CHAPTER-4

CERTAIN TRANSFORMATIONS FOR BASIC HYPERGEOMETRIC SERIES

4.1 Introduction :

In this chapter, we shall make use of the following Bailey's transformation:

if $$\beta_n = \sum_{r=0}^{n} \alpha_r u_{n-r} v_{n+r}, \tag{1}$$

and $$\gamma_n = \sum_{r=0}^{\infty} \delta_{r+n} u_r v_{2n+r}, \tag{2}$$

where α_r, δ_r, u_r and v_r are functions of r only such that the series for γ_n exists, then

$$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n. \tag{3}$$

In order to establish certain transformation and summation formulae for Basic Hypergeometric functions, we shall be in need of the following known results:

$$ {}_3\Phi_2\left[\begin{matrix} a, & b, & q; & q; & q \\ & e, & f & & \end{matrix}\right]_N = \frac{(q-e)(e-abq)}{(aq-e)(e-bq)} \left\{ 1 - \frac{[a, b; q]_{N+1}}{\left[\frac{e}{q}, \frac{abq}{e}; q\right]_{N+1}} \right\}, \tag{4}$$

provided $ef = abq^2$.[1;(4.2)]

$$ {}_4\Phi_3\left[\begin{matrix} a, & q\sqrt{a}, & -q\sqrt{a}, & e; & q; \frac{1}{e} \\ & \sqrt{a}, & -\sqrt{a}, & \frac{aq}{e} & \end{matrix}\right]_N = \frac{[aq,eq;q]_N}{[q,aq/e;q]_N e^N}. \tag{5} $$

[1; App II(23)]

$$ {}_6\Phi_5\left[\begin{matrix} a, & q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d; & q; q \\ & \sqrt{a}, & -\sqrt{a}, & \frac{aq}{b}, & \frac{aq}{c}, & \frac{aq}{d} & \end{matrix}\right]_N $$

$$ = \frac{[aq,bq,cq,dq;q]_N}{[q,aq/b,aq/c,aq/d;q]_N}, \tag{6} $$

with $a = bcd$.

[1;App. II. (25)].

$$ {}_2\Phi_1\left[\begin{matrix} a, & y; & q; & q \\ & ayq & & \end{matrix}\right]_N = \frac{[aq,yq;q]_N}{[q,ayq;q]_N}, \tag{7} $$

[1; App. II (18)]

$$ \sum_{k=0}^{n} \frac{(1-ap^k q^k)[a;p]_k [c;q]_k c^{-k}}{(1-a)[q;q]_k \left[\frac{ap}{c};p\right]_k} = \frac{[ap;p]_n [cq;q]_n c^{-n}}{[q;q]_n \left[\frac{ap}{c};p\right]_n} \tag{8} $$

[8;App. II(34)]

$$ \sum_{k=0}^{n} \frac{(1-ap^k q^k)(1-bp^k q^{-k})[a,b;p]_k [c,a/bc;q]_k q^k}{(1-a)(1-b)\left[q,\frac{aq}{b};q\right]_k \left[\frac{ap}{c},bcp;p\right]_k} $$

$$= \frac{[ap;bp;p]_n [cq,aq/bc;q]_n}{[q,aq/b;q]_n \left[\frac{ap}{c},bcp;p\right]_n} \tag{9}$$

[8; App. II(35)]

$$\sum_{k=0}^{n} \frac{(1-adp^k q^k)\left(1-\frac{b}{d}p^k q^{-k}\right)[a,b;p]_k \left[c,\frac{ad^2}{bc};q\right]_k q^k}{(1-a)\left(1-\frac{b}{d}\right)\left[dq,\frac{adq}{b};q\right]_k \left[\frac{adp}{c};\frac{bcp}{d};p\right]_k}$$

$$= \frac{(1-a)(1-b)(1-c)\left(1-\frac{ad^2}{bc}\right)}{d(1-ad)\left(1-\frac{b}{d}\right)\left(1-\frac{c}{d}\right)\left(1-\frac{ad}{bc}\right)} \times$$

$$\times \frac{[ap,bp;p]_n \left[cq,\frac{adq^2}{bc};q\right]_n}{\left[dq,\frac{adq}{b};q\right]_n \left[\frac{adp}{c},\frac{bcp}{d};p\right]_n} - \frac{(b-ad)(c-ad)(d-bc)(1-d)}{d(1-a)(1-b)(1-c)(bc-ad^2)}, \tag{10}$$

which is the $m = 0$ case of [14; eq. 12. P. 83].

