac{1}{2\pi} \int_0^\pi \psi(t) \frac{\cos \frac{t}{2} - \cos \left( \frac{n + \frac{1}{2}}{2} \right) t}{\sin \frac{t}{2}} dt$$

If \( \overline{T_n^\alpha(x)} \) be the \((N, p^\alpha_n)\) mean of the sequence \( \{\overline{\sigma_n(x)}\} \), then we have by the application of (3.13)

$$\overline{T_n^\alpha(x)} - \frac{1}{2\pi} \int_0^\pi \psi(t) \cot \frac{t}{2} dt$$

$$= \frac{1}{p^{\alpha_n}} \sum_{\nu=0}^{n} p^{\nu} \overline{\sigma_{n-\nu}(x)} - \frac{1}{2\pi} \int_0^\pi \psi(t) \cot \frac{t}{2} dt$$

[69]
\[
2^n p = \int_0^\pi \psi(t) \sum_{n=0}^\infty p^n_v \frac{\cos t - \cos \left( n - \nu + \frac{1}{2} \right) t}{\sin \frac{1}{2} t} \, dt - \frac{1}{2\pi} \int_0^\pi \psi \cot \frac{t}{2} \, dt
\]

\[
= -\frac{1}{2\pi} \int_0^\pi \psi(t) \sum_{n=0}^\infty p^n_v \frac{\cos \left( n - \nu + \frac{1}{2} \right) t}{\sin \frac{1}{2} t} \, dt
\]

\[
= \int_0^\pi \psi(t) M_n(t) \, dt, \quad \text{say},
\]

where \( M_n(t) = -\frac{1}{2\pi} \sum_{n=0}^\infty p^n_v \frac{\cos \left( n - \nu + \frac{1}{2} \right) t}{\sin \frac{1}{2} t} \)

In order to prove theorem 2, we have to show that

(3.5.8) \( P = \int_0^\pi \psi(t) M_n(t) dt = o(1), \quad \text{as } n \to \infty \)

Let us write for \( 0 < \delta < \pi \),

\[
P = \int_0^\pi \psi(t) M_n(t) dt
\]

\[
= \left\{ \int_0^\nu + \int_\nu^\pi + \int_\delta^\pi \right\} \psi(t) M_n(t) dt
\]

(3.5.9) \( = J_1 + J_2 + J_3 \), say,

Since by our assumption \( \lim_{n \to \infty} \overline{f}_n(x) = \overline{f}(x) \) exists we have

(3.5.10) \( \frac{1}{2\pi} \int_0^\nu \psi(t) \cos \frac{t}{2} = o(1), \quad \text{as } n \to \infty \),

[70]
Also for all $t$ such that $0 < t \leq \pi$

\[
\frac{1}{2\pi p_n^2} \sum_{v=0}^{n} \frac{p_v^\alpha \cos \frac{t}{2} - \cos \left(n - v + \frac{1}{2}\right)t}{\sin \frac{1}{2}t}
\]

\[
= \frac{1}{2\pi p_n^2} \sum_{v=0}^{n} \frac{p_v^\alpha \sum_{k=0}^{n-v} 2\sin kt}{\sum_{k=0}^{n-v} |\sin kt|}
\]

\[
= O\left[ \frac{1}{p_n^\alpha} \sum_{v=0}^{n} \frac{p_v^\alpha (n - v)}{\sum_{k=0}^{n-v} |\sin kt|} \right]
\]

\[
= O\left[ \frac{1}{p_n^\alpha} \sum_{v=0}^{n} p_v^\alpha (n - v) \right]
\]

(3.5.11) \quad = O(n)

Therefore

\[
|J_1| = \left| \int_0^{\pi/2} \psi(t)M_n(t)dt \right|
\]

\[
= \left| \frac{1}{2\pi p_n^2} \int_0^{\pi/2} \psi(t) \sum_{v=0}^{n} \frac{p_v^\alpha \cos \frac{t}{2} - \cos \left(n - v + \frac{1}{2}\right)t}{\sin \frac{1}{2}t} dt \right|
\]

\[
- \left| \frac{1}{2\pi p_n^2} \sum_{v=0}^{n} p_v^\alpha \int_0^{\pi/2} \psi(t) \cos \frac{t}{2} dt \right|
\]

\[
= O(n)\int_0^{\pi/2} |\psi(t)|dt + o(1), \quad \text{by (3.5.10) and (3.5.11)}
\]

as $n \to \infty$,
\[
\begin{align*}
\frac{\lambda(n)p_n^\alpha}{\mu(p_n^\alpha)} + o(1), \text{ by (3.3.3)} \\
= o\left(\frac{\lambda(n)p_n^\alpha}{\mu(p_n^\alpha)}\right) + o(1)
\end{align*}
\]

(3.5.12) \(= o(1)\) as \(n \to \infty\)

Now, for \(\frac{1}{n} < t \leq \delta < \pi\), we have

\[
M_n(t) = O\left(\frac{p_n^\alpha}{t}\right),
\]

as in lemma 2, so that

\[
J_2 = O\left(\frac{1}{p_n^\alpha}\right) \int_{\frac{\delta}{n}}^{\delta} |\psi(t)| \frac{p_n^\alpha}{t} \, dt
\]

\(= o\ (1), \ \text{as} \ n \to \infty. \ \text{as in the case of} \ I_2 \ \text{in theorem 1.}\)

Finally

\(J_3 = o\ (1). \ \text{as} \ n \to \infty.\)

by Riemann-Lebesgue theorem and the regularity of method of summation.

This completes the proof of theorem 2.