4.2 Main Results :

In this chapter, we shall establish our main results:

Choosing

$u_r = v_r = 1$

and

$$\delta_r = \frac{[a;q]_r [b;q]_r q^r}{[e;q]_r \left[\frac{abq^2}{e};q\right]_r}$$

in (4.1.2) we get

$$\gamma_n = \sum_{r=0}^{\infty} \delta_{r+n} u_r v_{2n+r}$$

$$= \frac{[a;q]_n [b;q]_n q^n}{[e;q]_n \left[\frac{abq^2}{e};q\right]_n} \sum_{r=0}^{\infty} \frac{[aq^n;q]_r [bq^n;q]_r q^r}{[eq^n;q]_r \left[\frac{abq^{n+2}}{e};q\right]_r}$$

$$= \frac{[a;q]_n [b;q]_n q^n}{[e;q]_n \left[\frac{abq^2}{e};q\right]_n} {}_3\Phi_2 \left[\begin{matrix} aq^n, & bq^n, & q; & q; \; q \\ & eq^n, & \frac{abq^{2+n}}{e} & \end{matrix}\right], \qquad (4.2.1)$$

Now, using $N \to \infty$ case of (4.1.4) to sum the ${}_3\Phi_2$ series on the right hand side of (4.2.1) we get :

$$\gamma_n = \frac{\left(1-\frac{q}{e}\right)\left(1-\frac{ab}{e}q\right)}{\left(1-\frac{aq}{e}\right)\left(1-\frac{bq}{e}\right)} \left\{ \frac{[e;q]_n \left[\frac{abq^2}{e};q\right]_n}{q^n \left[\frac{e}{q};q\right]_n \left[\frac{abq}{e};q\right]_n} - \frac{[a,b;q]_\infty \left[e,\frac{abq^2}{e};q\right]_n}{\left[\frac{e}{q},\frac{abq}{e};q\right]_\infty [a,b;q]_n q^n} \right\}$$

..(4.2.2)

Substituting these values of γ_n and δ_n in (4.1.3) we get the following

Master Result :

$$\sum_{n=0}^{\infty} \alpha_n \left\{ \frac{[e;q]_n \left[\frac{abq^2}{e};q\right]_n}{q^n \left[\frac{e}{q};q\right]_n \left[\frac{abq}{e};q\right]_n} - \frac{[a,b;q]_\infty \left[e,\frac{abq^2}{e};q\right]_n}{\left[\frac{e}{q},\frac{abq}{e};q\right]_\infty [a,b;q]_n q^n} \right\}$$

$$=\frac{\left(1-\frac{aq}{e}\right)\left(1-\frac{bq}{e}\right)}{\left(1-\frac{q}{e}\right)\left(1-\frac{abq}{e}\right)}\sum_{n=0}^{\infty}\beta_n\frac{[a;q]_n[b;q]_n q^n}{[e;q]_n\left[\frac{abq^2}{e};q\right]_n}, \tag{4.2.3}$$

(i) Now, Setting.

$$\alpha_r=\frac{[R;p]_r[S;p]_r p^r}{[T;p]_r[U;p]_r},$$

where $UT=RSp^2$.

and $u_r=v_r=1$

in (4.1.1) we get :

$$\beta_n=\sum_{r=0}^{n}\alpha_r u_{n-r}v_{n+r}$$

$$=\sum_{r=0}^{n}\frac{[R;p]_r[S;p]_r p^r}{[T;p]_r[U;p]_r}$$

$$={}_3\Phi_2\begin{bmatrix}R, & S, & p; & p; & p\\ & T, & \frac{RSp^2}{T} & & \end{bmatrix}_n,$$

provided $UT=RSp^2$.

Now, summing ${}_3\Phi_2$ series with the help of (4.1.4) we get :

$$\beta_n=\frac{(p-T)(T-RSp)}{(Rp-T)(T-Sp)}\left\{1-\frac{(1-R)(1-S)}{\left(1-\frac{T}{p}\right)\left(1-\frac{RSp}{T}\right)}\frac{[Rp,Sp;p]_n}{\left[T,\frac{RSp^2}{T};p\right]_n}\right\}. \tag{4.2.4}$$

Substituting these values of α_n and β_n in Master Result (4.2.3) we have :

$$\sum_{n=0}^{\infty}\frac{[R;p]_n[S;p]_n p^n}{[T;p]_n[U;p]_n}\left\{\frac{[e;q]_n\left[\frac{abq^2}{e};q\right]_n}{q^n\left[\frac{e}{q};q\right]_n\left[\frac{abq}{e};q\right]_n}-\frac{[a,b;q]_\infty\left[e,\frac{abq^2}{e};q\right]_n}{\left[\frac{e}{q},\frac{abq}{e};q\right]_\infty[a,b;q]_n q^n}\right\}$$

$$=\frac{\left(1-\frac{aq}{e}\right)\left(1-\frac{bq}{e}\right)(p-T)(T-RSp)}{\left(1-\frac{q}{e}\right)\left(1-\frac{abq}{e}\right)(Rp-T)(T-Sp)}$$

$$\sum_{n=0}^{\infty}\left\{1-\frac{(1-R)(1-S)[Rp,Sp;p]_n}{\left(1-\frac{T}{p}\right)\left(1-\frac{RSp}{T}\right)\left[T,\frac{RSp^2}{T};p\right]_n}\right\}\frac{[a;q]_n[b;q]_n q^n}{[e;q]_n\left[\frac{abq^2}{e};q\right]_n}$$

$${}_5\Phi_4\left[\begin{matrix}R, & S, & p: & e, & \frac{abq^2}{e}: & p, & q; & \frac{p}{q}\\ & T, & \frac{RSp^2}{T}: & \frac{e}{q}, & \frac{abq}{e} & & & \end{matrix}\right]$$

$$-\frac{[a,b;q]_\infty}{\left[\frac{e}{q},\frac{abq}{e};q\right]_\infty}\,{}_5\Phi_4\left[\begin{matrix}R, & S, & p: & e, & \frac{abq^2}{e}; & p, & q; & \frac{p}{q}\\ & T, & \frac{RSp^2}{T}: & a, & b & & & \end{matrix}\right]$$

$$=\frac{\left(1-\frac{aq}{e}\right)\left(1-\frac{bq}{e}\right)(p-T)(T-RSp)}{\left(1-\frac{q}{e}\right)\left(1-\frac{abq}{e}\right)(Rp-T)(T-Sp)}\left\{{}_3\Phi_2\left[\begin{matrix}a, & b, & q; & q: & q\\ & e, & \frac{abq^2}{e} & & \end{matrix}\right]-\right.$$

$$\left.\frac{(1-R)(1-S)}{\left(1-\frac{T}{p}\right)\left(1-\frac{RSp}{T}\right)}\,{}_5\Phi_4\left[\begin{matrix} Rp, & Sp, & p; & a, & b; & p, & q; & q \\ & T, & \frac{RSp^2}{T}: & e, & \frac{abq^2}{e} & & & \end{matrix}\right]\right\},$$

....(4.2.5)

provided |p| and | q | < 1.

(ii) Further, taking

$$\alpha_r = \frac{[X;p]_r\left[p\sqrt{X};p\right]_r\left[-p\sqrt{X};p\right]_r[Y;p]_r}{[p;p]_r\left[\sqrt{X};p\right]_r\left[-\sqrt{X};p\right]_r\left[\frac{Xp}{Y};p\right]_r Y^r}$$

and $u_r = v_r = 1$

in (4.1.1) we get :

$$\beta_n = \sum_{r=0}^{n} \frac{[X;p]_r\left[p\sqrt{X};p\right]_r\left[-p\sqrt{X};p\right]_r[Y;p]_r}{[p;p]_r\left[\sqrt{X};p\right]_r\left[-\sqrt{X};p\right]_r\left[\frac{Xp}{Y};p\right]_r Y^r}$$

$$= {}_4\Phi_3\left[\begin{matrix} X, & p\sqrt{X}, & -p\sqrt{X}, & Y; & p; & \frac{1}{Y} \\ & \sqrt{X}, & -\sqrt{X}, & \frac{Xp}{Y} & & \end{matrix}\right]_n$$

Now, summing ${}_4\Phi_3$ series with the help of (4.1.5) we get :

$$\beta_n = \frac{[Xp;Yp;p]_n}{\left[p,\frac{Xp}{Y};p\right]_n Y^n} \qquad (4.2.6)$$

Substituting these values of α_n and β_n in Master Result (4.2.3) we get :

$$
{}_6\Phi_5\left[\begin{matrix} X, & p\sqrt{X}, & -p\sqrt{X}, & Y: & e, & \frac{abq^2}{e}; & p, & q; & \frac{1}{Yq} \\ & \sqrt{X}, & -\sqrt{X}, & \frac{Xp}{Y}: & \frac{e}{q}, & \frac{abq}{e} & & & \end{matrix}\right]
$$

$$
-\frac{[a,b;q]_\infty}{\left[\frac{e}{q},\frac{abq}{p};q\right]_\infty}\,{}_6\Phi_5\left[\begin{matrix} X, & p\sqrt{X}, & -p\sqrt{X}, & Y: & e, & \frac{abq^2}{e}; & p, & q; & \frac{1}{Yq} \\ & \sqrt{X}, & -\sqrt{X}, & \frac{Xp}{Y}: & a, & b & & & \end{matrix}\right]
$$

$$
=\frac{\left(1-\frac{aq}{e}\right)\left(1-\frac{bq}{e}\right)}{\left(1-\frac{q}{e}\right)\left(1-\frac{abq}{e}\right)}\times{}_4\Phi_3\left[\begin{matrix} Xp, & Yp: & a, & b; & p; & q; & \frac{q}{Y} \\ & \frac{Xp}{Y}: & e, & \frac{abq^2}{e} & & & \end{matrix}\right] \quad (4.2.7)
$$

(iii) Again, taking

$$
\alpha_r=\frac{[X;p]_r\left[p\sqrt{X};p\right]_r\left[-p\sqrt{X};p\right]_r[Y;p]_r[Z;p]_r[U;p]_r p^r}{[p;p]_r\left[\sqrt{X};p\right]_r\left[-\sqrt{X};p\right]_r\left[\frac{Xp}{Y};p\right]_r\left[\frac{Xp}{Z};p\right]_r\left[\frac{Xp}{U};p\right]_r}
$$

and $u_r=v_r=1$

in (4.1.1) we get :

$$
\beta_n=\sum_{r=0}^{n}\frac{[X;p]_r\left[p\sqrt{X};p\right]_r\left[-p\sqrt{X};p\right]_r[Y;p]_r[Z;p]_r[U;p]_r p^r}{[p;p]_r\left[\sqrt{X};p\right]_r\left[-\sqrt{X};p\right]_r\left[\frac{Xp}{Y};p\right]_r\left[\frac{Xp}{Z};p\right]_r\left[\frac{Xp}{U};p\right]_r}
$$

$$={}_6\Phi_5\left[\begin{matrix} X, & p\sqrt{X}, & -p\sqrt{X}, & Y, & Z, & U; & p; & p \\ & \sqrt{X}, & -\sqrt{X}, & \frac{Xp}{Y}, & \frac{Xp}{Z}, & \frac{Xp}{U} & & \end{matrix}\right]_n,$$

provided $X = YZU$

Now, summing ${}_6\Phi_5$ series with the help of (4.1.6) we get

$$\beta_n = \frac{[Xp, Yp, Zp, Up; p]_n}{\left[p, \frac{XP}{Y}; \frac{XP}{Z}, \frac{Xp}{U}; p\right]_n} \qquad (4.2.7)$$

Substituting these values of α_n and β_n in Master Result (4.2.3) we get :

$${}_8\Phi_7\left[\begin{matrix} X, & p\sqrt{X}, & -p\sqrt{X}, & Y, & Z, & U: & e, & \frac{abq^2}{e}; & p, & q; & \frac{p}{q} \\ & \sqrt{X}, & -\sqrt{X}, & \frac{Xp}{Y}, & \frac{Xp}{Z}, & \frac{Xp}{U}: & \frac{e}{q}, & \frac{abq}{e} & & & \end{matrix}\right]$$

$$-\frac{[a, b; q]_\infty}{\left[\frac{e}{q}, \frac{abq}{e}; q\right]_\infty}$$

$$\times {}_8\Phi_7\left[\begin{matrix} X, & p\sqrt{X}, & -p\sqrt{X}, & Y, & Z, & U: & e, & \frac{abq^2}{e}; & p, & q; & \frac{p}{q} \\ & \sqrt{X}, & -\sqrt{X}, & \frac{Xp}{Y}, & \frac{Xp}{Z}, & \frac{Xp}{U}: & a, & b & & & \end{matrix}\right]$$

$$= \frac{\left(1-\frac{aq}{e}\right)\left(1-\frac{bq}{e}\right)}{\left(1-\frac{q}{e}\right)\left(1-\frac{abq}{e}\right)} \times {}_6\Phi_5\left[\begin{matrix} Xp, & Yp, & Zp & Up: & a, & b; & p, & q: & q \\ & \frac{Xp}{Y}, & \frac{Xp}{Z}, & \frac{Xp}{U}: & e, & \frac{abq^2}{e} & & & \end{matrix}\right]$$

....(4.2.8)

(iv) Next, Setting

$$\alpha_r = \frac{[X;p]_r[Y;p]_r p^r}{[p;p]_r[XYp;p]_r}$$

and $u_r = v_r = 1$

in (4.1.1) we get:

$$\beta_n = \sum_{r=0}^{n} \frac{[X;p]_r[Y;p]_r p^r}{[p;p]_r[XYp;p]_r}$$

$$= {}_2\Phi_1\begin{bmatrix} X, & Y; & p; & p \\ & XYp & & \end{bmatrix}_n$$

Now summing ${}_2\Phi_1$ series with the help of (4.1.7) we get

$$\beta_n = \frac{[Xp, Yp; p]_n}{[p, XYp; p]_n} \tag{4.2.9}$$

Substituting these values of α_n and β_n in Master Result (4.2.3) we get :

$${}_4\Phi_3\begin{bmatrix} X, & Y: & e, & \frac{abq^2}{e}; & p, & q; & \frac{p}{q} \\ & XYp: & \frac{e}{q}, & \frac{abq}{e} & & & \end{bmatrix}$$

$$\frac{[a,b;q]_\infty}{\left[\frac{e}{q}, \frac{abq}{e}; q\right]_\infty} {}_4\Phi_3\begin{bmatrix} X, & Y: & e, & \frac{abq^2}{e}; & p, & q; & \frac{p}{q} \\ & XYp: & a, & b & & & \end{bmatrix}$$

$$=\frac{\left(1-\frac{aq}{e}\right)\left(1-\frac{bq}{e}\right)}{\left(1-\frac{q}{e}\right)\left(1-\frac{abq}{e}\right)}\,{}_4\Phi_3\left[\begin{matrix}Xp, & Yp: & a, & b; & p, & q; & q\\ & XYp: & e, & \frac{abq}{e} & & & \end{matrix}\right] \qquad (4.2.10)$$

provided $|X|<|Y|<1$.

(v) Now, setting

$$\alpha_r=\frac{(1-AP^rQ^r)[A;P]_r[C;Q]_rC^{-r}}{(1-A)[Q;Q]_r\left[\frac{AP}{C};P\right]_r}$$

and $u_r=v_r=1$

in (4.1.1) we get :

$$\beta_n=\sum_{r=0}^{n}\frac{(1-AP^rQ^r)[A;P]_r[C;Q]_rC^{-r}}{(1-A)[Q;Q]_r\left[\frac{AP}{C};P\right]_r}$$

Now, using (4.1.8) to sum the above series on the right hand side, we get :

$$\beta_n=\frac{[AP;P]_n[CQ;Q]_nC^{-n}}{[Q;Q]_n\left[\frac{AP}{C};P\right]_n}$$

Substituting these values of α_n and β_n in Master Result (4.2.3) we get :

$${}_5\Phi_4\left[\begin{matrix}C: & A: & APQ: & e, & \frac{abq^2}{e}; & Q, & P, & PQ, & q; & \frac{1}{Cq}\\ \frac{AP}{C}: & & A: & \frac{e}{q}, & \frac{abq}{e} & & & & & \end{matrix}\right]$$

$$\frac{[a,b;q]_\infty}{\left[e,\frac{abq}{e};q\right]_\infty} {}_5\Phi_4\left[\begin{matrix} C: & A: & APQ: & e, & \frac{abq^2}{e}; & Q, & P; & PQ; & q; & \frac{1}{Cq} \\ & \frac{AP}{C}: & A: & a, & b & & & & & \end{matrix}\right]$$

$$=\frac{\left(1-\frac{aq}{e}\right)\left(1-\frac{bq}{e}\right)}{\left(1-\frac{q}{e}\right)\left(1-\frac{abq}{e}\right)} {}_4\Phi_3\left[\begin{matrix} CQ: & AP: & a, & b; & Q, & P, & q; & \frac{Q}{c} \\ : & \frac{AP}{C}: & e, & \frac{abq^2}{e} & & & & \end{matrix}\right], \quad (4.2.12)$$

(vi) Now, taking

$$\alpha_r = \frac{(1-Ap^r q_1^r)(1-Bp^r q_1^{-r})[A,B;P]_r\left[C,\frac{A}{BC};q_1\right]_r q_1^r}{(1-A)(1-B)\left[q_1,\frac{Aq_1}{B};q_1\right]_r\left[\frac{Ap}{C},BCp;p\right]_r}$$

and $u_r = v_r = 1$

in (4.1.1) we get :

$$\beta_n = \sum_{r=0}^{n}\frac{(1-Ap^r q_1^r)(1-Bp^r q_1^{-r})[A,B;P]_r\left[C,\frac{A}{BC};q_1\right]_r q_1^r}{(1-A)(1-B)\left[q_1,\frac{Aq_1}{B};q_1\right]_r\left[\frac{Ap}{C},BCp;p\right]_r}$$

Now using (4.1.9) to sum the above series on the right hand side, we get :

$$\beta_n = \frac{[Ap,Bp;p]_n\left[Cq_1,\frac{Aq_1}{BC};q_1\right]_n}{\left[q_1,\frac{Aq_1}{B};q_1\right]_n\left[\frac{Ap}{C};BCp;p\right]_n} \quad (4.2.13)$$

Substituting these values of α_n and β_n in Master Result (4.2.3) we get :

$$
{}_8\Phi_7\left[\begin{matrix} C, & \frac{A}{BC}: & A, & B: & Apq_1: & \frac{Bp}{q_1}; & e, & \frac{abq^2}{e}; & q_1, p, pq_1; & \frac{p}{q_1}; \frac{q_1}{q} \\ & \frac{Aq_1}{C}: & \frac{Ap}{C}, & BCp: & A: & B: & \frac{e}{q}, & \frac{abq}{e} & & \end{matrix}\right]
$$

$$
\frac{[a,b;q]_\infty}{\left[\frac{e}{q},\frac{abq}{e};q\right]_\infty}
$$

$$
{}_8\Phi_7\left[\begin{matrix} C, & \frac{A}{BC}: & A, & B: & Apq_1: & \frac{Bp}{q_1}: & e, & \frac{abq^2}{e}; & q_1, & p, & pq_1, & \frac{p}{q_1}; & \frac{q_1}{q} \\ & \frac{Aq_1}{C}: & \frac{Ap}{C}, & BCp: & A: & B: & a, & b & & & & & \end{matrix}\right]
$$

$$
=\frac{\left(1-\frac{aq}{e}\right)\left(1-\frac{bq}{e}\right)}{\left(1-\frac{q}{e}\right)\left(1-\frac{abq}{e}\right)}\times
$$

$$
\times{}_6\Phi_5\left[\begin{matrix} Cq_1, & \frac{Aq_1}{BC}: & Ap, & Bp: & a, & b; & q_1, & p, & q; & q \\ & \frac{Aq_1}{B}: & \frac{Ap}{C}, & BCp: & e, & \frac{abq^2}{e} & & & & \end{matrix}\right] \quad (4.2.14)
$$

(viii) Now, Setting

$$
\alpha_r=\frac{(1-ADp^rq_1^r)\left(1-\frac{B}{D}p^rq_1^{-r}\right)[A,B;P]_r\left[C,\frac{AD^2}{BC};q_1\right]_r q_1^r}{(1-A)\left(1-\frac{B}{D}\right)\left[Dq_1,\frac{ADq_1}{B};q_1\right]_r\left[\frac{ADp}{C},\frac{BCp}{D};p\right]_r}
$$

and $u_r = v_r = 1$

in (4.1.1) we get:

$$\beta_n = \sum_{r=0}^{n} \frac{\left(1-ADp^r q_1^r\right)\left(1-\frac{B}{D}p^r q_1^{-r}\right)[A,B;P]_r\left[C,\frac{AD^2}{BC};q_1\right]_r q_1^r}{(1-A)\left(1-\frac{B}{D}\right)\left[Dq_1,\frac{ADq_1}{B};q_1\right]_r\left[\frac{ADp}{C},\frac{BCp}{D};p\right]_r}$$

Now, using (4.1.10) to sum the above series on the right hand side, we get :

$$\beta_n = \frac{(1-A)(1-B)(1-C)\left(1-\frac{AD^2}{BC}\right)}{D(1-AD)\left(1-\frac{B}{D}\right)\left(1-\frac{C}{D}\right)\left(1-\frac{AD}{BC}\right)} \times$$

$$\times\left\{\frac{[Ap,Bp;p]_n\left[Cq_1,\frac{AD^2}{BC}q_1;q_1\right]_n}{\left[Dq_1,\frac{ADq_1}{B};q_1\right]_n\left[\frac{ADp}{C},\frac{BCp}{D};p\right]_n}\right.$$

$$\left.-\frac{(B-AD)(C-AD)(D-BC)(1-D)}{D(1-A)(1-B)(1-C)\left(BC-AD^2\right)}\right\} \qquad (4.2.15)$$

Substituting these values of α_n and β_n in Master Result (4.2.3), we get :

$${}_8\Phi_8\left[\begin{matrix} A, & B: & C, & \frac{AD^2}{BC}: & ADpq_1: & \frac{BP}{Dq_1}: & e, \\ \frac{ADp}{C}, & \frac{BCp}{D}: & Dq_1, & \frac{ADq_1}{B}: & AD: & \frac{B}{D}: & \frac{e}{q} \end{matrix}\right.$$

$$\left.\begin{matrix} \frac{abq^2}{e}; & p, & q_1, & pq_1, & \frac{p}{q_1}, & q: & \frac{q_1}{q} \\ \frac{abq}{e} & & & & & & \end{matrix}\right]$$

$$-\frac{[a,b;q]_\infty}{\left[\frac{e}{q},\frac{abq}{e};q\right]_\infty}\times {}_8\Phi_8\left[\begin{matrix} A, & B: & C, & \frac{AD^2}{BC}: & ADpq_1: & \frac{BP}{Dq_1}: & e, & \frac{abq^2}{e}; & p, & q_1, & pq_1, & \frac{p}{q_1}, & q; & \frac{q_1}{q} \\ \frac{ADp}{C}, & \frac{BCp}{D}: & Dq_1, & \frac{ADq_1}{B}: & AD: & \frac{B}{D}: & a, & b & & & & & & \end{matrix}\right]$$

$$=\frac{\left(1-\frac{aq}{e}\right)\left(1-\frac{bq}{e}\right)(1-A)(1-B)(1-C)\left(1-\frac{AD^2}{BC}\right)}{\left(1-\frac{q}{e}\right)\left(1-\frac{abq}{e}\right)D(1-AD)\left(1-\frac{B}{D}\right)\left(1-\frac{C}{D}\right)\left(1-\frac{AD}{BC}\right)}\times$$

$$\times\left\{{}_6\Phi_6\left[\begin{matrix} Ap, & Bp: & Cq_1, & \frac{AD^2}{BC}q_1: & a, & b; & p,q_1, & q;q \\ \frac{AD}{C}p, & \frac{BC}{D}p: & Dq_1: & \frac{AD}{B}q_1: & e, & \frac{abq^2}{e} & & \end{matrix}\right]\right.$$

$$\left.-\frac{(B-AD)(C-AD)(D-BC)(1-D)}{D(1-A)(1-B)(1-C)\left(BC-AD^2\right)}\times {}_3\Phi_2\left[\begin{matrix} a, & b, & q; & q; & q \\ & e, & \frac{abq^2}{e} & & \end{matrix}\right]\right\}. \quad (4.2.16)$$

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